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1. Introduction

The concept of a laminated near-ring was introduced by Magill [1]. Let N be an arbitrary near-ring. Each element a of N yields a new near-ring N_a whose additive group coincides with that of N and whose multiplication $*$ is defined by $x * y = xay$ for any two elements x and y of N, where a product in the original near-ring is denoted by juxtaposition. The N_a is called a laminated near-ring of N with respect to the laminator a.

On the other hand, the concept of a generalized near-field was introduced by Murty [2]. A near-ring is called a generalized near-field if its multiplicative semigroup is an inverse semigroup.

The aim of this paper is to characterize those near-rings with the property that some laminated near-ring is a generalized near-field and also those near-rings with the property that each laminated near-ring is a generalized near-field.

Throughout this paper, all near-rings will be right near-rings containing at least two elements, and all laminators will be assumed to be non-zero. For the basic terminology and notation we refer to [3].

2. Near-rings with a laminated generalized near-field

In [2], it is shown that for a near-ring N the following are equivalent.

(i) N is a generalized near-field.

(ii) N is regular and each idempotent is central.

(iii) N is regular and $xN = Nx$ for each element x of N.

Using this result, we have :

LEMMA 1. *Let a be a non-zero element of a near-ring N. Then the laminated near-ring N_a of N is a generalized near-field if and only if N is a* *generalized near-field and* $Na=N$.

RROOF. Suppose that N_a is a generalized near-field. First, we show that N is regular. Let x be any element of N. Since N_a is regular, $x=x*y*x$ for some element y of N, that is, $x=x(aya)x$. So N is regular. Secondly, we show that $xN=Nx$ for any element x of N. Since $x*N=N*x$ and $x=x*y*x$ for some element y of N, we have $xN=x(aya)xN\subseteq xaN=Nax\subseteq Nx$. Dually, we obtain $Nx\subseteq xN$. So $xN=Nx$ for any element x of N. Thirdly, we show that $Na=N$. Let x be any element of N. Since $aN=Na$ and $x=x*y*x$ for some element y of N, x is in $x*N$, and $x*N=xaN=xNa\subseteq Na$. So we have $Na=N$. Thus N is a generalized near-field and $Na=N$.

Conversely, suppose that N is a generalized near-field and $Na=N$. Let x be any element of N. Since N is regular, $x=xzx$ for some element z of N. On the other hand, we have $N=aNa$ by $aN=Na=N$. Hence $z=aya$ for some element y of N. Therefore $x=x*y*x$ for the element y. So N_a is regular. Moreover, since $xN=Nx$ and $aN=Na=N$, we have $xaN=Nax$, that is, $x*N=N*x$ for any element x of N. Thus N_a is a generalized near-field.

In [2], it is shown that a strongly regular ring is a generalized near-field, and a generalized near-field which is a ring is a strongly regular ring. Using this property, we have :

LEMMA 2. *Let a be a non-zero element of a near-ring N. If the laminated near-ring N_a is a strongly regular ring, then N is a strongly regular ring.*

RROOF. By Lemma 1, N is a generalized near-field and $Na=N$. We show that the left distributive law holds in N. Let x, y and z be any elements of N. Since $Na=N$, $x=ma$ for some element m of N. Then we have $x(y+z)=ma(y+z)=m*(y+z)=m*y+m*z=xy+xz$. Thus N is a strongly regular ring.

We are now ready to prove one of our two theorems.

THEOREM 1. *Some laminated near-ring of a near-ring N is a generalized* *near-field if and only if N is a generalized near-field with identity.*

PROOF. Suppose that, for some non-zero element a of N, the laminated near-ring N_a is a generalized near-field. By Lemma 1, N is a generalized near-field and $Na=N$. Since N is regular, $Na=Ne$ for some idempotent element e of N. We show that the e is an identity element of N. Let x be any element of N. Since $Ne=N$, $x=me$ for some element m of N. Hence $xe=mee=me=x$, and $ex=x$ by the centrality of idempotents. So e is an identity element of N. Thus N is a generalized near-field with identity.

The converse is evident.

Combining Theorem 1 and Lemma 2, we have the following :

COROLLARY 1. *Some laminated near-ring of a near-ring N is a strongly regular ring if and only if N is a strongly regular ring with identity.*

3. Near-rings with all laminated generalized near-fields

In [2], It is shown that a near-ring N is a near-field if and only if N is a generalized near-field and integral. Using this property, we are able to prove the other of our two main theorems.

THEOREM 2. *The following conditions on a near-ring N are equivalent :*

(i) *Each laminated near-ring of N is a generalized near-field.*

(ii) *N is a near-field.*

(iii) *Each laminated near-ring of N is a near-field.*

(iv) *Some laminated near-ring of N is a near-field.*

PROOF. (i) $\Rightarrow$ (ii). By Lemma 1, N is a generalized near-field and $Na=N$ for each non-zero element a of N. By the latter property, N is integral. So N is a near-field.

(ii) $\Rightarrow$ (iii). Since N is a near-field, $Na=N$ for each non-zero element a of N. So each laminated near-ring N_a is a generalized near-field by Lemma 1. We show that N_a is integral. Let x and y be elements of N with $x * y=0$ and $x\neq$

0. Then $x(ay)=0$ and $x \neq 0$. Hence $ay=0$ by the integrality of N, and $a \neq 0$. Therefore we have $y=0$ by the same reason. So each N_a is integral. Thus each N_a is a near-field.

(iii) $\Rightarrow$ (iv). Evident.

(iv) $\Rightarrow$ (ii). Suppose that, for some non-zero element a of N, the laminated near-ring N_a is a near-field. Then N is a generalized near-field and $Na=N$ by Lemma 1. We show that N is integral. Let x and y be elements of N with $xy=0$ and $x \neq 0$. Since $aN=Na=N$, $y=az$ for some element z of N. Hence we have $x * z=0$ and $x \neq 0$. Therefore we obtain $z=0$ by the integrality of N_a, whence $y=0$. So N is integral. Thus N is a near-field.

(iii) $\Rightarrow$ (i). Evident. This completes our proof.

The following corollary is an immediate consequence of Theorem 2 and Lemma 2 :

COROLLARY 2. *Each laminated near-ring of a near-ring N is a strongly regular ring if and only if N is a division ring.*

References

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