Regular abstract affine near-rings

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1. Introduction

In ring theory, it is well known that regular rings are characterized in terms of quasi-ideals. The purpose of this paper is to extend the above result to a class of abstract affine near-rings.

In Sections 2 and 3, we deal with some properties of regular elements and quasi-ideals of abstract affine near-rings, which will be used in the following sections. In Section 4, we give some characterizations of regular abstract affine near-rings by quasi-ideals and generalize the results of Kovács [1], Luh [2] and Steinfeld [4]. In Section 5, we apply the preceding results to characterize regular abstract affine near-rings without non-zero nilpotent elements in terms of quasi-ideals and generalize the result of Kovács [1].

For the basic terminology and notation we refer to [3].

2. Regular elements of abstract affine near-rings

Let $N$ be a near-ring, which always means right one throughout this paper. For an element $n$ of $N$, we denote $n-n_0$ by $n_0$ and $n0$ by $n_c$. Then $n = n_0 + n_c$ with $n_0 \in N_0$, $n_c \in N_c$, where $N_0$ and $N_c$ are the zero-symmetric and constant parts of $N$, respectively.

A near-ring $N$ is called an abstract affine near-ring if $N$ is abelian and $N_0 = N_d$, where $N_d$ is the set of all distributive elements of $N$.

An element $n$ of a near-ring $N$ is called regular if $n = nxn$ for some element $x$ of $N$. The near-ring $N$ is called regular if every element of $N$ is regular.

These definitions lead immediately to:

**Proposition 2.1.** The following assertions on an element $n$ of an abstract affine near-ring $N$ are equivalent:
(1) \( n \) is a regular element of \( N \).
(2) \( n_0 \) is a regular element of \( N_0 \).
(3) \( n = nmn_0 \) for some element \( m \) of \( N_0 \).

**Proof.** (1) \( \Rightarrow \) (2) : First we remark that \((ab)_0 = a_0b_0\) for elements \( a, b \) of \( N \). In fact, since \( a_0 \) is distributive, we have
\[
ab = (a_0 + a_c) b = a_0(b_0 + b_c) + a_c = a_0b_0 + a_0b_c + a_c,
\]
whence \((ab)_0 = a_0b_0\).

Now the assumption (1) implies that \( n = nxn \) for some element \( x \) of \( N \). So, by the above remark \( n_0 = n_0x_0n_0 \), that is, \( n_0 \) is a regular element of \( N_0 \).

(2) \( \Rightarrow \) (3) : The assumption (2) implies that \( n_0 = n_0m_0n_0 \) for some element \( m \) of \( N_0 \). Then we have
\[
nmn_0 = (n_0 + n_c)m_0n_0 = n_0mn_0 + n_c = n_0 + n_c = n.
\]

(3) \( \Rightarrow \) (1) : By the assumption (3), \( m \) is distributive. So we have
\[
n(m - mn_c)n = n(mn - mn_c) = n(mn_0 + mn_c - mn_c) = nmn_0 = n,
\]
that is, \( n \) is a regular element of \( N \).

The equivalence of (2) and (3) is true without the assumption that \( N \) is an abstract affine near-ring. The following examples show that the other implications do not hold for arbitrary near-rings.

**Example 2.2.** Let \( V = \{0, a, b, c\} \) be the near-ring due to [3, Near-rings of low order (E-19)] defined by the tables
\[
+ \begin{array}{cccc}
0 & a & b & c \\
0 & 0 & a & b & c \\
a & a & 0 & c & b \\
b & b & c & 0 & a \\
c & c & b & a & 0 \\
\end{array} \quad \cdot \begin{array}{cccc}
0 & a & b & c \\
0 & 0 & 0 & 0 \\
a & a & a & a \\
b & 0 & 0 & 0 \\
c & a & a & a & c \\
\end{array}
\]
Then \( V_o = \{0, b\} \), \( V_c = \{0, a\} \) and \( c = b + a \). Since \( c^3 = c \), \( c \) is a regular element of \( V \). But \( c_0 = b \) is not a regular element of \( V_0 \), because \( bxb = 0 \) for all elements \( x \) of \( V_0 \).

**Example 2.3.** Let \( K = \{0, a, b, c\} \) be the near-ring due to [3, Near-rings of low order (E-21)] , whose addition coincides with that of \( V \) in Example 2.2 and
whose multiplication is defined by the table

\[
\begin{array}{c|cccc}
\cdot & 0 & a & b & c \\
\hline
0 & 0 & 0 & 0 & 0 \\
a & a & a & a & a \\
b & 0 & 0 & b & 0 \\
c & a & a & c & a \\
\end{array}
\]

Then \( K_0 = \{0, b\} \), \( K_c = \{0, a\} \) and \( c = b + a \). Since \( b^3 = b \) and \( c_0 = b, \ c_0 \) is a regular element of \( K_0 \). But \( c \) is not a regular element of \( K \), because \( c y c = a \) for all elements \( y \) of \( K \).

As an immediate consequence of Proposition 2.1, we have:

**Corollary 2.4.** An abstract affine near-ring \( N \) is regular if and only if the subring \( N_0 \) of \( N \) is regular.

3. Quasi-ideals of abstract affine near-rings

Let \( A, B \) and \( C \) be three non-empty subsets of a near-ring \( N \). Then \( AB \) (\( ABC \)) denotes the set of all finite sums of the form \( \sum a_k b_k \) with \( a_k \in A, b_k \in B \) (\( \sum a_k b_k c_k \) with \( a_k \in A, b_k \in B, c_k \in C \)), and \( A \ast B \) denotes the set of all finite sums of the form \( \sum (a_k (a'_k + b_k) - a_k a'_k) \) with \( a_k, a'_k \in A, b_k \in B \).

A right \( N \)-subgroup (left \( N \)-subgroup) of a near-ring \( N \) is a subgroup \( S \) of \( (N, +) \) such that \( SN \subseteq S \) (\( NS \subseteq S \)). Note that left \( N \)-subgroups are called \( N \)-subgroups in [3], and that in case \( N \) is an abstract affine near-ring, right \( N \)-subgroups coincide with right ideals.

A quasi-ideal of a near-ring \( N \) is a subgroup \( Q \) of \( (N, +) \) such that \( N \ast Q \cap NQ \cap QN \subseteq Q \). In case that \( N \) is zero-symmetric, a subgroup \( Q \) of \( (N, +) \) is a quasi-ideal of \( N \) if and only if \( NQ \cap QN \subseteq Q \) (see [5, Proposition 3]).

For a non-empty subset \( A \) of a near-ring \( N \), we denote \( A \cap N_0 \) by \( A_0 \) and \( A \cap N_c \) by \( A_c \). A quasi-ideal \( Q \) of \( N \) is said to be of the first kind if \( Q = Q_0 + Q_c \). A quasi-ideal \( Q \) of \( N \) is of the first kind if and only if \( Q \) is a subnear-ring of \( N \) (see [5, Theorem 1]). Right \( N \)-subgroups, left \( N \)-sub-
groups, $N_0$ and $N_c$ are quasi-ideals of the first kind, and the intersection of a family of quasi-ideals of the first kind is again a quasi-ideal of the first kind.

We state here some known results on quasi-ideals of abstract affine near-rings which will be used later. In the following remarks, $N$ denotes an abstract affine near-ring.

**REMARKS 3.1.** A subgroup $R$ of $(N, +)$ is a right $N$-subgroup of $N$ if and only if $R = R_0 + R_c$ where $R_0$ is a right $N_0$-subgroup of $N_0$, $R_c$ is a subgroup of $(N_c, +)$ and $R_0 N_c \subseteq R_c$ (see [3, Proposition 9.73]).

3.2. A subgroup $L$ of $(N, +)$ is a left $N$-subgroup of $N$ if and only if $L = L_0 + L_c$ where $L_0$ is a left $N_0$-subgroup of $N_0$ and $L_c = N_c$ (see [3, Proposition 9.73]).

3.3 A subgroup $Q$ of $(N, +)$ is a quasi-ideal of the first kind of $N$ if and only if $Q = Q_0 + Q_c$ where $Q_0$ is a quasi-ideal of $N_0$, $Q_c$ is a subgroup of $(N_c, +)$ and $N_0 Q_c \cap (Q_0 N_c + Q_c) \subseteq Q_c$ (see [6, Theorem]).

The following will be also used later:

**PROPOSITION 3.4.** Let $N$ be an abstract affine near-ring. Then the following assertions hold:

1. For every quasi-ideal $Q$ of the first kind of $N$, $QN_0 Q \subseteq Q$.
2. For every right $N$-subgroup $R$ of $N$, $R^2 = R_0^2 + R_c$.
3. For every left $N$-subgroup $L$ of $N$, $L^2 = L_0^2 + N_c$.
4. For every right $N$-subgroup $R$ and left $N$-subgroup $L$ of $N$,

   \[ RL = R_0 L_0 + R_c \quad \text{and} \quad R \cap L = R_0 \cap L_0 + R_c. \]

**PROOF.** (1) For a quasi-ideal $Q$ of the first kind of $N$, by Remark 3.3, we have

\[ Q_0 N_0 Q_0 \subseteq N_0 Q_0 \cap Q_0 N_0 \subseteq Q_0 \quad \text{and} \quad Q_0 N_c Q_c \subseteq N_0 Q_c \cap Q_0 N_c \subseteq Q_c. \]

So $Q N_0 Q = Q_0 N_0 Q_0 + Q_0 N_0 Q_c + Q_c \subseteq Q_0 + Q_c = Q$.

(2) For a right $N$-subgroup $R$ of $N$, by Remark 3.1, we have $R_0 N_c \subseteq R_c$. So $R^2 = R_0 R_0 + R_0 R_c + R_c = R_0^2 + R_c$.

(3) For a left $N$-subgroup $L$ of $N$, by Remark 3.2, we have $L_c = N_c$. So $L^2 = L_0 L_0 + L_0 N_c + N_c = L_0^2 + N_c$.

(4) Similarly, Remarks 3.1 and 3.2 imply that for a right $N$-subgroup $R$
and left $N$-subgroup $L$ of $N$,

$$RL = R_0L_0 + R_0N_c + R_c = R_0L_0 + R_c.$$  

Moreover, $R \cap L$ is a quasi-ideal of the first kind of $N$ with $(R \cap L)_0 = R_0 \cap L_0$ and $(R \cap L)_c = R_c$. So we have $R \cap L = R_0 \cap L_0 + R_c$.

4. Characterizations of regular abstract affine near-rings

Now we state the main result of this paper:

**Theorem 4.1.** The following conditions on abstract affine near-ring $N$ are equivalent:

1. $N$ is regular.
2. Every quasi-ideal $Q$ of the first kind of $N$ has the form $Q \cap N_0 = Q$.
3. For every right $N$-subgroup $R$ and left $N$-subgroup $L$ of $N$,
   a. $R^2 = R$,
   b. $L^2 = L$,
   c. $RL$ is a quasi-ideal of the first kind of $N$.
4. For every right $N$-subgroup $R$ and left $N$-subgroup $L$ of $N$,

   $$RL = R \cap L.$$  

**Proof.** (1) $\implies$ (2): Let $Q$ be a quasi-ideal of the first kind of $N$. Then the assumption (1) and Proposition 2.1 imply $Q \subseteq QN_0 \subseteq QN_0Q$. Moreover, by Proposition 3.4 (1), we get $QN_0Q \subseteq Q$. So $Q = QN_0Q$.

(2) $\implies$ (3): Let $R$, $L$ be right and left $N$-subgroups of $N$, respectively. Since $R$ is a quasi-ideal of the first kind of $N$, by the assumption (2) we get $R = RN_0R \subseteq RR \subseteq R$, that is, $R^2 = R$.

The statement $L^2 = L$ can be proved dually.

Finally, we show that $RL$ is a quasi-ideal of the first kind of $N$. In view of Proposition 3.4 (4) and Remark 3.3, we have to prove that $R_0L_0$ is a quasi-ideal of $N_0$ and that $N_0R_c \cap ((R_0L_0)N_c + R_c) \subseteq R_c$.

It is clear that $R_0L_0$ is a subgroup of $(N_0, +)$. Since $R_0$ and $L_0$ are right and left $N_0$-subgroups of $N_0$ respectively, we get $N_0(R_0L_0) \subseteq N_0L_0 \cap R_0N_0 \subseteq L_0 \cap R_0$.

Moreover, by [6, Corollary], $L_0 \cap R_0$ is a quasi-ideal of the first kind of $N$. So, by the assumption (2) we get $L_0 \cap R_0 = (L_0 \cap R_0)N_0(L_0 \cap R_0) \subseteq R_0N_0L_0 \subseteq R_0L_0$.  

Hence \( N_0 (R_0 L_0) \cap (R_0 L_0) N_0 \subseteq R_0 L_0 \), that is, \( R_0 L_0 \) is a quasi-ideal of \( N_0 \).

On the other hand, from the relation \( (R_0 L_0) N_c \subseteq R_0 N_c \subseteq R_0 \), it follows that \( N_0 R_c \cap ((R_0 L_0) N_c + R_c) \subseteq N_0 R_c \cap R_c \subseteq R_c \). Thus \( RL \) is a quasi-ideal of the first kind of \( N \).

(3)\( \Rightarrow \) (4) : Let \( R, L \) be right and left \( N \)-subgroups of \( N \), respectively. In view of Proposition 3.4 (4), we have to prove that \( R_0 L_0 = R_0 \cap L_0 \).

First we show that

\[
(4.2) \quad Q_0 = N_0 Q_0 \cap Q_0 N_0,
\]

for every quasi-ideal \( Q \) of the first kind of \( N \).

By Remark 3.1, \( Q_0 + Q_0 N_0 + N_c \) is a right \( N \)-subgroup of \( N \) containing \( Q \). So the condition (a) implies that

\[
Q_0 Q_0 + Q_0 N_0 + N_c = (Q_0 + Q_0 N_0 + N_c)^2
\]

\[
= Q_0^2 + Q_0 (Q_0 N_0) + Q_0 N_0 + (Q_0 N_0) Q_0 + (Q_0 N_0)^2 + (Q_0 N_0) N_c + N_c
\]

\[
\subseteq Q_0 N_0 + N_c,
\]

whence \( Q_0 \subseteq Q_0 N_0 \). Moreover, by Remark 3.2, \( Q_0 + N_0 Q_0 + N_c \) is a left \( N \)-subgroup of \( N \) containing \( Q \). So the condition (b) implies dually that \( Q_0 \subseteq N_0 Q_0 \). Hence \( Q_0 \subseteq N_0 Q_0 \cap Q_0 N_0 \subseteq Q_0 \), that is, \( Q_0 = N_0 Q_0 \cap Q_0 N_0 \).

Now we show that \( R_0 L_0 = R_0 \cap L_0 \). As \( R_0 L_0 \subseteq R_0 \cap L_0 \), always holds, we have only to prove that \( R_0 \cap L_0 \subseteq R_0 L_0 \).

Since \( R \cap L \) is a quasi-ideal of the first kind of \( N \) with \( (R \cap L)_0 = R_0 \cap L_0 \), by the relation (4.2) we get

\[
(4.3) \quad R_0 \cap L_0 = N_0 (R_0 \cap L_0) \cap (R_0 \cap L_0) N_0.
\]

Put \( R^* = (R_0 \cap L_0) N_0 + N_c \). Then, by Remark 3.1, \( R^* \) is a right \( N \)-subgroup of \( N \) with \( (R^*)_0 = (R_0 \cap L_0) N_0 \). So, by the condition (a) and Proposition 3.4 (2), we get

\[
(4.4) \quad (R_0 \cap L_0) N_0 = ((R_0 \cap L_0) N_0)^2 \subseteq (R_0 L_0) N_0.
\]

Dually put \( L^* = N_0 (R_0 \cap L_0) + N_c \). Then, by Remark 3.2, \( L^* \) is a left \( N \)-subgroup of \( N \) with \( (L^*)_0 = N_0 (R_0 \cap L_0) \). So, by the condition (b) and Proposition 3.4 (3), we get

\[
(4.5) \quad N_0 (R_0 \cap L_0) = (N_0 (R_0 \cap L_0))^2 \subseteq N_0 (R_0 L_0).
\]

On the other hand, the condition (c) and Proposition 3.4 (4) imply that \( R_0 L_0 \) is a quasi-ideal of \( N_0 \). So, from the relations (4.3), (4.4) and (4.5), it follows that
\[ R_0 \cap L_0 \subseteq N_0 (R_0 L_0) \cap (R_0 L_0) N_0 \subseteq R_0 L_0. \]

Thus \( RL = R \cap L \).

(4)\( \Rightarrow \) (1) : Let \( a \) be an arbitrary element of \( N \). By \( [a_0] \) we denote the subgroup of \( (N_0, +) \) generated by \( a_0 \). Then, by Remark 3.1, \([a_0] + a_0 N_0 + N_c \) is a right \( N \)-subgroup of \( N \) containing \( a \). Since \( N \) is a left \( N \)-subgroup of \( N \), the assumption (4) implies that

\[ a \in ([a_0] + a_0 N_0 + N_c) \cap N = ([a_0] + a_0 N_0 + N_c) N = a_0 N_0 + a_0 N_c + (a_0 N_0) N_c + N_c \subseteq a_0 N_0 + N_c, \]

whence \( a_0 \in a_0 N_0 \). Dually, \( [a_0] + N_0 a_0 + N_c \) is a left \( N \)-subgroup of \( N \) containing \( a \). A similar argument shows that \( a_0 \in N_0 a_0 \). Hence we get

(4.6) \[ a_0 \in a_0 N_0 \cap N_0 a_0. \]

On the other hand, put \( R' = a_0 N_0 + N_c \) and \( L' = N_0 a_0 + N_c \). Then, by Remarks 3.1 and 3.2, \( R' \) and \( L' \) are right and left \( N \)-subgroups of \( N \) with \( (R')_o = a_0 N_0 \) and \( (L')_o = N_0 a_0 \), respectively. From the condition (4) and Proposition 3.4 (4), it follows

(4.7) \[ a_0 N_0 \cap N_0 a_0 = (a_0 N_0) (N_0 a_0) \subseteq a_0 N_0 a_0. \]

By the relations (4.6) and (4.7), we get \( a_0 \in a_0 N_0 a_0 \), that is, \( a_0 \) is a regular element of \( N_0 \). So, by Proposition 2.1, \( a \) is a regular element of \( N \). Thus \( N \) is regular.

Theorem 4.1 can not be extended to arbitrary near-rings. The following example shows that neither of the conditions (2), (3) and (4) implies the condition (1) in general.

EXAMPLE 4.8. All quasi-ideals of the first kind of the near-ring \( K \) in Example 2.3 are \( \{0\} \), \( K_0 \), \( K_c \) and \( K \). All of them are right \( K \)-subgroups of \( K \), and all left \( K \)-subgroups of \( K \) are \( K_c \) and \( K \). It can be easily seen that each of the conditions (2), (3) and (4) holds for \( K \). But \( K \) is not regular.

5. Regular abstract affine near-rings without non-zero nilpotents

As an application of Theorem 4.1, we characterize regular abstract affine near-rings without non-zero nilpotent elements, in terms of quasi-ideals.

THEOREM 5.1. The following conditions on abstract affine near-ring \( N \) are
equivalent:

(1) \( N \) is a regular near-ring without non-zero nilpotent elements.

(2) Every quasi-ideal of the first kind of \( N \) is idempotent.

(3) For every right \( N \)-subgroup \( R \) and left \( N \)-subgroup \( L \) of \( N \),
\[
RL = R \cap L \subseteq LR.
\]

PROOF. (1)\(\Rightarrow\)(2): Let \( Q \) be a quasi-ideal of the first kind of \( N \). Since \( Q \) is a subnear-ring of \( N \), \( Q^2 \subseteq Q \). So we have only to prove \( Q \subseteq Q^2 \).

Let \( q \) be an arbitrary element of \( Q \). By the regularity of \( N \) and Proposition 2.1, we have \( q = q m q_0 \) for some element \( m \) of \( N_0 \). Then \( q_0 = q_0 m q_0 \). So \( m q_0 \) is an idempotent element of \( N_0 \). Moreover, the condition (1) and Corollary 2.4 imply that \( N_0 \) is a regular ring without non-zero nilpotent elements. Hence \( m q_0 \) is in the center of \( N_0 \). Using also \( Q = Q N_0 Q \) by Theorem 4.1, we get
\[
q = (q m) q_0 (m q_0) = (q m) (m q_0) q_0 = (q m^2 q_0) q_0 \in (Q N_0 Q) Q = Q^2,
\]
that is, \( Q \subseteq Q^2 \). Thus \( Q \) is idempotent.

(2)\(\Rightarrow\)(3): Since \( R \cap L \) is a quasi-ideal of the first kind of \( N \), the condition (2) implies that
\[
R \cap L = (R \cap L)^2 \subseteq RL \cap LR.
\]
On the other hand, the relation \( RL \subseteq R \cap L \) always holds. So, we get \( RL = R \cap L \subseteq LR \).

(3)\(\Rightarrow\)(1): The condition (3) and Theorem 4.1 imply that \( N \) is regular.

Let \( n \) be an element of \( N \). By induction on \( k \), we can show that
\[
n^k = n_0^k + n_c + \sum_{j=1}^{k-1} n_0^j n_c,
\]
for all integers \( k \geq 2 \).

Now assume that \( n \) is a nilpotent element of \( N \) with \( n^k = 0 \) for an integer \( k \geq 2 \). Then, by the relation (5.2), we get
\[
n_0^k = 0 \quad \text{and} \quad n_c + \sum_{j=1}^{k-1} n_0^j n_c = 0.
\]

On the other hand, by the regularity of \( N \), we get
\[
n_0^{k-1} \in n_0^{k-1} N \cap N n_0^{k-1}.
\]
Here \( n_0^{k-1} \) is a left \( N \)-subgroup of \( N \). Moreover, since \( n_0^{k-1} \) is distributive, \( n_0^{k-1} N \) is a right \( N \)-subgroup of \( N \). So, by the condition (3) and the first relation in (5.3), we get
\[
n_0^{k-1} \in (N n_0^{k-1}) (n_0^{k-1} N) = N n_0^{k+1} N = N_c.
\]
whence \( n_0^{-1} = 0 \).

Repeating this procedure, we can get \( n_0 = 0 \). This result and the second relation in (5.3) imply that \( n_e = 0 \), that is, \( n = 0 \). Thus \( N \) has no non-zero nilpotent elements.

Theorem 5.1 cannot be extended to arbitrary near-rings. The following example shows that neither the condition (2) nor (3) implies the condition (1) in general.

**EXAMPLE 5.4.** It is easily seen that each of the conditions (2) and (3) holds for the near-ring \( K \) in Example 2.3. But \( K \) is not regular.

**References**