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Quasi-ideals in abstract affine near-rings

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1. Introduction

In [**1**, Theorem] and [**2**, Proposition 9.73], N-subgroups, right ideals, ideals and two-sided invariant subgroups of an abstract affine near-ring N are characterized. All of these substructures of N are quasi-ideals of N which are subnear-rings of N. Our aim is to extend the above results to a class of quasi-ideals of N which are subnear-rings of N.

In this note, a near-ring is a right near-ring. For the basic terminology and notation we refer to [**2**].

2. Quasi-ideals

If A and B are two non-empty subsets of a near-ring N, then AB denotes the set of all finite sums of the form

$$\Sigma a_i b_i \ (a_i \in A,\ b_i \in B),$$

and $A * B$ denotes the set of all finite sums of the form

$$\Sigma\ (a_i\ (a_i' + b_i) - a_i a_i') \quad (a_i,\ a_i' \in A,\ b_i \in B).$$

By a quasi-ideal of N we mean a subgroup Q of $(N, +)$ such that $N * Q \cap NQ \cap QN \subseteq Q$.

These definitions lead immediately to

PROPOSITION 1. *Let A be a non-empty subset of an abstract affine near-ring N. Then $N * A = N_0 A$, where N_0 is the zero-symmetric part of N.*

PROOF. For arbitrary near-ring, $N_0 A \subseteq N * A$ is always true. So we show that $N * A \subseteq N_0 A$.

Let x be any element of $N * A$. Then x has the form

$$x = \Sigma\ (n_i(n_i' + a_i) - n_i n_i') \quad (n_i,\ n_i' \in N,\ a_i \in A),$$

and n_i has the form

$$n_i = n_{i0} + n_{ic} \quad (n_{i0} \in N_0,\ n_{ic} \in N_c),$$

where N_c is the constant part of N.

Since N is abelian and every element of N_0 is distributive in N, we have

$$\begin{aligned} & n_i\ (n_i' + a_i) - n_i n_i' \\ &= (n_{i0} + n_{ic})(n_i' + a_i) - (n_{i0} + n_{ic})\ n_i' \\ &= n_{i0} n_i' + n_{i0} a_i + n_{ic} - n_{i0} n_i' - n_{ic} \\ &= n_{i0} a_i \end{aligned}$$

whence $x \in N_0 A$. Hence $N * A \subseteq N_0 A$.

PROPOSITION 2. *Let N be an abstract affine near-ring. Then a subgroup Q of $(N, +)$ is a quasi-ideal of N if and only if $N_0 Q \cap QN \subseteq Q$.*

PROOF. By Proposition 1, $N * Q = N_0 Q \subseteq NQ$. So we have

$$N * Q \cap NQ \cap QN = N_0 Q \cap QN,$$

by which this proposition is easily seen.

3. Main result

In general the zero-symmetric part N_0 and the constant part N_c of a near-ring N are subnear-rings of N, and every subgroup of $(N_c, +)$ is a quasi-ideal of both N_c and N.

Now we are ready to state the main result of this note.

THEOREM. *Let N be an abstract affine near-ring. Then all of those quasi-ideals Q of N whith are subnear-rings of N are of the form $Q = Q_0 + Q_c$ where Q_0 is a quasi-ideal of N_0, Q_c is a subgroup of $(N_c, +)$ and $N_0 Q_c \cap (Q_0 N_c + Q_c) \subseteq Q_c$.*

PROOF. Suppose that Q is a quasi-ideal of N which is a subnear-ring of N. Let $Q_0 = Q \cap N_0$ and $Q_c = Q \cap N_c$. Then, by [3, Theorem 1], Q_0 is a quasi-ideal of N_0, Q_c is a subgroup of $(N_c, +)$ and $Q = Q_0 + Q_c$. This and $Q_c = Q_c N_c$ imply $Q_0 N_c + Q_c = QN_c$. So, by Proposition 2, we have

$$N_0 Q_c \cap (Q_0 N_c + Q_c) = N_0 Q_c \cap QN_c \subseteq N_0 Q \cap QN \subseteq Q.$$

Hence $N_0Q_c \cap (Q_0N_c+Q_c) \cap N_c \subseteq Q \cap N_c$, that is, $N_0Q_c \cap (Q_0N_c+Q_c) \subseteq Q_c$.

Conversely, suppose that Q_0 is a quasi-ideal of N_0 and Q_c is a subgroup of $(N_c, +)$ such that $N_0Q_c \cap (Q_0N_c+Q_c) \subseteq Q_c$.

Now we show that Q_0+Q_c is a quasi-ideal of N. Let x be any element of $N_0(Q_0+Q_c) \cap (Q_0+Q_c)N$. Then x has the form

$$x=\sum_{i=1}^{s} n_{0i}\ (q_{0i}+q_{ci})=\sum_{k=1}^{t}\ (p_{0k}+p_{ck})\ (m_{0k}+m_{ck}),$$

where n_{0i}, $m_{0k} \in N_0$, q_{0i}, $p_{0k} \in Q_0$, q_{ci}, $p_{ck} \in Q_c$ and $m_{ck} \in N_c$. Since N is abelian and every element of N_0 is distributive in N, we have

$$n_{0i}\ (q_{0i}+q_{ci})=n_{0i}q_{0i}+n_{0i}q_{ci}$$

and

$$(p_{0k}+p_{ck})\ (m_{0k}+m_{ck})=p_{0k}m_{0k}+p_{0k}m_{ck}+p_{ck}.$$

Hence we have

$$x0=\sum_{i=1}^{s} n_{0i}q_{ci}=\sum_{k=1}^{t}\ (p_{0k}m_{ck}+p_{ck})$$

and

$$x-x0=\sum_{i=1}^{s} n_{0i}q_{0i}=\sum_{k=1}^{t} p_{0k}m_{0k}.$$

These imply that $x0 \in N_0Q_c \cap (Q_0N_c+Q_c) \subseteq Q_c$ and $x-x0 \in N_0Q_0 \cap Q_0N_0 \subseteq Q_0$, whence $x \in Q_0+Q_c$. So Q_0+Q_c is a quasi-ideal of N by Proposition 2.

Finally, we show that Q_0+Q_c is a subnear-ring of N. We have $(Q_0+Q_c)0=Q_c \subseteq Q_0+Q_c$. So, by [**3**, Theorem 1], Q_0+Q_c is a subnear-ring of N.

COROLLARY. *If N is an abstract affine near-ring, then every quasi-ideal of N_0 is a quasi-ideal of N.*

PROOF. Let Q_0 be any quasi-ideal of N_0. Take the subgroup $Q_c=\{0\}$ of $(N_c, +)$. Then $N_0Q_c \cap (Q_0N_c+Q_c)=Q_c$. So, by Theorem, $Q_0=Q_0+Q_c$ is a quasi-ideal of N.

4. Remarks

REMARK1. Not every quasi-ideal of an abstract affine near-ring is a subnear-ring : take for instance the abstract affine near-ring $N=\{0, a, b, c\}$ [**2**, Near-rings of low order (E 20)] defined by the tables

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	a	a	a	a
b	0	a	b	c
c	a	0	c	b

Then $N_0=\{0, b\}$. Let $Q=\{0, c\}$. Since $N_0Q=Q$, Q is a quasi-ideal of N. But Q is not a subnear-ring of N, for $Q0=\{0, a\} \nsubseteq Q$.

REMARK 2. Not every quasi-ideal of the zero-symmetric part of a near-ring is a quasi-ideal of the near-ring : take the near-ring $N=\{0, a, b, c, x, y, z, w\}$ [**2**, Near-rings of low order (K 132)] defined by the tables

+	0	a	b	c	x	y	z	w
0	0	a	b	c	x	y	z	w
a	a	b	c	0	y	z	w	x
b	b	c	0	a	z	w	x	y
c	c	0	a	b	w	x	y	z
x	x	w	z	y	0	c	b	a
y	y	x	w	z	a	0	c	b
z	z	y	x	w	b	a	0	c
w	w	z	y	x	c	b	a	0

$\cdot$	0	a	b	c	x	y	z	w
0	0	0	0	0	0	0	0	0
a	x	z	x	z	x	a	x	a
b	0	0	0	0	0	b	0	b
c	x	z	x	z	x	c	x	c
x	x	x	x	x	x	x	x	x
y	0	b	0	b	0	y	0	y
z	x	x	x	x	x	z	x	z
w	0	b	0	b	0	w	0	w

Then $N_0=\{0, b, y, w\}$. Let $Q_0=\{0, y\}$. Since $Q_0N_0=Q_0$, Q_0 is a quasi-ideal of N_0. But Q_0 is not a quasi-ideal of N, for $N * Q_0 \cap NQ_0 \cap Q_0N=N_0 \nsubseteq Q_0$.

REMARK 3. The converse of Corollary does not hold : let $N=\{0, a, b, c, x, y, z, w\}$ be the near-ring [**2**, Near-rings of low order (K 1)], whose addition is one in Remark 2 and whose multiplication is defined by the table

$\cdot$	0	a	b	c	x	y	z	w
0	0	0	0	0	0	0	0	0
a	0	a	a	a	0	a	a	0
b	0	b	b	b	0	b	b	0
c	0	c	c	c	0	c	c	0
x	x	x	x	x	x	x	x	x
y	x	y	y	y	x	y	y	x
z	x	z	z	z	x	z	z	x
w	x	w	w	w	x	w	w	x

Then $N_0=\{0, a, b, c\}$ has three quasi-ideals: $\{0\}$, $\{0, b\}$ and N_0. Evidently, all of them are quasi-ideals of N. But N is not an abstract affine near-ring, for N is not abelian.

References

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