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Theory of the $\mathscr{L}^{(p,\lambda)}$ spaces of strong type and its applications to elliptic partial differential equations

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Introduction

The $\mathscr{L}^{(p,\lambda)}$ spaces were first studied by C. B. Morrey [15] and afterwards by F. John and L. Nirenberg [12]. Motivated by these researches, S. Campanato, G. Stampacchia and others have given general definition of these spaces and the structures of them have been characterized by them, G. N. Meyers [14] and others (see [27]-[30] for bibliography). The elliptic partial differential equations in these spaces were first studied by C. B. Morrey [15] (see also [16]) and afterwards mainly by S. Campanato [4]-[10].

On the other hand, G. Stampacchia has introduced in [29] the $\mathscr{L}^{(p,\lambda)}$ spaces of strong type the structures of which are far more general and complicated than those of the $\mathscr{L}^{(p,\lambda)}$ spaces mentioned above and greater part of them have been characterized by him, L. C. Piccinini, A. Ono, Y. Furusho and others (see [11], [18]-[22], [24], [26], [28] and [29]). The elliptic partial differential equations in these spaces have been studied by A. Ono in [23] and [25].

In this paper, we shall describe at first the theory of the $\mathscr{L}^{(p,\lambda)}$ spaces of strong type, namely, isomorphism theorems, Morrey-Sobolev type imbedding theorems and inclusion property and secondly interior and boundary estimates for elliptic partial differential equations in these spaces.

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Capter I The theory of the $\mathscr{L}^{(p,\lambda)}$ spaces of strong type

1. Relevant definitions

We denote a fixed bounded closed subdomain of Euclidean n-space E^n with sufficiently smooth boundary by $\bar{D}$. And we denote a subcube of $\bar{D}$ with sides parallel to the axes by Q (from now on, "subcube" means such a subcube without loss of generality). We consider subfamilies of integrable functions on $\bar{D}$ and denote the mean value of function $v(x)=v(x_1, \cdots, x_n)$ on subcube Q by v_Q : $v_Q=|Q|^{-1}\int_Q v(x)\,dx$.

DEFINITION 1. A function $v\in L^p(\bar{D})$ is said to belong to the space $\mathscr{L}_r^{(p,\lambda)}(\bar{D})$ (the $\mathscr{L}^{(p,\lambda)}$ space of strong type r) if the following estimate holds for v :

$$K(Q_j)=[v]_{\mathscr{L}^{(p,\lambda)}(Q_j)}=\sup_{Q\subset Q_j}(|Q|^{\frac{\lambda}{n}-1}\int_Q |v(x)-v_Q|^p dx)^{\frac{1}{p}}<\infty$$

and furthermore, there exists a constant $L=L(v)$ such that

$$\sup_{\{Q_j\}\in\bar{S}}\Big\{\sum_j K^r(Q_j)\Big\}^{\frac{1}{r}}=L$$

where $1\leqq p<\infty$, $-p<\lambda\leqq n$, $1\leqq r<\infty$, $\frac{n}{r}-\frac{\lambda}{p}\leqq 1$ [1)] and $\bar{S}$ is the family of all systems of finite number subcubes of $\bar{D}$ no two of which have common interior point.

By taking as a norm of the space $\mathscr{L}_r^{(p,\lambda)}(\bar{D})$

$$\|v\|_{\mathscr{L}_r^{(p,\lambda)}(\bar{D})}=[v]_{\mathscr{L}_r^{(p,\lambda)}(\bar{D})}+\|v\|_{L^p(\bar{D})}=L+\|v\|_{L^p(\bar{D})}$$

the space $\mathscr{L}_r^{(p,\lambda)}(\bar{D})$ is a Banach space.

As special cases of this definition, we state the following :

DEFINITION 2. The space $\mathscr{L}_r^{(p,0)}(\bar{D})$ is isomorphic to the space $\mathscr{L}_r^{(1,0)}(\bar{D})$ with their corresponding norms, where $1<p<\infty$ and $n\leqq r<\infty$[12]. Therefore, we call the space $\mathscr{L}_r^{(1,0)}(\bar{D})$ the John-Nirenberg space of strong type r.

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1) If $\frac{n}{r}-\frac{\lambda}{p}>1$, then the space $\mathscr{L}_r^{(p,\lambda)}(\bar{D})$ consists of only constant functions. Therefore, we exclude the trivial case.

DEFINITION 3. In the case of $1 \leqq p < \infty$, $-p < \lambda < 0$, $0 < \alpha = -\frac{\lambda}{p} < 1$ and $\frac{n}{1-\alpha} \leqq r < \infty$, the seminorm $[v]_{\mathcal{L}_r^{(p,\lambda)}(\bar{D})}$ is equivalent to the seminorm

$$[v]_{\mathcal{H}_r^\alpha(\bar{D})} = \sup_{\{Q_j\} \in \bar{S}} \left\{ \sum_j [v]^r_{C^\alpha(Q_j)} \right\}^{\frac{1}{r}}$$

by [2] and [14], and therefore we call the space $\mathcal{H}_r^\alpha(\bar{D})$ the Hölder space with exponent α of strong type r. Similarly, we can define the space $\mathcal{H}_r^{l+\alpha}(\bar{D})$ (l : natural number).

2. Isomorphism theorems

We state the following :

DEFINITION 4. The Sobolev space $H^{l,p}(\bar{D})$ is the completion of the space $C^l(\bar{D})$ with respect to the norm

$$\|v\|_{H^{l,p}(\bar{D})} = \sum_{|\beta| \leqq l} \|D^\beta v\|_{L^p(\bar{D})}$$

where l is a natural number and $1 < p < \infty$.

Now, Stampacchia has proved in [29] the following :

THEOREM 1.1. *The space $\mathcal{H}^\alpha_{\frac{n}{1-\alpha}}(\bar{D})$ is isomorphic to the Sobolev space $H^{1,\frac{n}{1-\alpha}}(\bar{D})$ with their corresponding norms, where $0 < \alpha < 1$.*

This theorem is generalized in [18], [19] and we have obtained the most general result in [21] concerning these spaces. To state this result, we make the following :

DEFINITION 5. A function $v \in L^p(\bar{D})$ is said to belong to the space Lip $(a, p, \bar{D})$ (the Lipschitz space of order a in the space $L^p(\bar{D})$, where $0 < a \leqq 1$ and $1 < p < \infty$), if the following estimate holds for v:

$$[v]_{\mathrm{Lip}(a, p, \bar{D})} = \sup_{x,\, x+h \in \bar{D}} |h|^{-a} \left\{ \int_{\bar{D}} |v(x+h) - v(x)|^p dx \right\}^{\frac{1}{p}} < \infty$$

We define a norm of this space by

$$[v]_{\mathrm{Lip}(a, p, \bar{D})} + \|v\|_{L^p(\bar{D})}$$

endowed with this norm the space $\mathrm{Lip}(a, p, \bar{D})$ is a Banach space. Furthermore, we remark that the space $\mathrm{Lip}\,(1, p, \bar{D})$ is isomorphic to the Sobolev space $H^{1,p}(\bar{D})$ with their corresponding norms.

Now, our result obtained in [21] is the following :

THEOREM 1.2. *The space* $\mathcal{L}_r^{(p,\lambda)}(\bar{D})$ *is isomorphic to the space* $Lip\,(\frac{n}{r}-\frac{\lambda}{p}, r, \bar{D})$ *and we have*

$$C^{-1}\|v\|_{\mathcal{L}_r^{(p,\lambda)}(\bar{D})} \leqq \|v\|_{\mathrm{Lip}(\frac{n}{r}-\frac{\lambda}{p}, r, \bar{D})} \leqq C\|v\|_{\mathcal{L}_r^{(p,\lambda)}(\bar{D})} \tag{1.1}$$

where $1<p<\infty$, $-p<\lambda\leqq n$, $1<r<\infty$, $0<\frac{n}{r}-\frac{\lambda}{p}\leqq 1$ *and* C *is a positive constant independent of the function* v.[2)]

On the other hand , as direct conclusions of John, Nirenberg [12] and Campanato, Meyers [2], [14] we know the following :

THEOREM 1.3.

1. *The space* $\mathcal{L}_r^{(p,0)}(\bar{D})$ *is isomorphic to the space* $\mathcal{L}_r^{(1,0)}(\bar{D})=\mathcal{L}_r^{(1,\frac{0}{p})}(\bar{D})$ *with their corresponding norms, where* $1<p<\infty$ *and* $n\leqq r\leqq\infty$.

2. *The space* $\mathcal{L}_r^{(p,\lambda)}(\bar{D})$ *is isomorphic to the space* $\mathcal{H}_r^{(-\frac{\lambda}{p})}(\bar{D})\cong\mathcal{L}_r^{(1,\frac{\lambda}{p})}(\bar{D})$ *with their corresponding norms, where* $1<p<\infty$, $-p<\lambda<0$ *and*

$$\frac{n}{1+\frac{\lambda}{p}}\leqq r<\infty.$$

Now, we have proved in [26] similar result as this theorem. Namely,

THEOREM 1.4 *The space* $\mathcal{L}_r^{(p,\lambda)}(\bar{D})$ *is isomorphic to the space* $\mathcal{L}_r^{(1,\frac{\lambda}{p})}(\bar{D})$ *and we have.*

$$C^{-1}\|v\|_{\mathcal{L}_r^{(q,\lambda)}(\bar{D})} \leqq \|v\|_{\mathcal{L}_r^{(1,\frac{\lambda}{p})}(\bar{D})} \leqq C\|v\|_{\mathcal{L}_r^{(p,\lambda)}(\bar{D})} \tag{1.2}$$

where $1<p<\infty$, $0<\lambda<n$, $1<r<\infty$, *and* $0\leqq\frac{n}{r}-\frac{\lambda}{p}\leqq 1$.

2) Throughout this paper, we always denote by C the positive constants independent of functions u, v, w, f, g etc. appearing in the inequalities.

3. Morrey-Sobolev type imbedding theorems

Stampacchia has proved in [29] the following :

THEOREM 1.5 *Let v be a function such that the derivatives $\{v_x\}$ belong to the space $\mathscr{L}^{(p,\lambda)}(\bar{D})=\mathscr{L}_{\infty}^{(p,\lambda)}(\bar{D})$ (for the definition, see Definition 1), where $1<p<\infty$ and $0<\lambda<n$.*

Then, the following estimates hold for v.

1. *$p<\lambda$: v belongs to the space $\mathscr{L}^{(\tilde{q},\mu)}(\bar{D})$ and we have*

$$(1.3)\quad [v]_{\mathscr{L}^{(\tilde{q},\mu)}(\bar{D})}\leqq C\|v_x\|_{\mathscr{L}^{(p,\lambda)}(\bar{D})}$$

where q, μ are arbitrary constants satisfying $1<q<p$, $\frac{\mu}{q}=\frac{\lambda}{p}$ and $\frac{1}{\tilde{q}}=\frac{1}{q}-\frac{1}{\mu}$.[3)]

2. *$p=\lambda$: v belongs to the space $\mathscr{L}^{(1,0)}(\bar{D})$ and we have*

$$(1.4)\quad [v]_{\mathscr{L}^{(1,0)}(\bar{D})}\leqq \text{the right hand side of (1.3)}$$

3. *$p>\lambda$: v belongs to the space $C^{1-\frac{\lambda}{p}}(\bar{D})$ and we have*

$$(1.5)\quad [v]_{C^{1-\frac{\lambda}{p}}(\bar{D})}\leqq \text{the right hand side of (1.3)}$$

Concerning the $\mathscr{L}^{(p,\lambda)}$ spaces of strong type, we have proved in [21], [22] and [24] (see also [20] and [26]) the following :

THEOREM 1.6. *Let v be a function such that the derivatives $\{v_x\}$ belong to the space $\mathscr{L}_r^{(p,\lambda)}(\bar{D})$, where $1<p<\infty$, $0<\lambda<n$, $1<r<\infty$ and $\frac{n}{r}-\frac{\lambda}{p}<1$.*

Then, the following estimates hold for v.

1. *$p<\lambda$ and*

(1) *$r\leqq\frac{n}{\lambda}p$; v belongs to the space $\mathscr{L}_s^{(\tilde{p},\lambda)}(\bar{D})$ and we have*

$$(1.6)\quad [v]_{\mathscr{L}_s^{(\tilde{p},\lambda)}(\bar{D})}\leqq C\|v_x\|_{\mathscr{L}_r^{(p,\lambda)}(\bar{D})}$$

where s is an arbitrary constant greater than $\frac{n}{\lambda}p$.

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3) Throughout this paper, we always denote by $\tilde{p}$, $\tilde{q}$, where $\frac{1}{\tilde{p}}=\frac{1}{p}-\frac{1}{\lambda}$, $\frac{1}{\tilde{q}}=\frac{1}{q}-\frac{1}{\mu}$ in Chapter I and $\frac{1}{\tilde{q}}=\frac{1}{q}-\frac{1}{\lambda}$ in Chapter II respectively.

(2) $\frac{n}{\lambda}p<r\leqq\frac{n}{\lambda}\tilde{p}$; *v belongs to the space* $\mathscr{L}_r^{(\tilde{p},\lambda)}(\bar{D})$ *and we have*

(1.7) $[v]_{\mathscr{L}_r^{(\tilde{p},\lambda)}(\bar{D})}\leqq$ the right hand side of (1.6)

(3) $\frac{n}{\lambda}\tilde{p}<r<\infty$; *v belongs to the space* $\mathscr{L}_r^{(\tilde{q},\mu)}(\bar{D})$ *and we have*

(1.8) $[v]_{\mathscr{L}_r^{(\tilde{q},\mu)}(\bar{D})}\leqq$ the right hand side of (1.6)

where q, μ *are arbitrary constants satisfying* $1<q<p$ *and* $\frac{\mu}{q}=\frac{\lambda}{p}$.

2. $p=\lambda$ *and*

(1) $\frac{n}{2}<r\leqq n$; *v belongs to the space* $\mathscr{L}_s^{(1,0)}(\bar{D})$ *and we have*

(1.9) $[v]_{\mathscr{L}_s^{(1,0)}(\bar{D})}\leqq$ the right hand side of (1.6)

where s is as in 1 (1).

(2) $n<r<\infty$; *v belongs to the space* $\mathscr{L}_r^{(1,0)}(\bar{D})$ *and we have*

(1.10) $[v]_{\mathscr{L}_r^{(1,0)}(\bar{D})}\leqq$ the right hand side of (1.6)

3. $p>\lambda$ *and*

(1) $r\leqq\frac{n}{\lambda}p$; *v belongs to the space* $\mathscr{H}_s^{1-\frac{\lambda}{p}}(\bar{D})$ *and we have*

(1.11) $[v]_{\mathscr{H}_s^{1-\frac{\lambda}{p}}(\bar{D})}\leqq$ the right hand side of (1.6)

where s is as in 1 (1).

(2) $\frac{n}{\lambda}p<r<\infty$; *v belongs to the space* $\mathscr{H}_r^{1-\frac{\lambda}{p}}(\bar{D})$ *and we have*

(1.12) $[v]_{\mathscr{H}_r^{1-\frac{\lambda}{p}}(\bar{D})}\leqq$ the right hand side of (1.6)

4. Inclusion property

A necessary and sufficient condition for the inclusion property between the spaces $L_r^{(p,\lambda)}(\bar{D})$ and $L_s^{(q,\mu)}(\bar{D})$ (the Morrey spaces of strong type, see [28] for the definition) has been proved by Piccinini.

On the other hand, a necessary and sufficient condition for inclusion property between the spaces $\mathscr{L}_r^{(p,\lambda)}(\bar{D})$ and $\mathscr{L}_s^{(q,\mu)}(\bar{D})$ is more complicated.

We have proved in [26] the following :

THEOREM 1.7. *A necessary and sufficient condition that the following relation holds*

(1.13) $\mathscr{L}_r^{(p,\lambda)}(\bar{D})\subset\mathscr{L}_s^{(q,\mu)}(\bar{D})$

and furthermore,

(1.14) $\|v\|_{\mathscr{L}_s^{(q,\mu)}(\bar{D})}\leqq C\|v\|_{\mathscr{L}_r^{(p,\lambda)}(\bar{D})}$

is that the following inequalities hold :

(1.15) $p\geqq q,\ \frac{\lambda}{p}\leqq\frac{\mu}{q}$ and $\frac{1}{s}\leqq\frac{1}{r}+\frac{1}{n}\left(\frac{\mu}{q}-\frac{\lambda}{p}\right)$

where $1<p,\ q<\infty, -p<\lambda<n,\ -q<\mu<n$ *and* $0\leqq\frac{n}{r}-\frac{\lambda}{p}\leqq 1.$

For the remainder of this theorem Furusho has proved in [11] the following result with the aid of Piccinini's technique :

THEOREM 1.8. *If* $\frac{n}{\lambda}p<r$ *and furthermore* (1.13), (1.14) *hold, then* $\frac{n}{\mu}q\leqq s$, *where* $p,\ q,\ r,\ s$ *are as in Theorem* 1.7 *and* $0<\lambda,\ \mu<n$.

Then, necessary and sufficient conditions that (1.13), (1.14) *hold are as follow* :

1. $\frac{n}{\mu}q<s$

(1.16) $p\geqq q,\ \frac{\lambda}{p}\leqq\frac{\mu}{q}$ and $\frac{1}{s}\left(1-\frac{\lambda}{n}\right)\leqq\frac{1}{r}\left(1-\frac{p}{q}\frac{\mu}{n}\right)+\frac{1}{n}\left(\frac{\mu}{q}-\frac{\lambda}{p}\right)$

2. $s=\frac{n}{\mu}q$

(1.17) $\frac{\lambda}{p}\leqq\frac{\mu}{q}$ and $\frac{1}{s}\left(1-\frac{\lambda}{n}\right)\leqq\frac{1}{r}\left(1-\frac{p}{q}\frac{\mu}{n}\right)+\frac{1}{n}\left(\frac{\mu}{q}-\frac{\lambda}{p}\right)$

Chapter II Applications to elliptic partial differential equations

Mainly Campanato has treated elliptic partial differential equations in the $\mathscr{L}^{(q,\lambda)}$ spaces [4]-[10] and deduced existence theorems and interior and boundary estimates for the solutions.

We have also deduced interior and boundary estimates for elliptic partial differential equations in the $\mathscr{L}^{(q,\lambda)}$ spaces of strong type in [23] and [25] respectively.

5. Interior estimates

We denote by S_R and C_R a fixed semisphere with radius R in the Euclidean n-space E^n and its flat boundary respectively. Namely,

$$S_R=\{x=(x_1, \cdots, x_n)\ ;\ |x|\leqq R,\ x_n\geqq 0\} \text{ and } C_R=S_R|x_n=0$$

Furthermore, we denote by $\bar{D}$ a fixed closed domain with boundary included in the interior of S_R ; $\overset{\circ}{S}_R$ and sufficiently smooth.

1. Strong Schauder estimates

We consider the following elliptic partial differential equations in the space $\mathscr{L}_p^{(q,\lambda)}(\bar{D})$, where $1<p,\ q<\infty,\ -q<\lambda<n$ and $0<a=\dfrac{n}{p}-\dfrac{\lambda}{q}<1.$

$$\text{(E)}\quad Lu=\sum_{|\beta|\leq 2m} a_\beta(x)D^\beta u=f(x)\quad x\in\overset{\circ}{S}_R$$

or of variational form :

$$\text{(E)}'\quad Lu=\sum_{\substack{|\beta|\leq l\\ |\gamma|\leq 2m-l}} D^\gamma[a_{\beta\gamma}(x)D^\beta u]=\sum_{|\gamma|\leq 2m-l} D^\gamma f_\gamma(x)\quad x\in\overset{\circ}{S}_R\quad l<2m$$

under the following fundamental assumptions :

(A)$_{\mathscr{H}_p^a}$I, (A)$'_{\mathscr{H}_p^a}$I Uniform ellipticity made in [1].

(A)$_{\mathscr{H}_p^a}$II The coefficients and function $\{a_\beta, f\}$ belong to the space $\mathscr{H}_p^{l-2m+\alpha}(S_R)$, where $l\geqq 2m$ and $0<\alpha<1$.

(A)$'_{\mathscr{H}_p^a}$II The coefficients and functions $\{a_{\beta\gamma}, f_\gamma\}$ belong to the space $\mathscr{H}_p^\alpha(S_R)$, where $0<\alpha<1$.

Then, we have proved the following :

THEOREM 2.1. *Let u be an $H^{2m,p}$-(or $H^{l,p}$-) solution of the equation* (E) (*or* (E)$'$) *under the assumptions* (A)$_{\mathscr{H}_p^\alpha}$ (*or* (A)$'_{\mathscr{H}_p^\alpha}$).

Then, in fact u belongs to the space $\mathscr{H}_p^{l+\alpha}(\bar{D})$ and we have

(2.1) $\|u\|_{\mathscr{H}_p^{l+a}(\bar{D})}$

$$\leqq C\{\|f\|_{\mathscr{H}_p^{l-2m+a}(\bar{D})}+\|u\|_{L^p(\bar{D})}\} \quad \text{for (E)}$$

or

$$\leqq C\{\sum_{|\gamma|\leqq 2m-l}\|f_\gamma\|_{\mathscr{H}_p^a(\bar{D})}+\|u\|_{L^p(\bar{D})}\} \quad \text{for (E)}'$$

2. Strong $\mathscr{L}^{(q,\lambda)}$ estimates

We make the following assumptions in place of $(A)_{\mathscr{H}_p^a}$ and $(A)'_{\mathscr{H}_p^a}$:

$(A)_{\mathscr{L}_p^{(q,\lambda)}}$ I, $(A)'_{\mathscr{L}_p^{(q,\lambda)}}$ I same as $(A)_{\mathscr{H}_p^a}$ I, $(A)'_{\mathscr{H}_p^a}$ I respectively.

$(A)_{\mathscr{L}_p^{(q,\lambda)}}$ II The coefficients and function $\{a_\beta, f\}$ are smooth. Namely, we assume

$$a_\beta\in C^{l-2m+a}(S_R),\ D^{l-2m}f\in\mathscr{L}_p^{(q,\lambda)}(S_R)$$

where $l\geqq 2m$, $0<\lambda<n$ and $0<a=\dfrac{n}{p}-\dfrac{\lambda}{q}<1$.

$(A)'_{\mathscr{L}_p^{(q,\lambda)}}$ II The coefficients and functions $\{a_{\beta\gamma}, f_\gamma\}$ are smooth. Namely, we assume

$$a_{\beta\gamma}\in C^a(S_R),\ f_\gamma\in\mathscr{L}_p^{(q,\lambda)}(S_R)$$

where λ, a are as in $(A)_{\mathscr{L}_p^{(q,\lambda)}}$II.

Now, our results for the strong $\mathscr{L}^{(q,\lambda)}$ estimates read as follow :

THEOREM 2.2 *Let u be an $H^{2m,p}$-(or $H^{l,p}$-) solution of the equation* (E) (*or* (E)′) *under the assumptions* $(A)_{\mathscr{L}_p^{(q,\lambda)}}$ (*or* $(A)'_{\mathscr{L}_p^{(q,\lambda)}}$).

Then, in fact the derivatives $\{D^\beta u\}_{|\beta|\leqq l}$ belong to the space $\mathscr{L}_p^{(q,\lambda)}(\bar{D})$ and we have

(2.2) $\sum_{|\beta|\leqq l}\|D^\beta u\|_{\mathscr{L}_p^{(q,\lambda)}(\bar{D})}$

$$\leqq C\{\sum_{|\gamma|\leqq l-2m}\|D^\gamma f\|_{\mathscr{L}_p^{(q,\lambda)}(\bar{D})}+\|u\|_{L^p(\bar{D})}\} \quad \text{for (E)}$$

or

$$\leqq C\{\sum_{|\gamma|\leqq 2m-l}\|f_\gamma\|_{\mathscr{L}_p^{(q,\lambda)}(\bar{D})}+\|u\|_{L^p(\bar{D})}\} \quad \text{for (E)}'$$

Here, we make the following important :

Remark 1. Although the norm of the space $\mathscr{L}_p^{(q,\lambda)}(\bar{D})$ is $[v]_{\mathscr{L}_p^{(q,\lambda)}(\bar{D})}+\|v\|_{L^q(\bar{D})}$, the space $\mathscr{L}_p^{(q,\lambda)}(\bar{D})$ is isomorphic to the space $\mathrm{Lip}\left(\frac{n}{p}-\frac{\lambda}{q}, p, \bar{D}\right)$ with their corresponding norms by Theorem 1.2 which is a subspace of $L^p(\bar{D})$. Therefore, we may always consider the equations in the L^p-framework.

6. Boundary estimates

1. Strong Schauder estimates

As was mentioned in 1, we denote by S_R and C_R a fixed semisphere with radius R in the Euclidean n-space E^n and its flat boundary respectively.

Now, we consider the following boundary value problems :

$$(\mathrm{E})-(\mathrm{B})\quad \begin{cases} Lu=\sum_{|\beta|\leq 2m} a_\beta(x)D^\beta u=f(x) & x\in S_R \\ B_k u=\sum_{|\beta'|\leq mk} b_{k,\beta'}(x')D^{\beta'}u\,|_{x_n=0}=g_k(x') & x'\in C_R \quad k=1,\cdots,m \end{cases}$$

or of variational form

$$(\mathrm{E})'-(\mathrm{B})'\quad \begin{cases} Lu=\sum_{\substack{|\beta|\leq l\\ |\gamma|\leq 2m-l}} D^\gamma[a_{\beta\gamma}(x)D^\beta u]=\sum_{|\gamma|\leq 2m-l} D^\gamma f_\gamma(x) \quad x\in S_R \\ B_k u=\sum_{\substack{|\beta'|\leq l-1\\ |\gamma'|\leq mk-l+1}} D_{x'}^{\gamma'}[b_{k,\beta',\gamma'}(x')D^{\beta'}u]\,|_{x_n=0}=\sum_{|\gamma'|\leq mk-l+1} D_{x'}^{\gamma'}g_{k,\gamma'}(x') \\ \qquad\qquad x'\in C_R \quad k=1,\cdots,m \end{cases}$$

where $l<\mathrm{Min}(2m, m_k+1)$ and greater than the maximum order of x_n differentiation occuring in B_k.

Now, we make the following fundamental assumptions on the problems (E)$-$(B) and (E)$'-$(B)$'$.

(A)$_{\mathscr{H}_p^\alpha}$ I , II ; (A)$'_{\mathscr{H}_p^\alpha}$ I , II Uniform ellipticity and complemeting condition on boundary operators made in [1].

(A)$_{\mathscr{H}_p^\alpha}$III The coefficients and functions $\{a_\beta, f\}$; $\{b_{k,\beta'}, g_k\}$ belong to the spaces $\mathscr{H}_p^{l-2m+\alpha}(S_R)$; $\mathscr{H}_p^{l-mk+\alpha}(C_R)$ respectively, where $l\geq l_1=\mathrm{Max}(2m, m_k+1)$,

$0<\alpha<1$ and $\dfrac{n}{1-\alpha}\leqq p<\infty$.

(A)$'_{\mathscr{H}_p^\alpha}$III The coefficients and functions $\{a_{\beta\gamma}, f_\gamma\}$; $\{b_{k,\beta',\gamma'}, g_{k,\beta'}\}$ belong to the space $\mathscr{H}_p^\alpha(S_R)$; $\mathscr{H}_p^{1+\alpha}(C_R)$ respectively, where α, p are as in (A)$_{\mathscr{H}_p^\alpha}$III.

Now, we have needed the characterization of the trace operator on certain $\mathscr{L}^{(q,\lambda)}$ spaces of strong type to deduce the boundary estimates. Therefore, we have proved in [25] with the aid of Nikol'skii's theorem [17] (see also [13] Kufner et al.) the following :

THEOREM 2.3.

1. *Let v be a function belonging to the space $\mathscr{L}_p^{\{q,\lambda)}(S_R)$.*

Then, there exists a function w such that belongs to the space $\mathscr{L}_p^{\{q,\lambda)}(C_R)$ and $v|_{x_n=0}=w$. Furthermore, we have

$$(2.3)\quad \|w\|_{\mathscr{L}_p^{\{q,\lambda)}(C_R)}\leqq C\|v\|_{\mathscr{L}_p^{\{q,\lambda)}(S_R)}$$

where $1<p$, $q<\infty$, $-q<\lambda<n$ and $\dfrac{1}{p}<a=\dfrac{n}{p}-\dfrac{\lambda}{q}<1$.

2. *Conversely, let w be a function belonging to the space $\mathscr{L}_p^{\{q,\lambda)}(C_R)$.*

Then, there exists a function v such that belongs to the space $\mathscr{L}_p^{\{q,\lambda)}(S_R)$ and $v|_{x_n=0}=w$. Furthermore, we can extend the function w so that the following estimate holds :

$$(2.4)\quad \|v\|_{\mathscr{L}_p^{\{q,\lambda)}(S_R)}\leqq C\|w\|_{\mathscr{L}_p^{\{q,\lambda)}(C_R)}$$

where p, q, λ and a are as in 1.

Applying this theorem for $-q<\lambda<0$, we have proved the following strong Schauder estimates :

THEOREM 2.4. *Let u be a $C^{l_1+\alpha}$-(or $C^{l+\alpha}$-) solution of the problem* (E)$-$(B) *(or* (E)$'-$(B)$'$) *vanishing near the curved boundary of S_R under the assumptions* (A)$_{\mathscr{H}_p^\alpha}$ *(or* (A)$'_{\mathscr{H}_p^\alpha}$).

Then, in fact u belongs to the space $\mathscr{H}_p^{l+\alpha}(S_R)$ and the following estimates hold for u :

(2.5) $\|u\|_{\mathscr{H}_p^{l+a}(S_R)}$

$$\leqq C\{\|f\|_{\mathscr{H}_p^{l-2m+a}(S_R)}+\sum_{k=1}^{m}\|g_k\|_{\mathscr{H}_p^{l-m_k+a}(C_R)}+\|u\|_{L^p(S_R)}\} \quad \text{for (E)}-\text{(B)}$$

or

$$\leqq C\{\sum_{|\gamma|\leqq 2m-l}\|f_\gamma\|_{\mathscr{H}_p^a(S_R)}+\sum_{k=1}^{m}\sum_{|\gamma'|\leqq m_k-l+1}\|g_{k,\gamma'}\|_{\mathscr{H}_p^{1+a}(C_R)}+\|u\|_{L^p(S_R)}\}$$

for (E)$'$−(B)$'$

2. Applications of Morrey-Sobolev type imbedding theorems

(1) Strong $\mathscr{L}^{(q,\lambda)}$ estimates

We make the following assumptions so as to obtain these estimates.

(A)$_{\mathscr{L}_p^{(q,\lambda)}}$ I, II : (A)$'_{\mathscr{L}_p^{(q,\lambda)}}$ I, II same as (A)$_{\mathscr{H}_p^a}$ I, II : (A)$'_{\mathscr{H}_p^a}$ I, II respectively.

(A)$_{\mathscr{L}_p^{(q,\lambda)}}$III The coefficients and functions $\{a_\beta, f\}$; $\{b_{k,\beta'}, g_k\}$ are smooth. Namely, we assume

$$a_\beta\in C^{l-2m+a}(S_R),\ D^{l-2m}f\in\mathscr{L}_p^{(q,\lambda)}(S_R)$$

$$b_{k,\beta'}\in C^{l-m_k+a}(C_R),\ D_{x'}^{l-m_k}g_k\in\mathscr{L}_p^{(q,\lambda)}(C_R)$$

where $l\geqq l_1=\mathrm{Max}(2m,\ m_k+1)$, $1<p,\ q<\infty$, $0<\lambda<$n and $0<a=\dfrac{n}{p}-\dfrac{\lambda}{q}<1$, $a\neq\dfrac{1}{p}$.

(A)$'_{\mathscr{L}_p^{(q,\lambda)}}$III The coefficients and functions $\{a_{\beta,\gamma},\ f_\gamma\}$; $\{b_{k,\beta',\gamma'},\ g_{k,\gamma'}\}$ are smooth. Namely, we assume

$$a_{\beta\gamma}\in C^a(S_R),\ f_\gamma\in\mathscr{L}_p^{(q,\lambda)}(S_R)$$

$$b_{k,\beta',\gamma'}\in C^{1+a}(C_R),\ D_{x'}g_{k,\gamma'}\in\mathscr{L}_p^{(q,\lambda)}(C_R)$$

where p, q, λ and a are as in (A)$_{\mathscr{L}_p^{(q,\lambda)}}$III.

With the aid of Theorem 2.3 and Theorem 1.6. 1(2), 2(2), 3(2), we have proved the following strong $\mathscr{L}^{(q,\lambda)}$ estimates :

THEOREM 2.5. *Let u be an* $H^{l_1,p}$-(*or* $H^{l,p}$-) *solution of the problem* (E)−(B) (*or* (E)$'$−(B)$'$) *vanishing near the curved boundary of* S_R *under the assumptions*

(A)$_{\mathscr{L}_p^{(q,\lambda)}}$ *(or* (A)$'_{\mathscr{L}_p^{(q,\lambda)}}$*).*

Then, in fact the derivatives $\{D^\beta u\}_{|\beta|\leq l}$ *belong to the space* $\mathscr{L}_p^{(q,\lambda)}(S_R)$ *and we have*

$$(2.6)\quad \sum_{|\beta|\leq l}\|D^\beta u\|_{\mathscr{L}_p^{(q,\lambda)}(S_R)}$$

$$\leqq C\{\sum_{|\gamma|\leq l-2m}\|D^\gamma f\|_{\mathscr{L}_p^{(q,\lambda)}(S_R)}+\sum_{k=1}^{m}\sum_{|\gamma'|\leq l-m_k}\|D_{x'}^{\gamma'}g_k\|_{\mathscr{L}_p^{(q,\lambda)}(C_R)}+\|u\|_{L^p(S_R)}\} \quad \text{for (E)}-\text{(B)}$$

or

$$\leqq C\{\sum_{|\gamma|\leq 2m-l}\|f_\gamma\|_{\mathscr{L}_p^{(q,\lambda)}(S_R)}+\sum_{k=1}^{m}\sum_{|\beta'|\leq 1}\sum_{|\gamma'|\leq m_k-l+1}\|D_{x'}^{\beta'}g_{k,\gamma'}\|_{\mathscr{L}_p^{(q,\lambda)}(C_R)} +\|u\|_{L^p(S_R)}\} \quad \text{for (E)}'-\text{(B)}'$$

Here, we make the following :

Remark 2. We have proved this theorem for the case of $\frac{1}{p}<a=\frac{n}{p}-\frac{\lambda}{q}<1$ by Theorem 2.3. On the other hand, we have needed furthermore Theorem 1.6. 1(2), 2(2), 3(2) to deduce the estimates for the case of $0<a<\frac{1}{p}$.

Now, for the exceptional case, that is, the case of $a=\frac{n}{p}-\frac{\lambda}{q}=\frac{1}{p}$, we have proved a proposition analogous to Theorem 2.5 under slightly stronger conditions. Namely,

PROPOSITION. *Let u be an $H^{l,p}$-(or $H^{1,p}$-) solution of the problem* (E)$-$(B) *(or* (E)$'-$(B)$'$*) vanishing near the curved boundary of S_R under the assumptions* (A)$_{\mathscr{L}_p^{(q,\lambda)}}$ *(or* (A)$'_{\mathscr{L}_p^{(q,\lambda)}}$*) with $a=\frac{n}{p}-\frac{\lambda}{q}=\frac{1}{p}$. Furthermore, we assume*

$$g_k\in H^{l-m_k,p}(C_R) \quad \text{for (E)}-\text{(B)}$$

or

$$g_{k,\gamma'}\in H^{1,p}(C_R) \quad \text{for (E)}'-\text{(B)}'$$

Then, in fact the derivatives $\{D^\beta u\}_{|\beta|\leq l}$ *belong to the space* $\mathscr{L}_p^{(q,\lambda)}(S_R)$ *and we have*

$$(2.7)\quad \sum_{|\beta|\leq l}\|D^{\beta}u\|_{\mathscr{L}_p^{(q,\lambda)}(S_R)}$$

$$\leq C\{\sum_{|\gamma|\leq l-2m}\|D^{\gamma}f\|_{\mathscr{L}_p^{(q,\lambda)}(S_R)}+\sum_{k=1}^{m}\|g_k\|_{H^{l-m_k,p}(C_R)}+\|u\|_{L^p(S_R)}\}\quad \text{for (E)}-\text{(B)}$$

or

$$\leq C\{\sum_{|\gamma|\leq 2m-l}\|f_\gamma\|_{\mathscr{L}_p^{(q,\lambda)}(S_R)}+\sum_{k=1}^{m}\sum_{|\gamma'|\leq m_k-l+1}\|g_{k,\gamma'}\|_{H^{1,p}(C_R)}+\|u\|_{L^p(S_R)}\}$$

$$\text{for (E)}'-\text{(B)}'$$

(2) **Estimates for the lower order derivatives of the solutions of the boundary value problems**

Applying Theorem 1.6. 1(1), 2(1), 3(1) to Theorem 2.5, we can prove easily the following :

THEOREM 2.6. *Let u be a solution as in Theorem 2.5.*

Then, the following estimates hold for u.

1. $q<\lambda$: *the derivatives $\{D^{\beta}u\}_{|\beta|\leq l-1}$ belong to the space $\mathscr{L}_r^{(\bar{q},\lambda)}(S_R)$ and we have*

$$(2.8)\quad \sum_{|\beta|\leq l-1}\|D^{\beta}u\|_{\mathscr{L}_r^{(\bar{q},\lambda)}(S_R)}$$

$$\leq C\{\sum_{|\gamma|\leq l-2m}\|D^{\gamma}f\|_{\mathscr{L}_p^{(q,\lambda)}(S_R)}+\sum_{k=1}^{m}\sum_{|\beta'|\leq l-m_k}\|D_{x'}^{\beta'}g_k\|_{\mathscr{L}_p^{(q,\lambda)}(C_R)}+\|u\|_{L^p(S_R)}\}$$

$$\text{for (E)}-\text{(B)}$$

or

$$\leq C\{\sum_{|\gamma|\leq 2m-l}\|f_\gamma\|_{\mathscr{L}_p^{(q,\lambda)}(S_R)}+\sum_{k=1}^{m}\sum_{|\beta'|\leq 1}\sum_{|\gamma'|\leq m_k-l+1}\|D_{x'}^{\beta'}g_{k,\gamma'}\|_{\mathscr{L}_p^{(q,\lambda)}(C_R)}$$

$$+\|u\|_{L^p(S_R)}\}\quad \text{for (E)}'-\text{(B)}'$$

where r is an arbitrary constant greater than $\frac{n}{\lambda}q$.

2. $q=\lambda$: *the derivatives $\{D^{\beta}u\}_{|\beta|\leq l-1}$ belong to the space $\mathscr{L}_r^{(1,0)}(S_R)$ and we have*

$$(2.9)\quad \sum_{|\beta|\leq l-1}\|D^{\beta}u\|_{\mathscr{L}_r^{(1,0)}(S_R)}\leq \text{the right hand side of (2.8)}$$

where r is as in 1.

3. $q>\lambda$; *u belongs to the space* $\mathscr{H}_r^{l-\frac{\lambda}{q}}(S_R)$ *and we have*

(2.10) $\quad \|u\|_{\mathscr{H}_r^{l-\frac{\lambda}{q}}(S_R)} \leqq$ the right hand side of (2.8)

where r is as in 1.

Next, we have proved similar estimates as Theorem 2.6, even if we replace the assumption "$0\leqq\frac{n}{p}-\frac{\lambda}{q}<1$" in $(A)_{\mathscr{L}_p^{(q,\lambda)}}$ by "$\frac{n}{p}<\frac{\lambda}{q}$". Namely,

THEOREM 2.7. *Let u be an* $H^{l_1,p}$*-solution* ($l_1=Max(2m,\ m_k+1)$) *of the problem* (E)$-$(B) *vanishing near the curved boundary of* S_R *under the assumptions* $(A)_{\mathscr{L}_p^{(q,\lambda)}}$ *with* $\frac{n}{\lambda}q<p$. *Furthermore, we assume that l is greater than* l_1.

Then, the following estimates hold for u.

1. $q<\lambda$ *and*

(1) $\frac{n}{\lambda}q<p\leqq\frac{n-1}{\lambda}\tilde{q}$; *the derivatives* $\{D^\beta u\}_{|\beta|\leqq l-1}$ *belong to the space* $\mathscr{L}_p^{(\tilde{q},\lambda)}(S_R)$ *and we have*

$$(2.11)\quad \sum_{|\beta|\leqq l-1}\|D^\beta u\|_{\mathscr{L}_p^{(\tilde{q},\lambda)}(S_R)} \leqq C\{\sum_{|\gamma|\leqq l-2m}\|D^\gamma f\|_{\mathscr{L}_p^{(q,\lambda)}(S_R)}$$

$$+\sum_{k=1}^{m}\sum_{|\beta'|\leqq l-m_k}\|D_{x'}^{\beta'}g_k\|_{\mathscr{L}_p^{(q,\lambda)}(C_R)}+\|u\|_{L^p(S_R)}\}$$

(2) $\frac{n-1}{\lambda}\tilde{q}<p<\frac{n}{\lambda}\tilde{q}$; *the derivatives* $\{D^\beta u\}_{|\beta|\leqq l-1}$ *belong to the space* $\mathscr{L}_p^{(\tilde{q}_1,\lambda_1)}(S_R)$ *and we have*

(2.12) $\quad \sum_{|\beta|\leqq l-1}\|D^\beta u\|_{\mathscr{L}_p^{(\tilde{q}_1,\lambda_1)}(S_R)} \leqq$ the right hand side of (2.11)

where q_1, λ_1 *are arbitrary constants satisfying* $1<q_1<q$, $\frac{\lambda_1}{q_1}=\frac{\lambda}{q}$ *and* $\frac{1}{\tilde{q}_1}=\frac{1}{q_1}-\frac{1}{\lambda_1}$.

2. $q=\lambda$ *and* $n<p<\infty$; *the derivatives* $\{D^\beta u\}_{|\beta|\leqq l-1}$ *belong to the space* $\mathscr{L}_p^{(1.0)}(S_R)$ *and we have*

(2.13) $\quad \sum_{|\beta|\leqq l-1}\|D^\beta u\|_{\mathscr{L}_p^{(1.0)}(S_R)} \leqq$ the right hand side of (2.11)

3. $q>\lambda$ *and* $\frac{n}{\lambda}q<p<\infty$; *u belongs to the space* $\mathscr{H}_p^{l-\frac{\lambda}{q}}(S_R)$ *and we have*

(2.14) $\|u\|_{\mathscr{H}_p^{l-\frac{\lambda}{q}}(S_R)} \leqq$ the right hand side of (2.11)

Finally, we make the following :

Remark 3. Similar estimates as Theorems 2.6 and 2.7 hold for interior estimates.

Bibliography

[1] S. AGMON, A. DOUGLIS and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I*, Comm. Pure Appl. Math., **12** (1959), 623-727.

[2] S. CAMPANATO, *Proprietă di Hölderianită di alcune claasi di funzioni*, Ann. Scuola Norm. Sup. Pisa, **17** (1963), 175-188.

[3] ———, *Proprietă di una famiglia di spazi funzionali*, Ann. Scuola Norm. Sup, Pisa, **18** (1964), 137-160.

[4] ———, *Equazioni ellitiche del* II^0 *ordine e spazi* $\mathscr{L}^{2,\lambda}$, Ann. Mat. Pura Appl, **69** (1965), 321-381.

[5] ———, *Equazioni ellittiche non variazionali a coefficienti continui*, Ann. Mat, Pura Appl., **86** (1970), 125-154.

[6] ———, *Sistemi ellittici in forma divergenza, Regolarita all'interno*, "Quaderni", Ann. Scuola Norm. Sup. Pisa, 1980.

[7] ———, *Generation of analytic semigroups by elliptic operators of second order in Hölder spaces*, Ann. Scuola Norm. Sup. Pisa, 8 (1981), 495-512.

[8] ———, *Differentiability of the solutions of non-linear elliptic systems with natural growth*, Ann. Mat. Pura Appl., **131** (1982), 75-106.

[9] ———, *Hölder continuity of the solutions of some non-linear elliptic systems*, Advances Math., **48** (1983), 16-43.

[10] S. CAMPANATO and P. CANNARSA, *Differentiability and partial Hölder continuity of the solutions of non-linear elliptic systems of order 2m with quadratic growth*, Ann. Scuola Norm. Sup. Pisa, 8 (1981), 285-309.

[11] Y. FURUSHO, *On inclusion property for certain* $\mathscr{L}^{(p,\lambda)}$ *spaces of strong type*, Funkcial. Ekvac., **23** (1980), 197-205.

[12] F. JOHN and L. NIRENBERG, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math., **14** (1961), 415-426.

[13] A. KUFNER et al., Function spaces, Noordhoff, 1977.

[14] G. N. MEYERS, *Mean oscillation over cubes and Hölder continuity*, Proc. Amer. Math. Soc., **15** (1964), 717-721.

[15] C. B. MORREY, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc., **43** (1938), 126-166.

[16] ———, Multiple integrals in the calculus of variations, Springer, 1966.

[17] S. M. NIKOL'SKII, Approximation of functions of several variables and imbedding theorems, Springer, 1975.

[18] A. ONO, *On imbedding between strong Hölder spaces and Lipschitz spaces in* L^p, Mem. Fac. Sci. Kyushu Univ., **24** (1970), 76-93.

[19] ———, *On isomorphism between certain strong* $\mathscr{L}^{(p,\lambda)}$ *spaces and the Sobolev spaces* $H^{1,p}$, Boll. Unione Mat. Ital., **5** (1972), 230-233.

[20] ———, *On imbedding theorems in strong $\mathcal{L}^{(q,\mu)}$ spaces*, Rend. Ist. Mat. Univ. Trieste, **4** (1972), 53-65.

[21] ———, *On isomorphism between certain strong $\mathcal{L}^{(p,\lambda)}$ spaces and the Lipschitz spaces and its applications*, Funkcial. Ekvac., **21** (1978), 261-270.

[22] ———, *Remarks on the Morrey-Sobolev type imbedding theorems in the strong $\mathcal{L}^{(p,\lambda)}$ spaces*, Math. Rep. Coll, Gen. Educ. Kyushu Univ., **11** (1978), 149-155.

[23] ———, *Interior estimates for elliptic partial differential equations in the $\mathcal{L}^{(q,\lambda)}$ spaces of strong type*, Canad. J. Math., **36** (1984), 385-404.

[24] ———, *Morrey-Soblev type imbedding theorems in the $\mathcal{L}^{(p,\lambda)}$ spaces of strong type*, Funkcial. Ekvac., **28** (1985), 83-92.

[25] ———, *Boundary estimates for elliptic partial differential equations in the $\mathcal{L}^{(q,\lambda)}$ spaces of strong type*, Funkcial. Ekvac., **30** (1987), 169-202.

[26] A. Ono and Y. Furusho, *On isomorphism and inclusion property for certain $\mathcal{L}^{(p,\lambda)}$ spaces of strong type*, Ann. Mat. Pura Appl., **114** (1977), 289-304.

[27] J. Peetre, *On the theory of $\mathcal{L}_{p,\lambda}$ spaces*, J. Func. Anal., **4** (1969), 71-87.

[28] L. C. Piccinini, *Proprietā di inclusione e di interpolazione tra spazi di Morrey e loro generalizzazioni*, Thesis Scuola Norm. Sup. Pisa, 1969.

[29] G. Stampacchia, *The spaces $\mathcal{L}^{(p,\lambda)}$, $N^{(p,\lambda)}$ and interpolation*, Ann. Scuola Norm. Sup. Pisa, **19** (1965), 443-462.

[30] ———, *The $\mathcal{L}^{(p,\lambda)}$ spaces and applications to the theory of partial differential equations*, Proc. Conf. Diff. Equ. and Appl., Bratislava, 1969, 129-141.