A note on the energy decay for some nonlinear dissipative wave equations in one space dimension

中尾, 慎宏
九州大学教養部数学教室

https://doi.org/10.15017/1449058

出版情報：九州大学教養部数学雑誌. 16 (2), pp.43-48, 1988-12. 九州大学教養部数学教室

Math. Rep.
16-2, 1988.

A note on the energy decay for some nonlinear dissipative wave equations in one space dimension

Mitsuhiro NAKAO

(Received September 16, 1988)

0. Introduction

In this note we give a further result on the decay property of the solutions to the initial-boundary value problem for the nonlinear wave equations in one space dimension :

$$u_{tt}-u_{xx}+\sigma(x, u_t)+\beta(x, u)=0 \text{ in } \Omega\times R^+ \tag{0.1}$$

with the initial-boundary conditions

$$u(x, 0)=u_0(x),\ u_t(x, 0)=u_1(x) \quad \text{and} \quad u(x, t)|_{\partial\Omega}=0, \tag{0.2}$$

where Ω is a bounded open interval in R, $R^+=(0, \infty)$ and $\sigma(x, u_t)$, $\beta(x, u)$ are functions like

$$\sigma(x, u_t)=a(x)|u_t|^r u_t,\ r>0, \quad \text{and} \quad \beta(x, u)=|u|^\alpha u,\ \alpha>0. \tag{0.3}$$

Concerning a (x) in the above we assume that $a\in L^\infty(\Omega)$, $a(x)\geqq 0$, $a(x)\neq 0$ a. e. $x\in\Omega$ and $a^{-1}(x)\equiv 1/a(x)\in L^p(\Omega)$ with some $0<p<\infty$.

This problem has been treated in our previous paper [**5**], but under some restrictive condition on p. Let us consider the case (0.3) for simplicity. Then, we can apply a result of [**5**] (cf. Theorem 4) to get

$$\begin{aligned} E(u(t))\equiv \frac{1}{2}\{\|u_t(t)\|^2_{L^2}+\|u_x(t)\|^2_{L^2}\}+\int_\Omega\int_0^u\beta(x, \eta)\,d\eta dx \\ \leqq C_1(1+t)^{-4p/(pr+2)} \end{aligned} \tag{0.4}$$

under the condition

$$\alpha+1>(pr+2)/2p \quad \text{and} \quad 2/r>p>2/(r+2), \tag{0.5}$$

where C_1 denotes a constant depending on $\|u_0\|_{H_2}+\|u_1\|_{H_1}$.

If a (x) is smooth we must take necessarily p such that $0<p<1/2$ (consider the simplest case $a(x)=(x-x_0)^2$, $x_0\in\Omega$). Therefore, the last inequality of (0.5) impose $r>2$, which seems to be rather restrictive because the linear dissipative case $(r=0)$ is completely excluded.

The object of this paper is to try to remove the condition $p>2/(r+2)$ in (0.

5). Roughly speaking our result shows that if $E(u(0))$ is sufficiently small the decay estimate (0.4) holds provided that

$$\alpha+1>(pr+2)/2p \quad \text{and} \quad 2/(r+2)>p>0.$$

As a corollary of our result we rediscover Theorem 4 in [**5**] mentioned before.

When $r=0$ and $a(x)$ is smooth we can expect a usual exponential decay, but we do not know at this time the way how to prove it.

1. Some lemmas and the result

Let Ω be a bounded open interval in the real line R. We use only the familiar function spaces $L^p(\Omega)$, $0<p<\infty$, $H_m(\Omega)$, $C([0, T], L^q(\Omega))$, $1<q<\infty$, etc. (strictly speaking, $L^p(\Omega)$, $0<p<1$, is not vector space and $\|u\|_{L^p}\equiv(\int_\Omega |u|^p dx)^{1/p}$ does not define a usual norm) and the definition of them are omitted. The functions considered are all real valued.

We make the following assumptions on $\sigma(x, v)$ and $\beta(x, u)$.

A_1. $\sigma(x, v)$ is measurable on $\Omega\times R$, continuously differentiable in $v\in R$ for each $x\in\Omega$ and satisfies the conditions:

$$\sigma(x, v)v\geqq k_0 a(x)|v|^{r+2},\ \frac{\partial}{\partial v}\sigma(x, v)\geqq 0 \tag{1.1}$$

and

$$|\sigma(x, v)|\leqq k_1\{|v|^{r+1}+|v|\} \tag{1.2}$$

with some $k_0>0$, $k_1>0$ and $r\neq 0$, where $a(x)$ is a function such that

$$a(\cdot)\in L^\infty(\Omega),\ a(x)\geqq 0,\ a(x)\neq 0 \text{ a. e. } x\in\Omega \text{ and } a^{-1}(\cdot)\in L^p(\Omega) \tag{1.3}$$

for some $0<p<\infty$.

A_2. $\beta(x, u)$ is measurable on $\Omega\times R$, continuously differentiable in u for each $x\in\Omega$ and satisfies

$$\beta(x, u)u\geqq k_0\int_0^u \beta(x, \eta)\, d\eta dx \geqq 0 \tag{1.4}$$

and

$$\left|\frac{\partial}{\partial u}\beta(x, u)\right|\leqq k_2(L)|u|^\alpha \text{ for } |u|\leqq L \tag{1.5}$$

with some $\alpha>0$, $k_0>0$ and $k_2(L)\geqq 0$, where $k_2(L)$ may be chosen for each $L>0$.

To state our result we introduce a set of initial data (u_0, u_1).

For $K>0$ we define

$$S_K=\{(u_0, u_1)\in H_2\cap \mathring{H}_1\times \mathring{H}_1 \mid I_1+Ck_2 I_0^{\alpha+1}(1+G_K)<K\} \tag{1.6}$$

where C is a certain positive constant depending on p, r and α, and we set

$$I_0^2 \equiv E(u(0)) \equiv \frac{1}{2}\{\|u_1\|_2^2 + \|u_{0x}\|_2^2\} + \int_\Omega\int_0^{u_0} \beta(x, \eta)\, d\eta dx,$$

$$I_1^2 \equiv E(u_t(0)) \equiv \frac{1}{2}\{\|u_{tt}(0)\|_2^2 + \|u_{tx}(0)\|_2^2\}$$

$$= \frac{1}{2}\{\|u_{0xx} - \sigma(\cdot, u_1) - \beta(\cdot, u_0)\|_2^2 + \|u_{1x}\|_2^2\},$$

$$G_K \equiv K^{(2-pr)/2p} + I_0^{((4p+2+pr)(r+1)-p(r+2))/p(r+2)} + I_0^{(2+pr)/2} \tag{1.7}$$

and

$k_2 \equiv k_2(\tilde{C} I_0)$ (with some $\tilde{C} > 0$).

Our result reads as follows.

THEOREM. *Suppose that the hypotheses* A_1 *and* A_2 *are fulfilled with* $p>0$, $r>0$ *and* $\alpha>0$ *such that*

$$\alpha + 1 > (pr+2)/2p \quad and \quad 2/r > p > 0 \tag{1.8}$$

Then, for each $(u_0, u_1) \in S = \bigcup_{K \geq 0} S_K$ *the problem* (0.1)–(0.2) *has a unique solution* u(t) *in the class*

$$C^2([0, \infty) : L^2(\Omega)) \cap C^1([0, \infty) ; \overset{\circ}{H}_1(\Omega)) \cap C([0, \infty) ; H_2(\Omega)) \tag{1.9}$$

and u(t) *meets the decay estimate* :

$$E(u(t)) \leq C_1 (1+t)^{-4p/(pr+2)}, \tag{1.10}$$

where C_1 *is a constant depending on* $\|u_0\|_{H_2}, \|u_1\|_{H_1}, \|a^{-1}\|_p, \|a\|_\infty$ *and other known quantities.*

REMARK. (i) Take any $K>0$ such that $K>I_1$. Then, we see $(u_0, u_1) \in S_K$ if I_0 is sufficiently small. Thus, roughly speaking, the estimate (1.10) holds if $\|u_0\|_{H_1} + \|u_1\|_2$ is sufficiently small.

(ii) If $\beta(x, u) \equiv 0$, i.e., $k_2 = 0$ (1.10) is valid for all $p>0$ and all $(u_0, u_1) \in H_2 \cap \overset{\circ}{H}_1 \times \overset{\circ}{H}_1$, which is the content of Theorem 2 in [**5**].

COROLLARY (*Theorem* 4 *in* [**5**]). *Under the condition* (0.5) *we have the estimate* (0.4) ((0.10)) *for all solutions* u(t) *with* $(u_0, u_1) \in H_2 \cap \overset{\circ}{H}_1 \times \overset{\circ}{H}_1$.

PROOF.

If $p > 2/(r+2)$ we see $(2-pr)/2p < 1$, and hence it is clear that $(u_0, u_1) \in S_K$ for sufficiently large $K>0$. Thus, the assertion follows from Theorem.

q. e. d.

To prove our Theorem we rely on the following lemmas.

LEMMA 1. (*Gagliardo & Nirenberg*). *For* $1\leqq r\leqq q\leqq\infty$ *we have*

$$\|u\|_q\leqq \mathrm{const.}\|u\|_{hm}^{\theta}\|u\|_r^{1-\theta} \text{ for } u\in H_m(\Omega),\ m\geqq 1,$$

with $\theta=(r^{-1}-q^{-1})(m+r^{-1}-2^{-1})^{-1}$, *where* $\|\cdot\|_q$ *denotes* $\|\cdot\|_{Lq}$.

LEMMA 2. *Let* $\phi(t)$ *be a nonnegative nonincreasing function on* $[0, T)$, $T>>1$, *satisfying*

$$\phi(t)^{1+r}\leqq g(t)(\phi(t)-(\phi(t+1)),\ 0\leqq t<T-1,$$

with a monotone continuous function $g(t)$ *on* $[0, T)$ *and a positive constant* r. *Then, we have*

$$\phi(t)\leqq \{\phi(0)^{-r}+r\int_1^{t-2} g(s)^{-1}ds\}^{-1/r} \tag{1.11}$$

for $2\leqq t<T$.

PROOF.

The cases $g(t)=C>0$ and $g(t)=C(1+t)^{\theta}(\theta>0)$ are proved in [**2**] and [**3**], respectively. The general case can be proved also quite similarly and the proof is omitted.

2. Proof of Theorem

Our treatment of the degeneracy of $a(x)$ is the same as in our previous papers [**4, 5**]. That is, our starting point is the following proposition.

PROPOSITION 1. *Let* $(u_0, u_1)\in H_2\cap \mathring{H}_1\times \mathring{H}_1$ *and let* u *be the solution of the problem* (0.1)–(0.2) *in the class* (1.9). *Assume that* $0<p<2/r$ *if* $r>0$. *Then, the following inequality holds.*

$$E(u(t))\leqq C\{D(t)^{4p(r+2)/(4p+2+pr)}\sup_{t\leqq s\leqq t+1}\|u_{tx}(s)\|_2^{2(2-pr)/(4p+2+pr)} + D(t)^{2(r+1)}+D(t)^{r+2}\},\ 0\leqq t<T-1, \tag{2.1}$$

where C *denotes a constant independent of* u *and we set*

$$D(t)^{r+2}\equiv E(u(t))-E(u(t+1)).$$

PROOF.

The proof is essentially included in [**4**, proposition 1] (see also [**5**, Proposition 1]) and omitted. (Lemma 1 is used here.)

We proceed to the proof of Theorem. The existence and uniqueness part is well known for all $(u_0, u_1)\in H_2\cap \mathring{H}_1\times \mathring{H}_1$ (cf. Lions [**1**], Strauss [**6**] etc.) and it suffices to derive the estimate (1.10) for $(u_0, u_1)\in S$.

Above all we notice that the energy equality

$$E(u(t))+\int_0^t\int_\Omega \sigma(x, u_t)\,u_t dxds=E(u(0)) \tag{2.2}$$

holds, and hence, we have in particular

$$\|u(t)\|_\infty \leqq \tilde{C}\|u_x(t)\|_2 \leqq \tilde{C}\sqrt{E(u(0))} \equiv \tilde{C}I_0 < \infty. \tag{2.3}$$

We set $k_2 \equiv k_2(\tilde{C}I_0)$ by taking account of (2.3) and (1.5).

Let us assume $(u_0, u_1) \in S_K$, $K>0$, and let

$$E(u_t(t)) \leqq K^2 \quad \text{for} \quad 0 \leqq t \leqq T \tag{2.4}$$

with some $T>0$. Note that (2.4) is valid if T is small. Then, by Proposition 1 and (2.2) we have

$$\begin{aligned} E(u(t))^{1+(2+pr)/4p} &\leqq C\{E(u(t))-E(u(t+1))\} \sup_{t\leqq s\leqq t+1}\|u_{tx}(s)\|_2^{(2-pr)/2p} \\ &\quad + C\{D(t)^{(4p+2+pr)(r+1)/2p} + D(t)^{(4p+2+pr)(r+2)/4p}\} \\ &\leqq CG_K, \quad 0 \leqq t \leqq T-1 \end{aligned} \tag{2.5}$$

where we recall the definition of G_K, (1.7).

Now, applying Lemma 2 to (2.5) we find

$$E(u(t)) \leqq \{E(u(0))^{-(2+pr)/4p} + G_K^{-1}(t-3)^+\}^{-4p/(2+pr)} \tag{2.6}$$

for $0 \leqq t \leqq T$.

On the other hand, differentiating the equation (0.1) with respect to t we have

$$u_{ttt} - u_{txx} + \sigma'(x, u_t)\, u_{tt} + \beta'(x, u)\, u_t = 0. \tag{2.7}$$

$(\beta'(x, u) \equiv \frac{\partial}{\partial u}\beta(x, u))$

From (2.7) we obtain (cf. Strauss [7]) the energy equality for u_t:

$$E(u_t(t)) + \int_0^t\int_\Omega \sigma'(x, u_t)\, u_{tt}^2 dxds = E(u_t(0)) - \int_0^t\int_\Omega \beta'(x, u)\, u_t u_{tt} dxds, \tag{2.8}$$

where we recall

$$E(u_t(t)) \equiv \frac{1}{2}\{\|u_{tt}(t)\|_2^2 + \|u_{tx}(t)\|_2^2\}.$$

It follows from (2.8), Hypotheses A_2 and Lemma 1 that

$$\begin{aligned} E(u_t(t)) &\leqq E(u_t(0)) + k_2\int_0^t\int_\Omega |u|^\alpha |u_t||u_{tt}| dxds \\ &\leqq I_1^2 + k_2\int_0^t \|u(s)\|_\infty^\alpha \|u_t(s)\|_2 \|u_{tt}(s)\|_2 ds \\ &\leqq I_1^2 + Ck_2\int_0^t E(u(s))^{(\alpha+1)/2}\sqrt{E(u_t(s))}\, ds. \end{aligned}$$

This inequality together with (2.6) yields

$$\begin{aligned} E(u_t(t)) &\leqq \{I_1 + \frac{1}{2}Ck_2\int_0^t E(u(s))^{(\alpha+1)/2} ds\}^2 \\ &\leqq \{I_1 + Ck_2\int_0^t (I_0^{-(2+pr)/2p} + G_K^{-1}(s-3)^+)^{-2p(\alpha+1)/(2+pr)} ds\}^2 \\ &\leqq \{I_1 + Ck_2 I_0^{\alpha+1} + Ck_2\int_0^\infty (I_0^{-(2+pr)/2p} + G_K^{-1}s)^{-2p(\alpha+1)/(2+pr)} ds\}^2 \\ &\leqq \{I_1 + Ck_2 I_0^{\alpha+1} + Ck_2 G_K I_0^{\alpha+1}\}^2, \quad 0 \leqq t \leqq T, \end{aligned} \tag{2.9}$$

with a certain $C>0$ independent of u, where we have used the assumption

$2p(\alpha+1) > 2+pr$ at the last step.

Thus, by the assumption $(u_0, u_1) \in S_K$ we obtain

$$E(u_t(t)) < K^2 \quad \text{for} \quad 0 \leqq t \leqq T. \tag{2.10}$$

From (2.4) and (2.10) we conclude that the estimate

$$E(u_t(t)) < K^2 \tag{2.11}$$

is valid for all $t \geqq 0$, and in turn we obtain the estimate (2.6) for all $t \geqq 0$. The proof of Theorem is now complete.

References

[1] J. L. LIONS, Quelques Méthodes de Résolution des Problèmes aux Limites Nonlinéares, Dunod et Gauthiers-Villars, Paris, 1969.

[2] M. NAKAO, Asymptotic stability of the bounded or almost periodic solution of the wave equation with nonlinear dissipative term, J. Math. Anal. Appl., **58** (1977), 336−343.

[3] M. NAKAO, A difference inequality and its applications to nonlinear evolution equations, J. Math. Soc. Japan, **30** (1978), 747−762.

[4] M. NAKAO, Energy decay for the wave equation with a degenerate dissipative term, Proc. Royal Soc. Edinburgh, **100 A** (1985), 19−27.

[5] M. NAKAO, Energy decay for nonlinear wave equations with degenerate dissipative terms, Patterns and Waves-Qualitative Analysis of Nonlinear Differential Equations-Studies in Math. and its applications 18 (1986), 583−596.

[6] W. A. STRAUSS, The Energy Method in Nonlinear Partial Differential Equations, Notas de Mat., No. 47, Rio de Janeiro 1969.