On the characteristic numbers of maps and Z_2-equivariant transversality to the fixed point set

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https://doi.org/10.15017/1449064

出版情報：九州大学教養部数学雑誌. 17 (1), pp.25-37, 1989-12. College of General Education, Kyushu University

Math. Rep.
17-1, 1989.

On the characteristic numbers of maps and Z_2-equivariant transversality to the fixed point set

Dedicated to Professor Shôrô Araki on his sixtieth birthday

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(Received September 18, 1989)

1. Introduction

In [8] R. E. Stong introduced the classification of maps of closed differentiable manifolds under the cobordism relation and determined the abelian group of all classes of the maps with the group structure imposed by disjoint union ; that is,

$$[M_1 \xrightarrow{f_1} N_1] + [M_2 \xrightarrow{f_2} N_2] = [M_1 \amalg M_2 \xrightarrow{f_1 \amalg f_2} N_1 \amalg N_2].$$

And he gave the characteristic numbers of a map which are algebraic invariants to determine the cobordism class. The classification of equivariant maps of G-manifolds were treated in [9]. In this paper we treat a Z_2-map $f : M^m \longrightarrow N^n$ of closed differentiable manifolds (M^m, T) and (N^n, T'), where T and T' are the differentiable involutions. In Section 2 we investigate the relation between the characteristic number of a Z_2-map $f : M^m \longrightarrow N^n$ and that of the restriction $f|F(M^m, T)$ to the fixed point set $F(M^m, T)$, and we study the mapping version of results obtained by Kosniowski and Stong [5]. In Section 3 we discuss the approximation by a Z_2-equivariant map which is transversal to the fixed point set.

2. The relation between the characteristic number of a Z_2-map and that of the restriction to the fixed point set

Let $f : M^m \longrightarrow N^n$ be a differentiable map of closed differentiable manifolds M^m and N^n of dimension m and n respectively. Let $e : M^m \longrightarrow R^k$ be an embedding. For the normal bundle $\nu(f)$ of the embedding $f \times e : M^m \longrightarrow N^n \times R^k$ we have the Thom isomorphism

$$\Phi_{\nu(f)} : H^*(M^m ; Z_2) \longrightarrow \widetilde{H}^{*+n+k-m}(T(\nu(f)) ; Z_2)$$

and the Gysin homomorphism

$$
\begin{aligned}
f_!: H^*(M^m : Z_2) &\xrightarrow{\Phi_{\nu(f)}} \tilde{H}^{*+n+k-m}(T(\nu(f)) ; Z_2) \\
&\xrightarrow{c^*} \tilde{H}^{*+n+k-m}((N^n)^+ \wedge S^k ; Z_2) \\
&\xrightarrow{\sigma^{-1}} H^{*+n-m}(N^n ; Z_2)
\end{aligned}
$$

where c^* is the homomorphism induced from the collapsing map $c : (N^n \times D^k)/(N^n \times S^{k-1}) \longrightarrow T(\nu(f))$, D^k : k-disk and S^{k-1} : $(k-1)$-sphere and σ is the suspension isomorphism. The Gysin homomorphism $f_!$ is coincident with the following composite

$$
(2.1) \quad H^*(M^m ; Z_2) \overset{D}{\cong} H_{m-*}(M^m ; Z_2) \xrightarrow{f_*} H_{m-*}(N^n ; Z_2) \overset{D^{-1}}{\cong} H^{*+n-m}(N^n ; Z_2)
$$

where D is the Poincare duality isomorphism.

Let $f : M^m \longrightarrow N^n$ be a differentiable map which is transversal to a submanifold B of N^n and let $A = f^{-1}(B)$. The differential df yields the bundle map $f : \nu(A) \longrightarrow \nu(B)$ of the normal bundles $\nu(A)$ and $\nu(B)$ of A and B respectively. Then the following commutative diagram is derived from the naturality of the Thom isomorphism

$$
(2.2) \quad \begin{array}{ccc}
H^*(N^n ; Z_2) & \xrightarrow{f^*} & H^*(M^m ; Z_2) \\
{\scriptstyle i_{B!}} \uparrow & & \uparrow {\scriptstyle i_{A!}} \\
H^*(B ; Z_2) & \xrightarrow{(f|A)^*} & H^*(A ; Z_2)
\end{array}
$$

where i_A and i_B are the inclusions.

Denote by $[M^m]$ the homology fundamental class of a closed manifold M^m. The degree $o(f)$ of a map $f : M^m \longrightarrow N^n$ of closed manifolds is defined by $f_*([M^m]) = o(f)[N^n]$.

LEMMA 2.3. *Let $w_j(\nu(B))$ be the j-th Stiefel-Whitney class. Then for any $b \in H^*(B ; Z_2)$, $o(f) < w_j(\nu(B)) \cdot b, [B] > = < w_j(\nu(A))(f|A)^*(b), [A] >$.*

PROOF. Using (2.1), we have

$$
\begin{aligned}
&< f^* i_{B!}(w_j(\nu(B)) \cdot b), [M^m] > \\
&= < i_{B!}(w_j(\nu(B)) \cdot b), o(f)[N^n] > \\
&= o(f) < 1, i_{B*}((w_j(\nu(B)) \cdot b) \cap [B]) > \\
&= o(f) < w_j(\nu(B)) \cdot b, [B] >.
\end{aligned}
$$

It follows from (2.2) that

$$<f^* i_{B!}(w_j(\nu(B))\cdot b),\ [M^m]>$$
$$=<i_{A!}(f|A)^*(w_j(\nu(B))\cdot b),\ [M^m]>=<w_j(\nu(A))(f|A)^*(b),\ [A]>.$$
Q. E. D.

We now recall the characteristic classes discussed in [7] and the Boardman map. Let X be a finite CW-complex. One has a natural transformation $w_t : Vect(X) \longrightarrow H^*(X)[[t_1, t_2, \cdots]]$ from the set of real vector bundles over X to the ring of formal power series in variables t_i over the graded ring $H^*(X)$, which is claimed

(1) $w_t(f^! \xi) = f^* w_t(\xi)$

(2) $w_t(\xi_1 \oplus \xi_2) = w_t(\xi_1) w_t(\xi_2)$

(3) for the canonical line bundle ξ_n over the n-dimensional projective space RP^n

$$w_t(\xi_n) = 1 + xt_1 + x^2 t_2 + \cdots + x^n t_n$$

where x is the generator of $H^1(RP^n; Z_2) \simeq Z_2$. The unoriented cobordism group $\mathcal{N}^k(X)$ is the stable homotopy group defined by the direct limit $\varinjlim_n [S^{n-k}X^+, MO(n)]$. The cobordism group yields the generalised cohomology theory with the characteristic classes $w_i^{\mathcal{N}} : Vect(X) \longrightarrow \mathcal{N}^i(X)$ which has the property of the naturality, the Cartan formula and the first Stiefel-Whitney class $w_1^{\mathcal{N}}(\xi_n)$ of the canonical line bundle ξ_n over RP^n is the homotopy class of the natural injection $RP^n \subset MO(1)$. The natural transformation

$$\beta : \mathcal{N}^*(X) \longrightarrow H^*(X)[[t_1, t_2, \cdots]]$$

is the multiplicative natural transformation with

$$\beta : (w_1^{\mathcal{N}}(\xi)) = x + x^2 t_1 + \cdots + x^{n+1} t_n + \cdots$$

for the canonical line bundle ξ over $BO(1) \simeq RP^\infty$. A differentiable map $f : M^m \longrightarrow N^n$ of closed manifolds gives a bordism class $[M^m \longrightarrow N^n]$ of the bordism group $\mathcal{N}_m(N^n)$ for the space N^n [1]. Apply the Poincare duality isomorphism $D : \mathcal{N}^*(N^n) \simeq \mathcal{N}_*(N^n)$ to $[M^m \longrightarrow N^n]$, and we have

$$D^{-1}[M^m \xrightarrow{f} N^n] = D^{-1} f_* [M^m \xrightarrow{id} M^m]$$
$$= D^{-1} f_* D(1)$$

where $1 \in \mathcal{N}^0(M^m)$. Then we have the Gysin homomorphism of $\mathcal{N}^*$-theory

$f_!(1)=D^{-1}[M^m \xrightarrow{f} N^n]$. Taking an embedding $e: M^m \longrightarrow R^k$ and the embedding $f\times e: M^m \longrightarrow N^n \times R^k$, it follows that

(2.4) $\beta(f_!(1))=f_! w_t(\nu(f\times e))$ (cf. [6])

where $\nu(f\times e)$ is the normal bundle of M^m in $N^n\times R^k$. A formal power series

$$F(x)=1+xt_1+x^2t_2+\cdots+x^nt_n+\cdots$$

is available to describe $w_t(\eta)$. If (N^n, T') is a differentiable closed manifold N^n with a differentiable involution T' whose fixed point set is union of connected manifolds

$$F(N^n, T')=F_1'\cup F_2'\cup\cdots\cup F_t'.$$

The normal bundle ν_{N^n} of N^n embedded in an adequate euclidean space has the characteristic class

$$w_t(\nu_{N^n})=\frac{1}{w_t(\tau(N^n))}$$

where $\tau(N^n)$ denotes the tangent bundle of N^n. We use the splitting principle of a vector bundle to describe the total Stiefel-Whitney class of $\tau(N^n)$ as

$$w(\tau(N^n))=(1+\tau_1')(1+\tau_2')\cdots(1+\tau_n')$$

and

$$w_t(\tau(N^n))=F(\tau_1')F(\tau_2')\cdots F(\tau_n').$$

For each normal bundle $\nu(F_j')$ of F_j' in N^n let

$$w(\nu(F_j'))=(1+\beta_1'^{(j)})(1+\beta_2'^{(j)})\cdots(1+\beta_{k_j}'^{(j)}).$$

Let $G(u, x_1, x_2, \cdots, x_k; y_1, y_2, \cdots, y_l; n)$ be the part of degree n of

$$\frac{1}{F(x_1+u)\cdots F(x_k+u)F(y_1)\cdots F(y_l)}$$

By making use of (2.4) and Proposition 4.1 of [3], we have

THEOREM 2.5 (cf. [4], [5])

$$<\frac{1}{F(\tau_1')F(\tau_2')\cdots F(\tau_n')}, [N^n]>$$
$$=\sum_{j=1}^{t}<G(1, \beta_1'^{(j)}, \cdots, \beta_{k_j}'^{(j)}; \tau_1'^{(j)}, \cdots, \tau_{l_j}'^{(j)}; n)/w(\nu(F_j')), [F_j']>$$

where $w(\tau(F_j'))=(1+\tau_1'^{(j)})(1+\tau_2'^{(j)})\cdots(1+\tau_{l_j}'^{(j)})$.

From now on we will treat an equivariant map

$$f: (M^m, T) \longrightarrow (N^n, T')$$

with the fixed point set

$$F(M^m, T)=F_1\cup F_2\cup\cdots\cup F_s$$

and

$$F(N^n, T')=F'_1\cup F'_2\cup\cdots\cup F'_t.$$

Although all of equivariant maps are not approximated by an equivariant map which is transversal to $F(N^n, T')$, we now claim that f is transversal to $F(N^n, T')$. Then $W_j=f^{-1}(F'_j)$ is the invariant submanifold of M^m with the normal bundle $\nu(W_j)$ in M^m which is isomorphic to the induced bundle $(f\mid W_j)^!$ $\nu(F'_j)$. By the splitting principle we describe the total Whitney classes as

$$w(\nu(F'_j))=(1+\beta_1'^{(j)})(1+\beta_2'^{(j)})\cdots(1+\beta_{k_j}'^{(j)})$$

$$w(\tau(F'_j))=(1+\tau_1'^{(j)})(1+\tau_2'^{(j)})\cdots(1+\tau_{l_j}'^{(j)})$$

$$w(\nu(W_j))=(1+\alpha_1^{(j)})(1+\alpha_2^{(j)})\cdots(1+\alpha_{k_j}^{(j)})$$

$$w(\tau(W_j))=(1+\tau_1^{(j)})(1+\tau_2^{(j)})\cdots(1+\tau_{w_j}^{(j)}).$$

Then we have the following

PROPOSITION 2.6.

$$o(f)<\frac{1}{F(\tau'_1)F(\tau'_2)\cdots F(\tau'_n)},\ [N^n]>$$

$$=\sum_{j=1}^{t}<G(1,\alpha_1^{(j)},\cdots,\alpha_{k_j}^{(j)};f_j^*(\tau_1'^{(j)}),\cdots,f_j^*(\tau_{l_j}'^{(j)});n)/w(\nu(W_j)),\ [W_j]>$$

where f_j denotes the restriction of f to W_j.

PROOF. Applying (2.2) to the commutative diagram

$$\begin{array}{ccc} W_j & \xrightarrow{f_j} & F'_j \\ \cap & & \cap \\ M^m & \xrightarrow{f} & N^n \end{array}$$

it follows from Lemma 2.3 that

$$o(f)<G(1,\beta_1'^{(j)},\cdots,\beta_{k_j}'^{(j)};\tau_1'^{(j)},\cdots,\tau_{l_j}'^{(j)};n)/w(\nu(F'_j)),\ [F'_j]>$$

$$=<G(1,f_j^*\beta_1'^{(j)},\cdots,f_j^*\beta_{k_j}'^{(j)};f_j^*\tau_1'^{(j)},\cdots,f_j^*\tau_{l_j}'^{(j)};n)/w(\nu(W_j)),\ [W_j]>$$

$$=<G(1,\alpha_1^{(j)},\cdots,\alpha_{k_j}^{(j)};f_j^*\tau_1'^{(j)},\cdots,f_j^*\tau_{l_j}'^{(j)};n)/w(\nu(W_j)),\ [W_j]>$$

We use Theorem 2.5 to complete the proof.

Proposition 2.6 derives the following

PROPOSITION 2.7. *For arbitrary symmetric polynomial $P(x_1, x_2, \cdots, x_n)$ of degree n*

$$o(f)<P(\tau'_1, \tau'_2, \cdots, \tau'_n), [N^n]>$$
$$=\sum_j < \frac{P(a_1^{(j)}+1, a_2^{(j)}+1, \cdots, a_{k_j}^{(j)}+1, f_j^*\tau_1'^{(j)}, \cdots, f_j^*\tau_{l_j}'^{(j)})}{(a_1^{(j)}+1)(a_2^{(j)}+1)\cdots(a_{k_j}^{(j)}+1)}, [W_j]>$$

Applying Proposition 2.7 to $P(x_1, x_2, \cdots, x_n)=x_1\cdots x_n$, we have

COROLLARY 2.8. *If $f:(M^m, T)\longrightarrow(N^n, T')$ is a Z_2-map which is transversal to $F(N^n, T')$ and $f^{-1}(F'_j)=\bigcup_{j(i)=j}F_i$ where $F(N^n, T')=\bigcup_j F'_j$, $F(M^m, T)=\bigcup_i F_i$ then*

$$o(f)\chi(N^n)=\sum_j\{\sum_{j(i)=j}o(f_i)\chi(F'_j)\}$$

where $f_i: F_i\longrightarrow F'_{j(i)}$ is the restriction of f and $\chi(N^n)$ denotes the Euler number module 2 and $j(i)=j$ if $f(F_i)\subset F'_j$.

We now turn to investigate the relation of the characteristic numbers for the pair $(W_j, \nu(W_j))$ of the invariant submanifold W_j and the normal bundle of $\nu(W_j)$ of W_j in M^m with $F(W_j, T)=\bigcup_{j(i)=j}F_i$, where $j(i)=j$ if $f(F_i)\subset F'_j$.

PROPOSITION 2.9. *Let $\nu(W_j)|F_j$ be the restriction of the normal bundle $\nu(W_j)$ to F_i, and let*

$$w(\nu(W_j)|F_i)=(\tilde{\alpha}_1^{(i)}+1)\cdots(\tilde{\alpha}_{k_j}^{(i)}+1).$$

Then

$$<F(a_1^{(j)})\cdots F(a_{k_j}^{(j)}), [W_j]>$$
$$=\sum_{j(i)=j}<\frac{F(1+\tilde{\alpha}_1^{(i)})\cdots F(1+\tilde{\alpha}_{k_j}^{(i)})}{w(\nu(F_i, W_j))}, [F_i]>$$

where $\nu(F_i, W_j)$ is the normal bundle of F_i in W_j.

PROOF. Let $h:(W_j, T)\longrightarrow(M^m, T)$ be the natural inclusion. The product $M^m\times I$, $I=[-1, 1]$, is the Z_2-manifold with the involution

$$\tilde{T}: M^m\times I\longrightarrow M^m\times I, \ \tilde{T}(x, t)=(T(x), -t)$$

whose fixed point set is $F(M^m, T)\times\{0\}$ and the normal bundle of $F_i\times\{0\}$ is $\nu(F_i)\oplus 1$. Let $\nu(F_i, W_j)$ be the normal bundle of F_i in W_j. Let $\mathrm{Int}(D(\nu(F_i)\oplus 1))$ be the interior of the disk bundle $\nu(F_i)\oplus 1$. The inclusion $\tilde{h}: W_j\times I$ —

$\to M^n \times I$, $\tilde{h}(x, t) = (h(x), t)$, induces

$$\tilde{h} : W_j \times I - \bigcup_{j(i)=j} \mathrm{Int}(D(\nu(F_i, W_j) \oplus 1)) \longrightarrow M^m \times I - \bigcup_i \mathrm{Int}(D(\nu(F_i) \oplus 1).$$

This is an equivariant map of the free Z_2-manifolds which bounds the maps

$$\tilde{h} : W_j \times \{-1, 1\} \longrightarrow M^m \times \{-1, 1\}$$

and

$$\bigcup_i h_i : \bigcup_{j(i)=j} S(\nu(F_i, W_j) \oplus 1) \longrightarrow \bigcup_i S(\nu(F_i) \oplus 1)$$

where $S(\ \ )$ denotes the sphere bundle and h_i is the map induced from the inclusion $\tilde{h} : W_j \times I \longrightarrow M^m \times I$. The classifying map $g_{\nu(\tilde{h})/Z_2}$ of the normal bundle $\nu(\tilde{h})/Z_2$ with respect to the inclusion $\tilde{h}$ and the collapsing map c yield the composite

$$\{M^m \times I - \bigcup_i Int(D(\nu(F_i) \oplus 1)\}/Z_2 \xrightarrow{\ \hat{c}\ } T(\nu(\tilde{h})/Z_2) \xrightarrow{T(g_{\nu(\tilde{h})/Z_2})} MO(k_j)$$

which bounds the following maps

$$M^m \xrightarrow{\ \hat{c}_j\ } T(\nu(W_j)) \xrightarrow{T(g_{\nu(W_j)})} MO(k_j)$$

and

$$\bigcup_i P(\nu(F_i) \oplus 1) \xrightarrow{\bigcup_i \hat{c}_i} \bigcup_{j(i)=j} T(\nu(\tilde{h}_i)) \xrightarrow{T(g_{\nu(\tilde{h}_i)})} MO(k_j)$$

where $P(\ \ )$ donotes the projective space bundle and $\nu(\tilde{h}_i)$ denote the normal bundle for the embedding $\tilde{h}_i : P(\nu(F_i, W_j) \oplus 1) \longrightarrow P(\nu(F_i) \oplus 1)$. Then we have a relation

$$[M^m \xrightarrow{T(g_{\nu(W_j)}) \circ \hat{c}_j} MO(k_j)] = \sum_i [P(\nu(F_i) \oplus 1) \xrightarrow{T(g_{\nu(\tilde{h}_i)}) \circ \hat{c}_i} MO(k_j)]$$

in $\mathcal{N}_*(MO(k_j))$. Let γ^{k_j} be the universal vector bundle on the classifying space $BO(k_j)$ for the k_j-dimensional vector bundles. By using the characteristic numbers given in [1] and [8] we have

$$< \hat{c}_j^* T(g_{\nu(W_j)})^* \Phi(w_t(\gamma^{k_j})), [M^m] >$$
$$= \sum_{j(i)=j} < \hat{c}_i^* T(g_{\nu(\tilde{h}_i)})^* \Phi(w_t(\gamma^{k_j})), [P(\nu(F_i) \oplus 1)] >$$

where Φ denotes the Thom isomorphism for γ^{k_j}. From the naturality of the Thom isomorphism and the fact that $\hat{c}_i^* \Phi_{\nu(h_i)}$ and $\hat{c}_j^* \Phi_{\nu(W_j)}$ are the Gysin homomorphism it follows that

$$< h_!(w_t(\nu(W_j))), [M^m] >$$
$$= \sum_{j(i)=j} < \tilde{h}_{i!}(w_t(\nu(\tilde{h}_i))), [P(\nu(F_i) \oplus 1)] >.$$

Let $\xi \longrightarrow X$ be a smooth vector bundle over a manifold X, and let η_ξ be the canonical line bundle over the projective space bundle $P(\xi)$. Then the tangent bundle $\tau(P(\xi))$ is stably isomorphic to $(\pi_\xi^!\xi)\otimes\eta_\xi\oplus\pi_\xi^!\tau(X)$. Since $h^!\nu(F_i)=\nu(F_i,\ W_j)\oplus h^!\nu(W_j)$ where $\nu(F_i,\ W_j)$ is the normal bundle of F_i in W_j, we have

$$w_t(\nu(\tilde{h}_i))=w_t(\pi_i^!(\nu(W_j)\,|\,F_i)\otimes(\eta_{\nu(F_i,\ W_j)\oplus 1})),$$

where $\pi_i : P(\nu(W_j)\,|\,F_i) \longrightarrow F_i$ is the bundle projection. Then

$$\begin{aligned}&<\tilde{h}_{i!}w_t(\nu(\tilde{h}_i)),\ [P(\nu(F_i)\oplus 1]>\\ &=<w_t(\nu(\tilde{h}_i)),\ [P(\nu(F_i,\ W_j)\oplus 1)]>\\ &=<\prod_r F(\tilde{\alpha}_r^{(i)}+w_1(\eta_{\nu(F_i,\ W_j)\oplus 1}),\ [P(\nu(F_i,\ W_j)\oplus 1)]>\end{aligned}$$

Let $\pi : P(\xi) \longrightarrow X$ be the projective space bundle of a smooth vector bundle ξ. Then the cohomology fundamental class of $P(\xi)$ is given as

$$[P(\xi)]^*=\pi^*([X]^*)\cdot c^{l-1},\quad c=w_1(\eta_\xi).$$

It follows from Proposition 2.2 of [3] that

$$\begin{aligned}&<\tilde{h}_{i!}w_t(\nu(\tilde{h}_i)),\ [P(\nu(F_i)\oplus 1]>\\ &=<\prod_r F(\tilde{\alpha}_r^{(i)}+1)/w(\nu(F_i,\ W_j)),\ [F_i]>.\end{aligned}$$

Since $<h_!w_t(\nu(W_j)),\ [M^m]>=<w_t(\nu(W_j)),\ [W_j]>$, we complete the proof.

It follows from Proposition 2.9 that

PROPOSITION 2.10. *Let $P(x_1,\ x_2,\ \cdots,\ x_{k_j})$ be a symmetric polynomial. Then*

$$<P(\alpha_1^{(j)},\ \cdots,\ \alpha_{k_j}^{(j)}),\ [W_j]>=\sum_{j(i)=j}<\frac{P(\tilde{\alpha}_1^{(i)}+1,\ \cdots,\ \tilde{\alpha}_{k_j}^{(i)}+1)}{w(\nu(F_i,\ W_j))},\ [F_i]>.$$

Let $g_r(x_1,\ x_2,\ \cdots,\ x_n)=\sum_{r_1+\cdots+r_n=r}(x_1+1)^{r_1}\cdots(x_n+1)^{r_n}$, where r is a non-negative integer and r_j runs over non-negative integers, and let $\pi(r,\ n)=g_r(0,\ \cdots,\ 0)$.

THEOREM 2.11. *Suppose that $f : (M^n,\ T) \longrightarrow (N^n,\ T')$, $n=2^s$, is a differentiable Z_2-map which is transversal to $F(N^n,\ T')$, $W_j=f^{-1}(F'_j)$, $f_j : W_j \longrightarrow F'_j$ is the restriction of f, and*

$$F(W_j,\ T)=\bigcup_{j(i)=j}F_i$$

with dim $F'_j<n$. *If $w(f_i^!\nu(F'_{j(i)}))=1$, $\nu(F'_{j(i)})$ the normal bundle of $F'_{j(i)}$ in N^n*

with $\dim \nu(F'_{j(i)})=k_{j(i)}$, *then for the symmetric polynomial* $S_{(n)}(x_1, \cdots, x_n)=x_1^n+\cdots+x_n^n$

$$o(f)<S_{(n)}(\tau(N^n)),\ [N^n]>=\sum_j\{<\sum_{j(i)=j}\frac{k_j\cdot\pi(n-k_j,\ k_j)}{w(\nu(F_i,\ W_j))},\ [F_i]>\}.$$

PROOF. Proposition 2.7 and Proposition 2.10 imply that

$$\begin{aligned}&o(f)<S_{(n)}(\tau(N^n)),\ [N^n]>\\&=\sum_j<\frac{k_j}{(\alpha_1^{(j)}+1)\cdots(\alpha_{k_j}^{(j)}+1)},\ [W_j]>\\&=\sum_i<\sum_{r_1+\cdots+r_{k_j}=\dim W_j}k_j(\alpha_1^{(j)})^{r_1}\cdots(\alpha_{k_j}^{(j)})^{r_{k_j}},\ [W_j]>\\&=\sum_j\{\sum_{j(i)=j}<\frac{\sum_{r_1+\cdots+r_{k_j}=\dim W_j}k_j(\tilde{\alpha}_1^{(i)}+1)^{r_1}\cdots(\tilde{\alpha}_{k_j}^{(i)}+1)^{r_{k_j}}}{w(\nu(F_i,\ W_j))},\ [F_i]>\}.\end{aligned}$$

Since $w(f_i^!\nu(F'_{j(i)}))=0$,

$$\sum_{r_1+\cdots+r_{k_j}=\dim W_j}(\tilde{\alpha}_1^{(i)}+1)^{r_1}\cdots(\tilde{\alpha}_{k_j}^{(i)}+1)^{r_{k_j}}=\pi(\dim\ W_j,\ k_j),$$

and $w(\nu(F_i,\ W_j))=w(\nu(F_i,\ M^n))$. Thus we complete the proof.

REMARK 2.12. We now consider a Z_2-manifold $(S^n,\ T')$, $n=2^s$, which is the n-sphere with the action $T'(x_0, \cdots, x_n)=(x_0, -x_1, \cdots, -x_n)$ and a Z_2-manifold $(RP^n,\ T)$ the n-dimensional real projective space with action $T([x_0, \cdots, x_n])=[-x_0, -x_1, x_2, \cdots, x_n]$. The fixed point set $F(RP^n,\ T)$ is $RP^1\cup RP^{n-2}$ with the normal bundle $\nu(RP^1)=(n-1)\xi_1$ and $\nu(RP^{n-2})=2\xi_{n-2}$ and

$$<\frac{1}{w(\nu(RP^1))},\ [RP^1]>=1.$$

We see that $\pi(n-1,\ 1)=1$, and that $S_{(n)}(\tau(S^n))[S^n]$ is zero. Thus it follows from Theorem 2.11 that there is no differentiable equivariant map which is transversal to the fixed point set $F(S^n,\ T')$.

3. The approximation by a Z_2-map which is transversal to the fixed point set

The equivaiant transversality theorms are discussed by Wasserman [10] and Hauschild [2] etc. In this Section we give a criterion of the existence of an equivariant map which is transversal to the fixed point set.

LEMMA 3.1. *Let* $\xi : E(\xi) \xrightarrow{\pi_\xi} M$ *and* $\eta : E(\eta) \xrightarrow{\pi_\eta} N$ *be smooth vector bundles over smooth manifolds with* $\dim \xi = l$ *and* $\dim \eta = k$. *Let* $f : E(\xi) \longrightarrow E(\eta)$ *be a smooth map which is linearly mapping from each fibre* $E(\xi_x)$ *into* $E(\eta_{f(x)})$. *If* $\dim M < l-k+1$, *then for any map* $\delta : E(\xi) \longrightarrow R^+ = \{x \mid x>0\}$ *there exists a smooth map* $g : E(\xi) \longrightarrow E(\eta)$ *such that*

(1) *g is epimorphic on each fiber.*

(2) *for any* $e \in E(\xi)$ $d(f(e), g(e)) < \delta(e)$, *where d is the metric.*

PROOF. Let $f^! \eta : E(f^! \eta) \longrightarrow M$ be the induced bundle by f. The map $\tilde{f} : E(\xi) \longrightarrow E(f^! \eta)$, $\tilde{f}(e) = (\pi_\xi(e), f(e))$, is a linear map on each fiber with the commutative diagram

$$\begin{tikzcd} E(\xi) \arrow[r, "f"] \arrow[d, "\tilde{f}"'] & E(\eta) \\ E(f^! \eta) \arrow[ur, "\hat{f}"'] & \end{tikzcd}$$

where $\hat{f}$ is the bundle map defined by $\hat{f}(x, e) = e$. Let $\mathrm{HOM}(\xi, f^! \eta)$ be a vector bundle over M whose total space consists of bundle homomorphisms from ξ to $f^! \eta$. Let $\Gamma(\xi, f^! \eta)$ be the space of cross-sections of $\mathrm{HOM}(\xi, f^! \eta) \longrightarrow M$ which is a Baire space. $\Gamma(\xi, f^! \eta)$ corresponds to the set of bundle homomorphisms. We prepare an open covering $\{U_i \mid i=1, 2, \cdots\}$ of M with the local triviality $\pi_\xi^{-1}(U_i) \cong U_i \times R^l$ and $\pi_{f^!\eta}^{-1}(U_i) \cong U_i \times R^k$. Thus the restriction $\tilde{f} \mid U_i$ gives an element of $\Gamma(\xi \mid U_i, f^! \eta \mid U_i) \cong \mathrm{Map}(U_i, M(l, k))$, where $\mathrm{Map}(U_i, M(l, k))$ consists of continuous maps from U_i to the space of matrices of type (l, k). The space $M(l, k\,;j)$ consisting of (l, k)-matrices with the rank j is a submanifold of $M(l, k)$ with the dimension $j(l+k-j)$. If $\dim U_i < lk - j(l+k-j)$, by the ordinary transversality theorem a cross section in $\Gamma(\xi \mid U_i, f^! \eta \mid U_i)$ is approximated by a cross section with the rank $\neq j$ on each fiber. Denote by $\Gamma(\xi \mid U_i, f^! \eta \mid U_i\,; j)$ the subspace of cross-sections with the rank j. Put

$$\Gamma(\xi \mid U_i, f^! \eta \mid U_i\,; j)^c = \Gamma(\xi \mid U_i, f^! \eta \mid U_i) - \Gamma(\xi \mid U_i, f^! \eta \mid U_i\,; j).$$

This is dense and open in $\Gamma(\xi \mid U_i, f^! \eta \mid U_i)$ if $\dim U_i < lk - j(l+k-j)$. We now suppose that $\dim M < l-k+1$. Then $\bigcup_{j=1}^{k-1} \Gamma(\xi \mid U_i, f^! \eta \mid U_i\,; j)^c$ is dense in $\Gamma(\xi \mid U_i, f^! \eta \mid U_i)$ and $\Gamma(\xi \mid U_i, f^! \eta \mid U_i\,; k)$ is dense. Let $\tilde{\Gamma}(\xi \mid U_i, f^! \eta \mid U_i\,; k)$ be the subspace consisting of cross-sections of $\mathrm{HOM}(\xi, f^! \eta)$ whose restriction to U_i is of rank k. $\tilde{\Gamma}(\xi|U_i, f'\eta|U_i ; k)$ is dense and open in $\Gamma(\xi, f\eta')$ and $\Gamma(\xi, f'\eta ; k)=\bigcap_i \tilde{\Gamma}(\xi|U_i, f'\eta|U_i ; k)$ is dense. Q. E. D.

THEOREM 3.2. (cf. [6]) *Let $f:(M, T)\longrightarrow(N, T')$ be a differentiable Z_2-map of compact Z_2-manifolds with $F(M, T)=F_1\cup F_2\cup\cdots\cup F_s$, $F(N, T')=F'_1\cup F'_2\cup\cdots\cup F'_t$ and $f_i=f|F_i : F_i\longrightarrow F'_{j(i)}$. If* codim F_i-codim $F'_{j(i)}+1>$ dim F_i *for all i and j, then f is equivariantly homotopic to a Z_2-map which is transversal to $F(N, T')$.*

PROOF. The open disk bundle $\mathrm{Int}(D(\nu(F_i)))$ of the normal bundle of F_i is identified with the ε-tubular neighborhood $N_\varepsilon(F_i)$, $\varepsilon>0$. The differential $df : \tau(M)\longrightarrow\tau(N)$ induces the homomorphism of normal bundles $\nu(F_i)$ and $\nu(F'_{j(i)})$ and Proposition 3.1 implies that $f|N_\varepsilon(F_i) : N_\varepsilon(F_i)\longrightarrow N_\varepsilon(F'_{j(i)})$ is equivariantly homotopic to a differentiable Z_2-map $g_i : N_\varepsilon(F_i)\longrightarrow N_\varepsilon(F'_{j(i)})$ which is transversal to $F'_{j(i)}$. For sufficiently small $\varepsilon>0$ the tubular neighborhoods $N_\varepsilon(F_i)$ do not intersect each other. Thus we have a differentiable Z_2-map $\tilde{g} : N_\varepsilon(F(M, T))\longrightarrow N$, $\tilde{g}|N_\varepsilon(F_i)=g_i$, which is transversal to $F(N, T')$ and $\tilde{g}|F(M, T)=f|F(M, T)$. By making use of Lemma 3.4 of [10], we have a δ-neighborhood $N_\delta(F(M, T))\subset N_\varepsilon(F(M, T))$, $0<\delta<\varepsilon$, and an equivariant homotopy F_t such that

$$F_0=f$$
$$F_t|M-N_\varepsilon(F(M, T))=f|M-N_\varepsilon(F(M, T))$$
$$F_1|N_\delta(F(M, T))=\tilde{g}.$$

For $x\in M-F(M, \mathrm{T})$ there is an open neighborhood $U(x)$ of x with $TU(x)\cap U(x)=\phi$. Let

$$\mathcal{U}=\{U(x)\cup TU(x)\,|\,x\in M-F(M, T)\}\cup\{N_\delta(F(M, T))\}$$

be an open covering consisting of above neighborhoods and the δ-neighborhood of $F(M, T)$. Then there is a finite open covering $\mathcal{O}=\{V_i\}$ of M which is required that $\mathcal{O}$ is a refinement of $\mathcal{O}$ and each open set V_i of $\mathcal{O}$ contained an open set W_i with $\overline{W}_i\subset V_i$ and $\bigcup_i W_i=M$, and we describe the index i of V_i as non-positive integer if $V_i\subset N_\delta(F(M, T))$. Let

$$\mathcal{O}=\{V_1, V_2, \cdots, V_k\}\cup\{V_j|j\leq 0, j\in Z\}.$$

We take $V_i=O_i\cup TO_i$, O_i an open set with $O_i\cap TO_i=\phi$. Suppose that

$\tilde{g}_r : M \longrightarrow N$ is a differntiable equivariant map which is transversal to $F(N, T')$ at each point of $A_r = \overline{W}_1 \cup \overline{W}_2 \cup \cdots \cup \overline{W}_r \cup F(M, T)$, and which is homotopic to $\tilde{g}_{r-1}$ and $\tilde{g}_0 = \tilde{g}$. By the ordinary transversality theorem we have a differentiable map $h_{r+1} : O_{r+1} \longrightarrow N$ which is transversal to $F(N, T')$, homotopic to $\tilde{g}_r | O_{r+1}$ and $h_{r+1} | O_{r+1} \cap A_r = \tilde{g}_r | O_{r+1} \cap A_r$. Then we have an equivariant map

$$\tilde{h} : V_{r+1} = O_{r+1} \cup TO_{r+1} \longrightarrow N$$

with $\tilde{h}_{r+1}(x) = h_{r+1}(x)$, $x \in O_{r+1}$ and $\tilde{h}_{r+1}(T(x)) = T' h_{r+1}(T(x))$, and $\tilde{h}_{r+1} | V_{r+1} \cap A_r = \tilde{g}_r | V_{r+1} \cap Ar$. The homotopy $H_{r+1,t} : V_{r+1} \longrightarrow N$ is given so that $H_{r+1,t} | V_{r+1} \cap A_r = \tilde{g}_r | V_{r+1} \cap A_r$, $H_{r+1,1} = \tilde{h}_{r+1}$ and $H_{r+1,0} = \tilde{g}_r$. Hence we have the homotopy $H'_{r+1,t} : \overline{W}_{r+1} \cup A_r \longrightarrow N$ with $H'_{r+1,t} | A_r = \tilde{g}_r | A_r$. The Lemma 3.2 of [10] implies that $H'_{r+1,t}$ can be extend to a differentiable equivariant homotopy $F_{r+1,t}$ such that $\tilde{g}_{r+1} = F_{r+1,1}$ is homotopic to $\tilde{g}_r$ and transversal to $F(N, T')$ at each point of $\overline{W}_{r+1} \cup A_r$. By induction on r we complete the proof.

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