Sofic Measures and Sofic Systems

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Kitchen and Tuncel [2] showed the existence and uniqueness of an equilibrium state of a sofic system using their semi-group measure. We give another proof of the existence and uniqueness using a sofic measure which was introduced by Fujiwara. Hamachi and Osikawa [1]. The author wish to thank Professors M. Osikawa and T. Hamachi for stimulating discussions.

Let S be a finite set and $P(a)$ for each $a \in S$ a $k \times k$ matrix whose entries are non-negative numbers. Put $P = \Sigma_{a \in S} P(a)$ and assume P is a stochastic matrix. Let p be a stochastic vector such that $pP = p$. For a cylinder set $[a_1, a_2, \cdots, a_n]$ of S^Z put $\mu([a_1, a_2, \cdots, a_n]) = pP(a_1)P(a_2) \cdots P(a_n)\mathbf{1}$, where $\mathbf{1}$ is a column vector whose entries are all 1. Then μ can be extended to a shift invariant Borel probability measure on S^Z, which we call a *sofic measure* defined by p and $P(a)$, $a \in S$. It is not difficult to see that a shift invariant Borel probability measure is a sofic measure if and only if it is an image of a shift invariant Markov measure by a one block map. We note that if we define $P^*(a)_{ij} = \frac{p_j}{p_i} \times P(a)_{ij}$ and $P^* = \Sigma_{a \in S} P^*(a)$ then P^* is a stochastic matrix and $\mu([a_1, a_2, \cdots, a_n]) = pP^*(a_n)P^*(a_{n-1}) \cdots p^*(a_1)\mathbf{1}$ for any cylinder set $[a_1, a_2, \cdots, a_n]$.

Let $\mathscr{G} = (W, V, i, t, S, \lambda)$ be a labeled graph, that is, W, V, S are finite sets, i and t are mappings from W onto V and λ is a mapping from W onto S. We call W an arc set, V a vertex set, i an initial map, t a terminal map, S a label set and λ a label map. Let $\Sigma(\mathscr{G})$ be the set of all elements (w_i) of W^Z such that $t(w_i) = i(w_{i+1})$ for all $i \in Z$, Λ a mapping from $\Sigma(\mathscr{G})$ into S^Z defined by $\Lambda(w_i) = (\lambda(w_i))$ for (w_i) in $\Sigma(\mathscr{G})$. An image $\Lambda(\Sigma(\mathscr{G}))$ of $\Sigma(\mathscr{G})$ is called a *sofic system* defined by $\mathscr{G}$.

Let f be a function defined on a sofic system $X = \Lambda(\Sigma(\mathscr{G}))$ such that

$f(x)$ depends on the zero-th coordinate of a point x in X. For each $a \in S$ $M(a)$ is a matrix defined by $M(a)_{ij} = \exp(f(a))$ if there exists an arc $w \in W$ such that $\lambda(w) = a$, $i(w) = i$ and $t(w) = j$, and $M(a)_{ij} = 0$ otherwise. Put $M = \Sigma_{a \in S} M(a)$ and assume M is irreducible. Define $P(a)_{ij} = \frac{\xi_j}{\beta \xi_i} \times M(a)_{ij}$ and $p_i = \xi_i \eta_i$, where β is the maximum eigenvalue of M and ξ and η are strictly positive vectors such that $M\xi = \beta\xi$, $\eta M = \beta\eta$ and $\sum_I \xi_i \eta_i = 1$. Then $P = \Sigma_{a \in S} P(a)$ is a stochastic matrix and $p = (p_i)$ is a stochastic vector with $pP = p$. A sofic measure defined by the above p and $P(a)$, $a \in S$ is called a sofic measure defined by $\mathscr{G}$ and f.

THEOREM. *Let $\mathscr{G} = (W, V, i, t, S, \lambda)$ be a right resolving and irreducible labeled graph, X a sofic system defined by $\mathscr{G}$, and f a function defined on X whose values depend only on one coordinate. Then a sofic measure μ defined by $\mathscr{G}$ and f is the unique equilibrium state of the sofic system, that is,*

$$\int (I_\mu + f)\, d\mu \geq \int (I_\nu + f)\, d\nu$$

for any shift invariant Borel probability measure ν on X, where

$$I_\nu(x) = -\lim_{n \to \infty} \log \frac{\nu([x_0, x_1, \cdots, x_n])}{\nu([x_1, x_2, \cdots, x_n])} \quad \text{is a cocycle function ([3]),}$$

and a measure μ satisfying the above inequality is unique.

PROOF. Put $\widetilde{M}(w)_{ij} = \exp(f(\lambda(w)))$ if $i(w) = i$ and $t(w) = j$. and $\widetilde{M}(w)_{ij} = 0$ otherwise. For each $w \in W$ $\widetilde{P}(w)$ is a matrix defined by $\widetilde{P}(w)_{ij} = \frac{\xi_j}{\beta \xi_i} \widetilde{M}(w)_{ij}$. Then we have $\Sigma_{w \in W} \widetilde{M}(w) = M$ and $\Sigma_{w \in W} \widetilde{P}(w) = P$. Let $\widetilde{\mu}$ be a sofic measure on $\Sigma(\mathscr{G})$ defined by p and $\widetilde{P}(w)$, $w \in W$. Then we have

$$\mu = \widetilde{\mu} \Lambda^{-1},$$

$$\widetilde{\mu}([w_1, w_2, \cdots, w_n]) = \frac{1}{\beta^n} \xi(t(w_n)) \eta(i(w_1)) \exp(f(\lambda(w_1)) + f(\lambda(w_2)) + \cdots + f(\lambda(w_n)))$$

and

$$\frac{\widetilde{\mu}([w_0, w_1, \cdots, w_n])}{\widetilde{\mu}([w_1, w_2, \cdots, w_n])} = \exp(f(\lambda(w_0))) \times \frac{\eta(i(w_0))}{\beta \eta(t(w_0))}.$$

Following the same method as one of Parry and Tuncel [3] we have

$$\int (I_{\widetilde{\mu}}-I_{\widetilde{\nu}})\,d\widetilde{\nu}\geq\int\left(-\log\frac{\widetilde{\mu}([w_0,w_1])}{\widetilde{\mu}([w_1])}+\log\frac{\widetilde{\nu}([w_0,w_1,\cdots,w_n])}{\widetilde{\nu}([w_1,\cdots,w_n])}\right)d\widetilde{\nu}$$

$$\geq-\sum_{[w_0,\cdots,w_n]}\log\frac{\widetilde{\mu}([w_0,w_1])\widetilde{\nu}([w_1,\cdots,w_n])}{\widetilde{\mu}([w_1])\widetilde{\nu}([w_0,w_1,\cdots,w_n])}\widetilde{\nu}([w_0,\cdots,w_n])$$

$$\geq\sum_{[w_0,\cdots,w_n]}\left(1-\frac{\widetilde{\mu}([w_0,w_1])\widetilde{\nu}([w_1,\cdots,w_n])}{\widetilde{\mu}([w_1])\widetilde{\nu}([w_0,w_1,\cdots,w_n])}\right)\widetilde{\nu}([w_0,w_1,\cdots,w_n])\geq 0.$$

Hence, we have

$$\begin{aligned}\int (I_{\widetilde{\mu}}+f\circ\Lambda)\,d\widetilde{\mu}&=\int (\log\beta+\log\eta(t(w_0))-\log\eta(i(w_0)))\,d\widetilde{\mu}\\&=\log\beta\\&=\int (I_{\widetilde{\mu}}+f\circ\Lambda)\,d\widetilde{\nu}\\&\geq\int (I_{\widetilde{\nu}}+f\circ\Lambda)\,d\widetilde{\nu}.\end{aligned}$$

Since Λ is finite-to-one from the right resolvingness of $\mathscr{G}$ we have

$$\begin{aligned}\int (I_{\mu}+f)\,d\mu&=\int (I_{\widetilde{\mu}}+f\circ\Lambda)\,d\widetilde{\mu}\\&\geq\int (I_{\widetilde{\nu}}+f\circ\Lambda)\,d\widetilde{\nu}\\&=\int (I_{\nu}+f)\,d\nu,\end{aligned}$$

where $\widetilde{\nu}$ is a measure such that $\nu=\widetilde{\nu}\Lambda^{-1}$

The uniqueness follows from considering the case of equalities in the above inequalities.

References

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