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Remarks on semilinear elliptic boundary value problems in exterior domains

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1. Introduction

In this note we shall consider the semilinear elliptic boundary value problems (BVP) of the form

(1.1) $Lu=f(x, u, \nabla u)$ in Ω,

$Bu=g(x, u)$ on $\partial\Omega$,

where L is a linear, second order, uniformly elliptic differential operator, B is a first order boundary operator and Ω is an exterior domain in R^N with suitably smooth boundary $\partial\Omega$. We denote $\overline{\Omega}=\Omega\cup\partial\Omega$.

In a recent paper [7], we obtained some results concerning the existence and multiplicity theory of bounded solutions (B-solutions) of the BVP (1.1). In particular, when f and g are respectively convex in u, ∇u and u, we showed the facts that the number of bounded positive solutions (BP-solutions) of the BVP (1.1) changes with the behavior of the maximal and the minimal BP-solutions.

The purpose of this paper, when f and g are sublinear in a sense as defined later, is to study the multiplicity of positive solutions (P-solutions) of the BVP (1.1). The main tools to discuss this matter are the method of super-subsolutions and the concept of degree of lower contact of two solutions of the BVP (1.1). Moreover, the idea of the investigation is the constructing a nonlinear operator equation whose fixed points are solutions of the BVP (1.1). In this case, it is very difficult to demand that the fixed point operator shall be increasing. For that reason, we refer the works of H. Amann and M. G. Crandall [1] and F. Inkmann [3]. The following section is composed of our assumptions, definitions and proposition. In the last section we discuss the multiplicity theorem of P-solutions of the BVP (1.1).

2. Preliminaries

Let u and v be arbitrarily continuous functions in $\overline{\Omega}$. We write $u \leqq v$ if $u(x) \leqq v(x)$ for every $x \in \overline{\Omega}$, $u < v$ if $u(x) \leqq v(x)$ but $u(x) \not\equiv v(x)$ for every $x \in \overline{\Omega}$, and $u \ll v$ if $u(x) < v(x)$ for every $x \in \overline{\Omega}$.

Throughout this paper, we denote by D an arbitrarily bounded subdomain of Ω. If $u \in C^{m+\alpha}(\bar{D})$, *we write* $u \in C^{m+\alpha}_{\mathrm{loc}}(\overline{\Omega})$, where $0 < \alpha < 1$ and m is a nonnegative integer. Moreover, for the functions u_1 and u_2 satisfying $u_1 \leqq u_2$, we define the interval :

$$[u_1, u_2]^{m+\alpha}_{\mathrm{loc}} = \{ w \mid u_1 \leqq w \leqq u_2,\ w \in C^{m+\alpha}_{\mathrm{loc}}(\overline{\Omega}) \}$$

Also, we denote $R^+ = [0, \infty)$ and $C^{2+\alpha}_+(\overline{\Omega}) = \{ u \mid u \in C^{2+\alpha}_{\mathrm{loc}}(\overline{\Omega}),\ u \gg 0 \}$.

The operator L and B of the BVP (1.1) are defined as follows, respectively, :

$$Lu := -\sum_{i,j=1}^{N} D_i(a_{ij}(x) D_j u)$$

$$Bu := \sum_{i,j=1}^{N} a_{ij}(x) \cos(\nu, x_j) D_i u + h(x) u,$$

where $D_i = \dfrac{\partial}{\partial x_i}$, ν is the outer normal to Ω at x (with respect to Ω), and (ν, x_j) is the angle between the normal ν and the positive x_j-axis.

Let $f : \Omega \times R \times R^N \to R$ and $g : \partial\Omega \times R \to R$ be given functions. We then consider the BVP (1.1), i.e.,

$$Lu = f(x, u, \nabla u) \quad \text{in } \Omega,$$

$$Bu = g(x, u) \qquad \text{on } \partial\Omega,$$

where $\nabla u = (D_1 u, D_2 u, \cdots, D_N u)$.

Assumptions.

For this BVP, we suppose the following :

(A-1) $a_{ij}(x) = a_{ji}(x)$, $a_{ij}(x)$ is bounded in $\overline{\Omega}$ and belongs to the class $C^{1+\alpha}_{\mathrm{loc}}(\overline{\Omega})$;

(A-2) L is uniformly elliptic in Ω, i.e., there exists a constant $\lambda > 0$ such that

$$\lambda \sum_{i=1}^{N} \xi_i^2 \leqq \sum_{i,j=1}^{N} a_{ij}(x) \xi_i \xi_j$$

for all $x \in \overline{\Omega}$ and all real vector $\xi = (\xi_1, \xi_2, \cdots, \xi_N)$;

(A-3) $h(x) \gg 0$ on $\partial\Omega$ and $h(x)$ is the trace on $\partial\Omega$ of a function of class $C^{1+\alpha}_{\mathrm{loc}}(\overline{\Omega})$;

(A-4) $\partial\Omega$ is of class $C^{2+\alpha}$;

(A-5) $f(x, 0, 0) \geqq 0$ in $\bar{\Omega}$, $f(x, s, t) \in C^1_{\mathrm{loc}}(\bar{\Omega} \times R \times R^N)$ and is bounded in $\bar{\Omega} \times R^+ \times R^N$. Moreover, $f_s(x, s, t)$ and $f_t(x, s, t)$ are bounded in s and t ;

(A-6) For each subdomain $D \subset \bar{\Omega}$, there exists an increasing function $\mu_{\bar{D}} : R^+ \to R^+$ such that

$$|f(x, s, t)| \leqq \mu_{\bar{D}}(|s|)(1+|t|^2)$$

for $(x, s, t) \in \bar{\Omega} \times R \times R^N$;

(A-7) $g(x, s)$ is the trace on $\partial\Omega \times R^+$ of a function of class $C^{1+\alpha}_{\mathrm{loc}}(\bar{\Omega} \times R)$. Moreover, $g(x, 0) \geqq 0$ on $\partial\Omega$, and there exists a function $w(x) \in C^{1+\alpha}_{\mathrm{loc}}(\bar{\Omega})$ such that $w(x) \gg 0$ on $\partial\Omega$ and

(2.1) $g(x, s) - g(x, s') \geqq -w(x)(s-s')$

for all $x \in \partial\Omega$ and s, s' satisfying $-\infty < s \leqq s' < \infty$.

This inequality (2.1) implies that

$$g^*(x, s) := g(x, s) + w(x)s$$

is nondecreasing in s. We now consider the boundary condition of the form :

$$Bu + w(x)u = g^*(x, u) \text{ on } \partial\Omega.$$

Then, this condition is obviously equivalent to the boundary condition of the BVP (1.1). Accordingly, we assume without loss of generality that

(B) $g(x, s)$ is nondecreasing in s for all $x \in \partial\Omega$.

Definition 2.1. Supersolution and subsolution. A function $\hat{v}(x)$ is called a *supersolution* for the BVP (1.1) if $\hat{v}(x) \in C^2_{\mathrm{loc}}(\bar{\Omega})$ and

$$L\hat{v} \geqq f(x, \hat{v}, \nabla\hat{v}) \text{ in } \Omega, \quad B\hat{v} \geqq g(x, \hat{v}) \text{ on } \partial\Omega.$$

A subsolution $\bar{v}(x)$ is defined by reversing the above inequality signs. Moreover, a supersolution (subsolution) is called a strict supersolution (strict subsolution) if it is not a solution of the BVP (1.1).

Degree of lower contact. Let u_1 and u_2 be P-solutions of the BVP (1.1) such that $u_1(x) \geqq u_2(x)$ in $\bar{\Omega}$, and let γ be an arbitrarily nonnegative constant. We consider the set

$$\Gamma = \{\gamma \mid u_2(x) \geqq \gamma u_1(x),\ x \in \bar{\Omega}\}.$$

Since $\gamma = 0$ is an element of Γ, Γ is obviously nonempty. Therefore, we call the value $\sup_{\gamma \in \Gamma} \Gamma$ the degree of lower contact of u_1 and u_2, and denote it by $\bar{\gamma}(u_1, u_2)$ or simply $\bar{\gamma}$.

Remark 2.2. Clearly, $0 \leqq \bar{\gamma}(u_1, u_2) \leqq 1$.

Sublinearity. If for all $\alpha \in [0, 1]$, $s \in R^+$ and $t \in R^N$

$f(x, \alpha s, \alpha t) \geqq \alpha f(x, s, t)$ in Ω,

we say that $f(x, s, t)$ is sublinear in s and t, or simply f is sublinear. Similarly, if for all $\alpha \in [0, 1]$ and $s \in R^+$,

$g(x, \alpha s) \geqq \alpha g(x, s)$ on Ω,

we say that $g(x, s)$ is sublinear in s, or simply g is sublinear.

Finally, for the sake of simplicity, we give the condition (C) and state an important proposition [7].

Condition (C). The BVP (1.1) possesses a supersolution $\hat{v}$ and a subsolution $\bar{v}$ such that $0 \ll \bar{v}(x) \leqq \hat{v}(x)$ in $\bar{\Omega}$.

PROPOSITION. *Suppose that the BVP* (1.1) *satisfies the assumptions* $(A\text{-}1)$-$(A\text{-}7)$ *and the condition* (C). *Then, the BVP* (1.1) *has a P-solution u such that* $u \in [\bar{v}, \hat{v}]^{2+\alpha}_{\text{loc}}$.

3. Multiplicity Theorem for P-solutions

From the assumption (A-5), we can choose a sufficiently small positive constant δ such that

(3.1) $\quad 1 - \delta f_s(x, s, t) \geqq k > 0 \qquad$ in $\bar{\Omega} \times R^+ \times R^N$,

where k is a suitable constant. By using this δ, we now consider the following boundary value problem :

(3.2) $$\begin{aligned} u + \delta(Lu - f(x, u, \nabla u)) &= \phi \qquad \text{in } \Omega, \\ Bu &= g(x, \phi) \qquad \text{on } \partial\Omega, \end{aligned}$$

where $\phi \in [\bar{v}, \hat{v}]^{2+\alpha}_{\text{loc}}$.

In order to simplify the representation of the functions $f(x, u, \nabla u)$ and $g(x, u)$, we shall introduce the Nemytskii-operators F and G for f and g, that is,

$F(u) := f(\cdot, u, \nabla u), \qquad G(u) := g(\cdot, u)$.

Here, we shall prepare some lemmas.

LEMMA 3.1. *Let the assumptions* $(A\text{-}1)$-$(A\text{-}7)$ *and the condition* (C) *be satisfied, and let* v_i *be a P-solution of the BVP* (3.2) *with* ϕ_i *in the right side of* (3.2), *where* $\phi_i \in [\bar{v}, \hat{v}]^{2+\alpha}_{\text{loc}}$, $i = 1, 2$. *If* $\phi_1 \leqq \phi_2$ *in* Ω, *then* $v_1 \leqq v_2$ *in* $\bar{\Omega}$. *Moreover, if* $\phi_1 \ll \phi_2$ *in* $\bar{\Omega}$, *then* $v_1 \ll v_2$ *in* $\bar{\Omega}$.

PROOF. Since $\phi_1 \leqq \phi_2$ in $\bar{\Omega}$, $G(\phi_1) \leqq G(\phi_2)$. Hence, we have

$v_2 + \delta(Lv_2 - F(v_2)) \geqq v_1 + \delta(Lv_1 - F(v_1))$ in Ω,

$$\mathrm{B}v_2 \geqq \mathrm{B}v_1 \qquad \text{on } \partial\Omega.$$

Now, set $w = v_2 - v_1$. Then, we see

$$Lw + \sum_{i=1}^{N} a_i D_i w + a_0 w \geqq 0 \qquad \text{in } \Omega,$$

$$\mathrm{B}w \geqq 0 \qquad \text{on } \partial\Omega,$$

where $\quad a_i = -\int_0^1 f_{t_i}(x,\ v_1 + \tau w,\ \nabla(v_1 + \tau w))\, d\tau \qquad (i = 1, 2, \quad , N)$

and

$$a_0 = \delta^{-1} \int_0^1 (1 - \delta f_s(x,\ v_1 + \tau w,\ \nabla(v_1 + \tau w)))\, d\tau.$$

From the assumption (A-5), a_i and a_0 are bounded in $\overline{\Omega}$. Moreover, from the condition (3.1), $a_0(x) \geqq k/\delta > 0$ in $\overline{\Omega}$. Therefore, it follows from the muximum principle (e.g., [4]) that $w \geqq 0$ in Ω, that is, $v_1 \leqq v_2$ in $\overline{\Omega}$. The later assertion is also the consequence of the maximum principle.

LEMMA 3.2. *Let the BVP* (1.1) *satisfy the assumptions* (*A*-1)-(*A*-7) *and the condition* (*C*), *and let* $\phi \in [\bar{v},\ \hat{v}]_{\mathrm{loc}}^{2+\alpha}$. *Then, the BVP* (3.2) *has a unique P-solution* $u \in [\bar{v},\ \hat{v}]_{\mathrm{loc}}^{2+\alpha}$.

PROOF. Since $G(u)$ is nondecreasing in u and $\bar{v} \leqq \phi \leqq \hat{v}$ in Ω, $G(\phi) \leqq G(\hat{v})$. Hence, we see easily that $\hat{v} + \delta(L\hat{v} - F(\hat{v})) \geqq \phi$ in Ω and $B\hat{v} \geqq G(\phi)$ on $\partial\Omega$. That is, $\hat{v}$ is also a supersolution of the BVP (3.2). Similarly, $\bar{v}$ is also a subsolution of the BVP. Therefore, the existence of P-solutions of the BVP (3.2) follows immediately from Proposition replaced the BVP (1.1) by the BVP (3.2). The assertion of uniqueness follows immediately from Lemma 3.1, and the proof is complete.

This lemma implies that the operator $T : [\bar{v},\ \hat{v}]_{\mathrm{loc}}^{2+\alpha} \to [\bar{v},\ \hat{v}]_{\mathrm{loc}}^{2+\alpha}$ is defined. That is, we have the operator equation induced by the BVP (3.2) :

(3.3) $\qquad u = T(\phi),\ \phi \in [\bar{v},\ \hat{v}]_{\mathrm{loc}}^{2+\alpha}.$

Concerning this equation, we can state the following :

LEMMA 3.3. *A fixed point of the equation* (3.3) *is also a P-solution of the BVP* (3.1).

Next, we shall state a result estimating locally the Hölder continuity of a P-solution of the BVP (3.2) by Ladyzhenskaya and Uraltseva [5].

LEMMA 3.4. *Let the assumptions* $(A\text{-}1)$-$(A\text{-}7)$ *be satisfied, and let* u *be a* P*-solution of the BVP* (3.2). *Then, there exists a positive constant* M_D *such that*

$$\|u\|_{1+\alpha}^{\bar{D}} \leqq M_D,$$

where M_D *depends only on* $\max_{x\in\bar{D}}|u(x)|$, α, λ, $\mu_{\bar{D}}$ *and on the boundary* ∂D, *if* $\partial D \cap \partial\Omega \neq \phi$.

With these preparations, we obtain the following lemma which is the basis for our theory.

LEMMA 3.5. *Let the assumptions of Lemma* 3.2 *be satisfied. Then the operator* T *is positive continuous, locally compact and strongly increasing* (*i.e.*, $\phi_1 \ll \phi_2$ *implies* $T(\phi_1) \ll T(\phi_2)$).

PROOF. We first show that the operator T is locally compact. However, as the discussion of the case for an interior subdomain of Ω is much easier, we make it in the neighborhood of the boundary $\partial\Omega$. Let $S=\partial\Omega\cap\partial D$. Since the P-solution of (3.2) is unique, we have from Schauder's boundary estimates for the BVP (3.2)

$$\|T(\phi)\|_{2+\alpha}^{\bar{D}} \leqq K_1\{\|\phi\|_\alpha^D + \|u+F(u)\|_\alpha^D + \|G(u)\|_{1+\alpha}^S\} \tag{3.4}$$

where the constant K_1 is independent of u and ϕ. By the assumptions of ϕ, F and G, and by Lemma 3.4, (3.4) means that

$$\|T(\phi)\|_{2+\alpha}^{\bar{D}} \leqq K_2$$

where the constant K_2 is independent of u and ϕ. This implies that $T(\phi)$ is uniformly bounded in $C^{2+\alpha}(\bar{D})$. Since D is an arbitrary subdomain of Ω, T is a locally compact mapping from $C_{\mathrm{loc}}^{2+\alpha}(\bar{\Omega})$ to $C_{\mathrm{loc}}^{2+\alpha}(\bar{\Omega})$. Furthermore, it follows from this compactness and uniqueness of Lemma 3.2 that T is continuous. Also, the assertion that T is positive and strongly increasing follows immediately from Lemma 3.2. The proof is complete.

We next state the multiplicity result due to H. Amann [2].

LEMMA 3.6. *Let* E *be ordered Banach space whose positive cone has a nonempty interior, and let* K *be a continuous, compact and strongly increasing mapping* $K: E \to E$. *Moreover, suppose that there exist four points* $\bar{y}_i$, $\hat{y}_i \in E$ $(i=1, 2)$ *such that* $\bar{y}_1 \ll \hat{y}_1 \ll \bar{y}_2 \ll \hat{y}_2$ *and* $\bar{y}_1 \leqq K(\bar{y}_1)$, $\hat{y}_1 \gg K(\hat{y}_1)$, $\bar{y}_2 \ll K(\bar{y}_2)$, $\hat{y}_2 \geqq K(\hat{y}_2)$. *Then,* K *has at least three different fixed points* x_1, x_2 *and* x_3 *such that* $x_1 \ll x_3 \ll x_2$ *and* $\bar{y}_i \leqq x_i \leqq \hat{y}_i$ $(i=1, 2)$.

With the aid of Lemma 3.5 and Lemma 3.6, we obtain the main lemma.

Namely,

LEMMA 3.7. *Suppose that there exist positive subsolutions* $\bar{v}_1$, $\bar{v}_2$ *and positive supersolutions* $\hat{v}_1$, $\hat{v}_2$ *of the BVP* (3.2) *such that* $0 \ll \bar{v}_1 \ll \hat{v}_1 \ll \bar{v}_2 \ll \hat{v}_2$ *and moreover* $\hat{v}_1$, $\bar{v}_2$ *are strict. Then, the BVP* (3.2) *has at least three distinct P-solutions* u_1, u_2 *and* u_3 *such that* $u_1 \ll u_3 \ll u_2$, $\bar{v}_1 \leqq u_1 \ll \hat{v}_1$, $\bar{v}_2 \ll u_2 \leqq \hat{v}_2$ *and* $u_i \in C^{2+\alpha}_{\mathrm{loc}}(\bar{\Omega})$ $(i=1, 2, 3)$.

PROOF. As has been in Lemma 3.3, a fixed point of the operator equation (3.3) is a solution of the BVP (3.2). Furthermore, since T is a positive operator, T is a mapping from $C^{2+\alpha}_+(\bar{\Omega})$ to $C^{2+\alpha}_+(\bar{\Omega})$. On the other hand, $C^{2+\alpha}_+(\bar{\Omega})$ is obviously a positive cone of ordered Banach space with respect to the natural ordering of $C^{2+\alpha}_+(\bar{\Omega})$, and $C^{2+\alpha}_+(\bar{\Omega})$ has a nonempty interior. Moreover, $T([\bar{v}_i, \hat{v}_i]^{2+\alpha}_{\mathrm{loc}}) \subset [\bar{v}_i, \hat{v}_i]^{2+\alpha}_{\mathrm{loc}}$ $(i=1, 2)$ and $\hat{v}_1$, $\bar{v}_2$ are strict. Hence, we have $\bar{v}_1 \leqq T(\bar{v}_1)$, $T(\hat{v}_1) \ll \hat{v}_1$, $\bar{v}_2 \ll T(\bar{v}_2)$ and $T(\hat{v}_2) \leqq \hat{v}_2$. These imply that four points $\bar{v}_i$, $\hat{v}_i$ $(i=1, 2)$ satisfy the conditions of $\bar{y}_i$, $\hat{y}_i$ $(i=1, 2)$ in Lemma 3.6, respectively. Therefore, it follows from Lemma 3.6 that equation (3.3) has at least three distinct fixed points u_1, u_2 and u_3 such that $u_1 \ll u_3 \ll u_2$ and $u_i \in [\bar{v}_i, \hat{v}_i]^{2+\alpha}_{\mathrm{loc}}$ $(i=1, 2)$. Accordingly, the BVP (3.2) has at least three distinct solutions u_1, u_2 and u_3 satisfying the relations of our assertions. This completes the proof.

Now, we are ready for proving the main result of this paper.

THEOREM. *Under the assumptions* $(A\text{-}1)$-$(A\text{-}7)$, *suppose that* f *and* g *are sublinear, and that the condition* (C) *is satisfied. Moreover, let* $\hat{u}$ *and* $\bar{u}$ *be the maximal and minimal P-solutions of the BVP* (1.1), *respectively. Then, it follows that*

(i) *if* $\bar{\gamma}(\hat{u}, \bar{u})=1$, *the BVP* (1.1) *has only one P-solution,*

(ii) *if* $0 \leqq \bar{\gamma}(\hat{u}, \bar{u})<1$, *the BVP* (1.1) *has an infinite number of P-solutions* $\{u_i\}$ *such that*

$$\bar{u} \ll u_1 \ll u_2 \ll \cdots \ll u_n \ll \cdots \ll \hat{u} \quad \text{in } \bar{\Omega}.$$

PROOF. Let $r>0$ be an arbitrary number such that $\{x \in R^N \mid |x|>r\} \subset \Omega$, and denote $\Omega_r = \{x \in \Omega \mid |x|<r\} \cap \Omega$ and $\partial\Omega_r = \{x \in \Omega \mid |x|=r\}$. Now, by using the lower contact $\bar{\gamma}$, we consider two functions $\bar{u}$ and $\bar{\gamma}\hat{u}$, of course, $\bar{u} \geqq \bar{\gamma}\hat{u}$ in $\bar{\Omega}_r$. Then, it follows from the sublinearity of f and Remark 2.2 that

$$L(\bar{\gamma}\hat{u}) = \bar{\gamma} f(x, \hat{u}, \nabla\hat{u}) \leqq f(x, \bar{\gamma}\hat{u}, \nabla(\bar{\gamma}\hat{u})) \qquad \text{in } \Omega_r,$$

and

$L(\bar{u})=f(x, \bar{u}, \nabla\bar{u})$ in Ω_r.

The comparison theorem (See, Proposition I in [6]) implies that

either $\bar{u}=\bar{\gamma}\hat{u}$ or $\bar{u}\gg\bar{\gamma}\hat{u}$ in $\bar{\Omega}_r$.

Here, we classify the value of $\bar{\gamma}$ into the two cases :

(3.5) (i) $\bar{\gamma}=1$ and (ii) $0\leqq\bar{\gamma}<1$,

and we discuss a number of P-solutions in each case.

Case (i). Obviously, either $\bar{u}=\hat{u}$ or $\bar{u}\gg\hat{u}$ in $\bar{\Omega}_r$. While, $\hat{u}\leqq\bar{u}$ in $\bar{\Omega}_r$. Hence, it holds that $\bar{u}=\hat{u}$ in $\bar{\Omega}_r$. Since r is an arbitrary number, the BVP (1.1) has only one P-solution.

Case (ii). As $\bar{\gamma}<1$, there exists a positive constant ε_1^1 such that $1-\bar{\gamma}\geqq\varepsilon_1^1$. We now define the two functions $\bar{w}_1^1=(1-\varepsilon_1^1)\hat{u}$ and $\hat{w}_1^1=(1+\varepsilon_1^1)\bar{u}$ in $\bar{\Omega}_r$. Then, it follows from sublinearities of f and g that

$$L\bar{w}_1^1=(1-\varepsilon_1^1)f(x, \hat{u}, \nabla\hat{u})\leqq f(x, \bar{w}_1^1, \nabla\bar{w}_1^1) \quad \text{in } \Omega_r,$$

$$B\bar{w}_1^1=(1-\varepsilon_1^1)g(x, \hat{u})\leqq g(x, \bar{w}_1^1) \quad \text{on } \partial\Omega,\ \bar{w}_1^1=(1-\varepsilon_1^1)\hat{u} \quad \text{on } \partial\Omega_r,$$

$$L\hat{w}_1^1=(1+\varepsilon_1^1)f(x, \bar{u}, \nabla\bar{u})\geqq f(x, \hat{w}_1^1, \nabla\hat{w}_1^1) \quad \text{in } \Omega_r,$$

$$B\hat{w}_1^1=(1+\varepsilon_1^1)g(x, \bar{u})\geqq g(x, \hat{w}_1^1) \quad \text{on } \partial\Omega,\ \hat{w}_1^1=(1+\varepsilon_1^1)\bar{u} \quad \text{on } \partial\Omega_r.$$

On the other hand, by the comparison theorem and $\bar{\gamma}<1$, $\hat{u}\gg\bar{u}$ in $\bar{\Omega}_r$. Hence, there exists a positive constant ν_r dependent on Ω_r such that $\hat{u}-\bar{u}\geqq\nu_r$ in $\bar{\Omega}_r$. Moreover, for this ν_r, we choose a small positive number ε_1^2 such that $\nu_r>\varepsilon_1^2(\hat{u}+\bar{u})$ in $\bar{\Omega}_r$, and we denote $\bar{w}_1^2=(1-\varepsilon_1^2)\hat{u}$ and $\hat{w}_1^2=(1+\varepsilon_1^2)\bar{u}$ in $\bar{\Omega}_r$. Then, we see that

$$\bar{w}_1^2-\hat{w}_1^2\geqq\nu_r-\varepsilon_1^2(\hat{u}+\bar{u})\gg 0 \quad \text{in } \bar{\Omega}_r.$$

Now, set $\varepsilon_1=\min[\varepsilon_1^1, \varepsilon_1^2]$, and define the two functions $\bar{w}_1=(1-\varepsilon_1)\hat{u}$ and $\hat{w}_1=(1+\varepsilon_1)\bar{u}$ in $\bar{\Omega}_r$. Obviously, $\hat{w}_1$ and $\bar{w}_1$ are respectively a supersolution and a subsolution of the BVP (1.1) in $\bar{\Omega}_r$, and, furthermore, $\hat{u}\gg\bar{w}_1\gg\hat{w}_1\gg\bar{u}$ in $\bar{\Omega}_r$. Here, if $\bar{w}_1$ or $\hat{w}$ is a P-solution of the BVP (1.1), the BVP (1.1) has clearly infinite P-solutions in $\bar{\Omega}_r$, since ε_1 is an arbitrarily small number. Therefore, we may assume that $\hat{w}_1$ and $\bar{w}_1$ are a strict supersolution and a strict subsolution of the BVP (1.1) respectively. This implies that the conditions in Lemma 3.7 are satisfied. Hence, it follows from Lemma 3.7 that the BVP (1.1) has at least three distinct P-solutions u_1^1, u_1^2 and u_1 such that $\bar{u}\leqq u_1^1\ll u_1\ll u_1^2\leqq\hat{u}$ in $\bar{\Omega}_r$. That is, there exists certainly the third P-solution u_1 such that

$\bar{u} \ll u_1 \ll \hat{u}$ in $\bar{\Omega}_r$.

Next, we consider again the degree of lower contact of $\hat{u}$ and u_1 in $\bar{\Omega}_r$, namely, $\bar{\gamma}_1 = \bar{\gamma}_1(\hat{u}, u_1)$ in $\bar{\Omega}_r$. Since $u_1 \leqq (1-\varepsilon_1)\hat{u} \ll \hat{u}$ in $\bar{\Omega}_r$, $0 < \bar{\gamma}_1 < 1-\varepsilon_1$. Hence, we choose a small positive number ε_2^1 such that $\varepsilon_2^1 < 1 - \bar{\gamma}_1$, and set $\bar{w}_2^1 = (1-\varepsilon_2^1)\hat{u}$ and $\hat{w}_2^1 = (1+\varepsilon_2^1)u_1$ in $\bar{\Omega}_r$. Then, it follows that

$$L\bar{w}_2^1 = (1-\varepsilon_2^1)f(x, \hat{u}, \nabla\hat{u}) \leqq f(x, \bar{w}_2^1, \nabla\bar{w}_2^1) \quad \text{in } \Omega_r,$$

$$B\bar{w}_2^1 = (1-\varepsilon_2^1)g(x, \hat{u}) \leqq g(x, \bar{w}_2^1) \text{ on } \partial\Omega_r,\ \bar{w}_2^1 = (1-\varepsilon_2^1)\hat{u} \text{ on } \partial\Omega_r,$$

$$L\hat{w}_2^1 = (1+\varepsilon_2^1)f(x, \bar{u}_1, \nabla\bar{u}_1) \geqq f(x, \hat{w}_2^1, \nabla\hat{w}_2^1) \quad \text{in } \Omega_r,$$

$$B\hat{w}_2^1 = (1+\varepsilon_2^1)g(x, \bar{u}_1) \geqq g(x, \hat{w}_2^1) \text{ on } \partial\Omega_r,\ \hat{w}_2^1 = (1+\varepsilon_2^1)\bar{u}_1 \text{ on } \partial\Omega_r.$$

On the other hand, since $\hat{u} \gg u_1$ in $\bar{\Omega}_r$, there exists a positive constant ν_r^2 such that $\hat{u} - u_1 \geqq \nu_r^2$ in $\bar{\Omega}_r$. For this ν_r^2, we take a positive number ε_2^2 such that $\nu_r^2 \gg \varepsilon_2^2(\hat{u}+u_1)$ in $\bar{\Omega}_r$, and denote $\bar{w}_2^2 = (1-\varepsilon_2^2)\hat{u}$ and $\hat{w}_2^2 = (1+\varepsilon_2^2)u_1$. Obviously, $\bar{w}_2^2 > \hat{w}_2^2$ in $\bar{\Omega}_r$. Here, we set $\varepsilon_2 = \min[\varepsilon_2^1, \varepsilon_2^2]$, and define the functions $\bar{w}_2 = (1-\varepsilon_2)\hat{u}$ and $\hat{w}_2 = (1+\varepsilon_2)u_1$ in $\bar{\Omega}_r$. Then, $\bar{w}_2$ and $\hat{w}_2$ are respectively a subsolution and a supersolution of the BVP (1.1) in $\bar{\Omega}_r$, and $u_1 \ll \hat{w}_2 \ll \bar{w}_2 \ll \hat{u}$ in $\bar{\Omega}_r$. By the same argument as above, we see also that there exist at least three distinct P-solutions u_2^1, u_2 and u_2^2 such that $u_1 \leqq u_2^1 \ll u_2 \ll u_2^2 \leqq \hat{u}$ in $\bar{\Omega}_r$. That is, there exists certainly the fourth P-solution u_2 of the BVP (1.1) such that $\bar{u} \ll u_1 \ll u_2 \ll \hat{u}$ in $\bar{\Omega}_r$.

By repetition of this argument, we obtain infinite P-solutions $\{u_i\}$ such as

$$\bar{u} \ll u_1 \ll u_2 \ll \cdots \ll u_n \ll \cdots \ll \hat{u} \quad \text{in } \bar{\Omega}_r.$$

Now, recall that r is an arbitrary number. Then, we see that there exists an infinite number of P-solutions $\{u_i\}$ such as

$$\bar{u} \ll u_1 \ll u_2 \ll \cdots \ll u_n \ll \cdots \ll \hat{u} \quad \text{in } \bar{\Omega}.$$

This completes the proof of this theorem.

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