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https://doi.org/10.15017/1449062

出版情報：九州大学教養部数学雑誌. 17 (1), pp.11-13, 1989-12. 九州大学教養部数学教室

Math. Rep.
17-1, 1989.

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(Received September 6, 1989)

1. Introduction

In his paper [2], Lajos proved the following theorems :

(1) An additive subgroup M of a regular ring A is a quasi-ideal of A if and only if $MAM \subseteq M$.

(2) Let Q_1, Q_2 be quasi-ideals of a regular ring A. Then the product $Q_1 Q_2$ is a quasi-ideal of A.

The purpose of this note is to extend the above theorems to a class of regular distributively generated near-rings.

For the basic terminology and notation we refer to [3].

2. Preliminaries

Let N be a near-ring, which always means right one throughout this note.

If A, B and C are three non-empty subsets of N, then AB (ABC) denotes the set of all finite sums of the form $\sum a_k b_k$ with $a_k \in A$, $b_k \in B$ $(\sum a_k b_k c_k$ with $a_k \in A$, $b_k \in B$, $c_k \in C)$. In case N is a ring, $ABC=(AB)C=A(BC)$. However, in case N is not a ring, $ABC=(AB)C \subseteq A(BC)$ in general.

A *right N-subgroup* (*left N-subgroup*) of N is a subgroup S of $(N, +)$ such that $SN \subseteq S$ $(NS \subseteq S)$. For every subgroup S of $(N, +)$, SN is a right N-subgroup of N. In fact, SN is a subgroup of $(N, +)$ and $(SN)N \subseteq S(NN) \subseteq SN$.

A *quasi-ideal* of a zero-symmetric near-ring N is a subgroup Q of $(N, +)$ such that $QN \cap NQ \subseteq Q$ (see [4, Proposition 3]).

A near-ring N is called *regular*, if for every element n of N there exists an element x in N such that $nxn=n$.

LEMMA. *Let N be a regular near-ring. Then, for every right N-sub-*

group R of N and for every non-empty subset S of N, $R \cap S \subseteq RS$.

PROOF. Suppose that R and S are a right N-subgroup and a non-empty subset of N, respectively. Let n be any element of $R \cap S$. By the regularity of N, there exists an element x in N such that $n = nxn$. Since $nx \in R$ and $n \in S$, we have $n = nxn \in RS$. Thus $R \cap S \subseteq RS$.

3. Main results

A near-ring N is *distributively generated* if there is a subsemigroup D of $(N_d, \cdot)$ generating $(N, +)$, where N_d is the set of all distributive elements of N.

We recall that a distributively generated near-ring is zero-symmetric, and that if N is a distributively generated near-ring, then by [1, 1.3.2] NS is a left N-subgroup of N for every subgroup S of $(N, +)$.

Now we are in a position to state the main results of this note.

THEOREM 1. *Let N be a regular distributively generated near-ring. Then a subgroup Q of $(N, +)$ is a quasi-ideal of N if and only if $Q(NQ) \subseteq Q$.*

PROOF. Suppose that Q is a quasi-ideal of N. Since NQ is a left N-subgroup of N, $Q(NQ) \subseteq N(NQ) \subseteq NQ$. Moreover, $Q(NQ) \subseteq QN$. So we have

$$Q(NQ) \subseteq QN \cap NQ \subseteq Q.$$

Conversely, suppose that Q is a subgroup of $(N, +)$ with $Q(NQ) \subseteq Q$. Then, since QN is a right N-subgroup of N, by Lemma, we have

$$QN \cap NQ \subseteq (QN)(NQ).$$

Moreover, since NQ is a left N-subgroup of N, we have

$$(QN)(NQ) \subseteq Q(N(NQ)) \subseteq Q(NQ).$$

Therefore it follows that

$$QN \cap NQ \subseteq Q,$$

that is, the subgroup Q is a quasi-ideal of N.

THEOREM 2. *Let Q_1, Q_2 be quasi-ideals of a regular distributively generated* *near-ring N. Then the product Q_1Q_2 is likewise a quasi-ideal of N.*

PROOF. It is easy to see that the product Q_1Q_2 is a subgroup of $(N, +)$. Moreover, since NQ_2 is a left N-subgroup of N, we have

$$N(Q_1Q_2) \subseteq N(NQ_2) \subseteq NQ_2.$$

So, it follows from Theorem 1 that

$$(Q_1Q_2)(N(Q_1Q_2)) \subseteq (Q_1Q_2)(NQ_2) \subseteq Q_1(Q_2(NQ_2)) \subseteq Q_1Q_2,$$

that is, the product Q_1Q_2 is a quasi-ideal of N.

Now, it is natural to ask whether Theorem 1 and Theorem 2 are true or not respectively for arbitrary regular zero-symmetric near-rings. This question is still open.

References

[1] A. FRÖHLICH : *Distributively generated near-rings* (I. *Ideal theory*), Proc. London Math. Soc., 8 (1958), 76-94.
[2] S. LAJOS : *On quasiideals of regular rings*, Proc. Japan Acad., **38** (1962), 210-211.
[3] G. PILZ : Near-rings, 2nd Edition, North-Holland, Amsterdam 1983.
[4] I. YAKABE : *Quasi-ideals in near-rings*, Math. Rep. Kyushu Univ., **14** (1983), 41-46.