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Sofic systems and flow equivalence

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1. Introduction

A sofic system is a certain type of symbolic dynamics introduced by B. Weiss [8, 9]. It can be determined by a labeled graph, but different labeled graphs may determine the same sofic system. R. Fischer [2, 3] and W. Krieger [4] gave ideas of canonical labeled graphs, which we call Fischer graph and Krieger graph respectively. Since Krieger graph (Fischer graph) is uniquely determined for a (transitive) sofic system, we can discuss sofic systems in terms of their Krieger graphs (Fischer graphs). In this paper we restrict ourselves to transitive sofic systems and Fischer graphs though similar results may be obtained for not necessarily transitive sofic systems: A subshift is said to be of finite type if it is conjugate to a topological Markov shift. A subshift is said to be almost of finite type if it is an image of a topological Markov shift by a continuous factor map which is one to one on a non-vacant open subset. We characterize subshifts of finite type and almost of finite type in terms of their Fischer graphs respectively and show that both properties of finite type and almost of finite type are invariants for flow equivalence, where dynamical systems are mutually flow equivalent if their suspension flows are conjugate. Flow equivalence of topological Markov shifts was studied by Parry and Sullivan [6] and Franks [4].

2. Definitions and Theorems

We consider the two sided infinite direct product set S^z of copies of a finite set S. We denote $\prod_{i=-\infty}^{\infty} x_i$ an element in S^z and define a shift σ by $\sigma(\prod_{i=-\infty}^{\infty} x_i) = \prod_{i=-\infty}^{\infty} x_{i+1}$. S^z is a compact space under the infinite direct product topology of discrete topolpgy on S and σ is a homeomorphism of S^z. If a closed subset Ω of S^z is σ-invariant (i. e. $\sigma\Omega = \Omega$), σ restricted on Ω is a dynamical system and called a *subshift*, where by a dynamical system we mean a homeomorphism from a compact space onto itself.

A *labeled graph* $\mathsf{G}=(W, V, i, t, S, \lambda)$ consists of finite sets W, V and S and maps i, t and λ from W onto V, V and S respectively. We call W, V and S an arc set, a vertex set and a label set, and i, t and λ an initial map, a terminal map and a label map respectively.

Let $\mathsf{G}=(W, V, i, t, S, \lambda)$ be a labeled graph. A sequence $\prod_{i=-\infty}^{\infty} w_i$ of arcs is said to be *G-admissible* if $t(w_i)=i(w_{i+1})$ for all $i \in \mathsf{Z}$. We denote $\Sigma(\mathsf{G})$ the set of all G-admissible sequences and $\Omega(\mathsf{G})$ the set of all sequences $\prod_{i=-\infty}^{\infty} \lambda(w_i)$ for $\prod_{i=-\infty}^{\infty} w_i \in \Sigma(\mathsf{G})$. $\Sigma(\mathsf{G})$ and $\Omega(\mathsf{G})$ are σ-invariant closed subsets of W^{z} and S^{z} respectively. Therefore $\Sigma(\mathsf{G})$ and $\Omega(\mathsf{G})$ are subshifts and called a *topological arc Markov shift* and a *sofic systems determined by* G, respectively. A map Λ from $\Sigma(\mathsf{G})$ onto $\Omega(\mathsf{G})$ defined by $\Lambda(\prod_{i=-\infty}^{\infty} w_i)=\prod_{i=-\infty}^{\infty} \mathrm{x}(w_i)$ for $\prod_{i=-\infty}^{\infty} \mathrm{w}_i \in \Sigma(\mathsf{G})$ is called a *factor map*. We note that a topological arc Markov shift $\Sigma(\mathsf{G})$ is determined by only a graph (W, V, i, t) and is a sofic system when we think that a label map is an identity map. A subshift Ω is called a *topological arc Markov shift* (a *sofic system*) if it is one determined by some graph (some labeled graph). We note that a topological arc Markov shift is a usual topological Markov shift determined by a $0-1$ matrix and that the two-higher block system $\Omega^{[2]}=\{\prod_{i=-\infty}^{\infty}(x_i, \mathrm{x}_{i+1}) \mid \prod_{i=-\infty}^{\infty} x_i \in \Omega\}$ of a usual topological Markov shift Ω is a topological arc Markov shift.

A labeled graph $\mathsf{G}=(W, V, i, t, S, \lambda)$ is called a *Fischer graph* if the following (1), (2) and (3) are satisfied:

(1) (*right resolving*)

$i(w)=i(w')$ and $\lambda(w)=\lambda(w')$ for $w, w' \in w$ imply $w=w'$,

(2) (*right reduced*)

$L_v(\mathsf{G})=L_{v'}(\mathsf{G})$ for $v, v' \in V$ implies $v=v'$, where

$$L_v(\mathsf{G})=\left\{\prod_{i=1}^{n} \lambda(w_i) \mid n \in \mathsf{N}, \prod_{i=1}^{n} w_i \text{ is a } \mathsf{G}\text{-admissible sequence of length } n \text{ with } i(w_1)=v\right\},\quad v \in V,$$

(3) (*irreducible*)

for any w and w' in W there exists a G-admissible finite sequence $\prod_{i=1}^{n} w_i$ with $w_1 = w$ and $w_n = w'$.

A dynamical system (X, T) is said to be *transitive* if for any non-vacant open subsets O_1 and O_2 there exists an integer n such that $T^n O_1 \cap O_2 \neq \phi$. An irreducible labeled graph determines a transitive sofic system. R. Fischer [2, 3] showed that for a transitive sofic system Ω there uniquely exists a Fischer graph G with $\Omega(\mathrm{G}) = \Omega$. The unique Fischer graph is denoted by $FG(\Omega)$ in the sequel.

For finite sequences $\alpha = \prod_{i=1}^{n} a_i$ and $\beta = \prod_{i=1}^{m} b_i$ we write $\alpha\beta = (a_1\ a_2 \cdots a_n\ b_1\ b_2 \cdots b_m)$. For a finite sequence α we write $\alpha^1 = \alpha$ and $\alpha^n = \alpha^{n-1}\alpha$ for $n = 2, 3, \cdots$. A finite sequence α is said to be *indecomposable* if there does not exist an integer $k \geqq 2$ and a finite sequence β such that $\alpha = \beta^k$.

Let $\mathrm{G} = (W, V, i, t, S, \lambda)$ be a Fischer graph. A G-admissible sequence $\prod_{i=-\infty}^{\infty} w_i$ is called a G-*cycle of length* n if $w_{i+n} = w_i$ for all $i \in \mathbf{Z}$. A G-cycle $\prod_{i=-\infty}^{\infty} w_i$ of length mn is called a G-*cycle of type* (m, n) if there is an indecomposable sequence α with $\prod_{i=1}^{mn} \lambda(w_i) = \alpha^n$ and $w_{i+mj} \neq w_i$ for all $i \in \mathbf{Z}$ and all $j\,(1 \leqq j \leqq n-1)$. G-cycles $\prod_{i=-\infty}^{\infty} w_i$ and $\prod_{i=-\infty}^{\infty} w'_i$ of type (m, n) are called *twin G-cycles of type* (m, n) if $\lambda(w_i) = \lambda(w'_i)$ and $w_{i+mj} \neq w'_i$ for all $i \in \mathbf{Z}$ and all $j \in \mathbf{Z}$.

Dynamical systems (X, T) and (X', T') are said to be *conjugate* if there exists a homeomorphism Φ from X onto X' with $\Phi T = T'\Phi$. The following theorem says that the existences of a cycle of type (m, n) and twin cycles of type (m, n) in Fischer graphs are invariants for conjugacy of sofic systems.

THEOREM 1. *Let Ω and Ω' be mutually conjugate transitive sofic systems and m and n be positive integers.*

(1) *$FG(\Omega)$ has a cycle of type (m, n) if and only if $FG(\Omega')$ has a cycle of type (m, n).*

(2) *$FG(\Omega)$ has twin cycles of type (m, n) if and only if $FG(\Omega')$ has twin cycles of type (m, n).*

A G-cycle of type n and twin cycles of type n mean ones of type (m, n) for some m.

THEOREM 2. *Let* Ω *be a transitive sofic system. Then the following* (1), (2) *and* (3) *are equivalent:*

(1) Ω *is conjugate to a topological arc Markov shift.*

(2) $FG(\Omega)$ *does not have a cycle of type* n $(n \geqq 2)$ *nor twin cycles of type 1.*

(3) *the factor map* Λ *from* $\Sigma(FG(\Omega))$ *onto* $\Omega(FG(\Omega))=\Omega$ *is one to one.*

A subshift which is conjugate to a topological arc Markov shift is said to be *of finite type*. A sofic system is said to be *proper* if it is not of finite type. Theorem 2 says that a transitive sofic system Ω is proper if and only if the Fischer graph $FG(\Omega)$ has a cycle of type $n(n \geqq 2)$ or twin cycles of type 1.

Let (W, V, i, t, S, λ) be a Fischer graph. A G-cycle $\prod_{i=-\infty}^{\infty} w_i$ of type (m, n) $(n \geqq 2)$ is said to *have a handle* if there exist G-admissible finite sequences $\prod_{i=1}^{k} z_i$ and $\prod_{i=1}^{k} z'_i$ such that $\lambda(z_i)=\lambda(z'_i)$ for $i=1,2,\cdots,k$, $t(z_k)=t(z'_k)$, $i(z_1)=t(w_i)$ and $i(z'_1)=t(w_{i+mj})$ for some i and some $j(1 \leqq j \leqq n-1)$. G-cycles $\prod_{i=-\infty}^{\infty} w_i$ and $\prod_{i=-\infty}^{\infty} w'_i$ of type (m, n_1) and (m, n_2) respectively for the same m are said to be *fraternal* G*-cycles with a handle* if $\prod_{i=1}^{m} \lambda(w_i)=\prod_{i=1}^{m} \lambda(w'_i)=\alpha$ for an indecomposable sequence α, $w_{i+mj} \neq w'_i$ for all $i \in \mathbf{Z}$ and all $j \in \mathbf{Z}$ and if there exist G-admissible finite sequences $\prod_{i=1}^{k} z_i$ and $\prod_{i=1}^{k} z'_i$ such that $\lambda(z_i)=\lambda(z'_i)$ for $i=1,2,\cdots,k$, $t(z_k)=t(z'_k)$, $i(z_1)=t(w_i)$ and $i(z'_1)=t(w'_{i+mj})$ for some i and some j.

A sofic system Ω is said to be *almost of finite type* if the factor map Λ from $\Sigma(FG(\Omega))$ onto $\Omega(FG(\Omega))=\Omega$ is one to one on a non-vacant open subset of $\Sigma(FG(\Omega))$ ([1]).

THEOREM 3. *Let* Ω *be a transitive sofic system. Then the following* (1), (2) *and* (3) *are equivalent.*

(1) Ω *is almost of finite type.*

(2) *There are no different $FG(\Omega)$-admissible sequences* $\prod_{i=-\infty}^{\infty} w_i$ *and* $\prod_{e=-\infty}^{\infty} w'_i$ *such that* $\prod_{i=-\infty}^{\infty} \lambda(w_i) = \prod_{i=-\infty}^{\infty} \lambda(w'_i)$ *and* $\prod_{i=n}^{\infty} w_i = \prod_{i=n}^{\infty} w'_i$ *for some n.*

(3) *$FG(\Omega)$ does not have a cycle with a handle nor fraternal cycles with a handle.*

The equivalence of (1) and (2) in Theorem 3 has already been shown by M. Boyle, B. Kitchen and B. Marcus [1]. The factor map Λ is said to be *left-closing* if (2) of Theorem 3 holds.

Let (X, T) be a dynamical system and $\xi(x)$ a continuous positive function on x. Let $\widetilde{T}$ be a homeomorphism of the direct product space $X \times \mathbf{R}$ defined by $\widetilde{T}(x, u) = (Tx, u - \xi(x))$ for (x, u) in $X \times \mathbf{R}$. We denote X_ξ the quotient topological space consisting of $\widetilde{T}$-orbits and by $\{(T, \xi)_t\}$ the suspension flow on X_ξ which is derived from a flow $\{T_t\}$ on $X \times \mathbf{R}$ defined by $T_t(x, u) = (x, u+t)$ for $(x, u) \in X \times \mathbf{R}$. Dynamical systems (X, T) and (X', T') are said to be *mutually flow equivalent* if there exist continuous functions $\xi(x)$ and $\xi'(x')$ and a homeomorphism Φ from X_ξ onto $X'_{\xi'}$ such that $\Phi(T, \xi)_t = (T', \xi')_t \Phi$ for any $t \in \mathbf{R}$ ([7]).

THEOREM 4. *Let Ω and Ω' be mutually flow equivalent transitive subshifts. Then Ω is a sofic system if and only if Ω' is a sofic system.*

The following theorem says that the existences of a cycle of type n, twin cycles of type n, a cycle with a handle and fraternal cycles with a handle in Fischer graphs are invariants for flow equivalence of sofic systems.

THEOREM 5. *Let Ω and Ω' be mutually flow equivalent transitive sofic systems and n be a positive integer.*

(1) *$FG(\Omega)$ has a cycle of type n if and only if $FG(\Omega')$ has a cycle of type n.*

(2) *$FG(\Omega)$ has twin cycles of type* n *if and only if $FG(\Omega')$ has twin cycles of type* n.

(3) *$FG(\Omega)$ has a cycle with a handle if and only if $FG(\Omega')$ has a cycle with a handle.*

(4) $FG(\Omega)$ *has fraternal cycles with a handle if and only if* $FG(\Omega')$ *has fraternal cycles with a handle.*

THEOREM 6. *Let* Ω *and* Ω' *be mutually flow equivalent transitive subshifts. Then* Ω *is of finite type or almost of finite type if and only if* Ω' *is of finite type or almost of finite type, respectively.*

3. Proofs

For a subshift Ω of $S^{\mathbf{Z}}$ *and a map* φ *from* S into a finite set S' *we define*

$$\Omega^{\varphi}=\{\prod_{i=-\infty}^{\infty}(x_i,\ \varphi(x_{i+1}))\mid \prod_{i=-\infty}^{\infty}x_i\in\Omega\}\quad\text{and}$$

$$^{\varphi}\Omega=\{\prod_{i=-\infty}^{\infty}(\varphi(x_{i-1}),\ x_i)\mid \prod_{i=-\infty}^{\infty}x_i\in\Omega\}.$$

Then Ω^{φ} and $^{\varphi}\Omega$ are subshifts conjugate to Ω.

For a labeled graph $\mathsf{G}=(W,\ V,\ i,\ t,\ S,\ \lambda)$ and a map φ from S into a finite set S' a labeled graph $\mathsf{G}^{\varphi}=(W^{\varphi},\ V^{\varphi},\ i^{\varphi},\ t^{\varphi},\ S^{\varphi},\ \lambda^{\varphi})$ is defined by the following way:

We define an equivalence relation in the arc set W by $w\sim w'$ *if* $i(w)=i(w')$ and $\varphi(\lambda(w))=\varphi(\lambda(w'))$, and denote $[w]$ an equivalence class containing an arc w.

$\mathrm{W}^{\varphi}=\{(w,\ [w'])\mid w,\ w'\in w,\ t(w)=i(w')\}$.

$\mathrm{V}^{\varphi}=\{[w]\mid w\in W\}$.

$\mathrm{i}^{\varphi}(w,\ [w'])=[w]$, $\quad t^{\varphi}\ (w,\ [w'])=[w']$,

$\lambda^{\varphi}(w,\ [w'])=(\lambda(w),\ \varphi(\lambda(w')))$ for $(w,\ [w'])\in W^{\varphi}$.

$S^{\varphi}=\lambda^{\varphi}(W^{\varphi})$.

Another labeled graph $^{\varphi}G=(^{\varphi}W,\ ^{\varphi}V,\ ^{\varphi}i,\ ^{\varphi}t,\ ^{\varphi}S,\ ^{\varphi}\lambda)$ is defined by the following way:

In this case we define another equivalence relation in the arc set W by $w\sim w'$ if $t(w)=t(w')$ and $\varphi(\lambda(w))=\varphi(\lambda(w'))$.

$^{\varphi}W=\{([w],\ w')\mid w,\ w'\in W,\ t(w)=i(w')\}$.

$^{\varphi}V=\{[w]\mid w\in W\}$.

$^{\varphi}i\ ([w],\ w')=[w]$, $\quad ^{\varphi}t([w],\ w')=[w']$

$^{\varphi}\lambda([w],\ w')=(\varphi(\lambda(w)),\ \lambda(w'))$ for $([w],\ w')\in{}^{\varphi}W$.

$^{\varphi}S={}^{\varphi}\lambda(^{\varphi}W)$.

It is not difficult to see that $\Omega(\mathsf{G}^{\varphi})=(\Omega(\mathsf{G}))^{\varphi}$, $\Omega(^{\varphi}\mathsf{G})={}^{\varphi}(\Omega(\mathsf{G}))$ and that if

G is a Fischer graph, G^{φ} and ${}^{\varphi}G$ are Fischer graphs ([6]).

LEMMA 1. Let G *be a Fischer graph and* n *and* m *positive integers. Then*
(1) G *has a cycle of type* (m, n) *if and only if* G^{φ} *has a cycle of type* (m, n).
(2) G *has a cycle of type* (m, n) *if and only if* ${}^{\varphi}G$ *has a cycle of type* (m, n).
(3) G *has twin cycles of type* (m, n) *if and only if* G^{φ} *has twin cycles of type* (m, n).
(4) G *has twin cycles of type* (m, n) *if and only if* ${}^{\varphi}G$ *has twin cycles of type* (m, n).

PROOF. Note that a G^{φ}-cycle (${}^{\varphi}G$-cycle) has a form $\prod_{i=-\infty}^{\infty}(w_i, [w_{i+1}])$ $(\prod_{i=-\infty}^{\infty}([w_{i-1}], w_i))$ and that for a G-admissible sequence $\prod_{i=-\infty}^{\infty} w_i$ and a positive integer k there exists a finite sequence α with $\prod_{i=1}^{n}\lambda(w_i)=\alpha^k$ if and only if there exists a finite sequence $\widetilde{\alpha}$ with $\prod_{i=1}^{n}(\lambda(w_i), \varphi(\lambda(w_{i+1}))=\widetilde{\alpha}^k$ $(\prod_{i=1}^{n}(\varphi(\lambda(w_{i-1})), \lambda(w_i))=\widetilde{\alpha}^k)$. Then Lemma 1 is a direct conclusion from the definitions.

PROOF of THEOREM 1.

N. Nasu [6] showed that starting from two different subshifts which are mutually conjugate we obtain the same subshift by finite steps of two kinds of deformations of subshifts : from Ω to Ω^{φ} and from Ω to ${}^{\varphi}\Omega$. These deformations correspond to deformations of labeled graphs : from G to G^{φ} and from G to ${}^{\varphi}G$ as seen before. Hence Theorem 1 follows from Lemma 1.

PROOF of THEOREM 2.

A graph which makes a topological arc Markov shift is a Fischer graph, as noted before when we think that a label map is an identity map, and does not have a cycle of type n $(n \geqq 2)$ nor twin cycles of type 1. Hence by Theorem 1 (1) implies (2). Assume that (3) does not hold. Then since $FG(\Omega)$ is right resolving, there exist $FG(\Omega)$-admissible sequences $\prod_{i=-\infty}^{\infty} w_i$ and $\prod_{i=-\infty}^{\infty} w'_i$ and an integer n such that $\lambda(w_i) = \lambda(w'_i)$ for all $i \in \mathbf{Z}$ and $w_i \neq w'_i$ for $i \leqq n$. Since $\{(w_i, w'_i) \mid i \leqq n\}$ is a subset of $W \times W$ and hence is finite, there exist integers k and m $(k < m < n)$ such that $(w_k, w'_k) = (w_m, w'_m)$. Then $\prod_{i=k}^{m} w_i$ and $\prod_{i=k}^{m} w'_i$ are cycles. In the case that at least one of them is a cycle of type $n (n \geqq 2)$ (2) does not hold. In the other case they must be twin cycles of type 1. This means that (2) implies (3). Assume that (3) holds, then Ω is conjugate to a topological arc Markov shift $\Sigma(FG(\Omega))$, hence (1) holds.

PROOF of THEOREM 3.

Assume that (1) does not hold. Then there exist different $FG(\Omega)$-admissible sequences $\prod_{i=-\infty}^{\infty} w_i$ and $\prod_{i=-\infty}^{\infty} w'_i$ with $\prod_{i=-\infty}^{\infty} \lambda(w_i) = \prod_{i=-\infty}^{\infty} \lambda(w'_i)$ and $w_0 \neq w'_0$. From the fact that $FG(\Omega)$ is right resolving $w_i \neq w'_i$ for any $i \leqq 0$, and hence (2) does not hold. Assume that (2) does not hold. Then from the fact that $FG(\Omega)$ is right resolving there exist $FG(\Omega)$-admissible sequences $\prod_{i=-\infty}^{\infty} w_i$ and $\prod_{i=-\infty}^{\infty} w'_i$ and an integer n such that $\prod_{i=-\infty}^{\infty} \lambda(w_i) = \prod_{i=-\infty}^{\infty} \lambda(w'_i)$, $w_i \neq w'_i$ for $i \leqq n$ and $w_i = w'_i$ for $i > n$. By the same argument as one in the proof of Theorem 2 we find integers k and m $(k < m < n)$ such that $\prod_{i=k}^{m} w_i$ and $\prod_{i=k}^{m} w'_i$ are $FG(\Omega)$-cycles. Since $\prod_{i=m+1}^{n} w_i$ together with $\prod_{i=m+1}^{n} w'_i$ is a handle of the cycles (3) does not hold. Assume that (3) does not hold, and let $\prod_{i=-\infty}^{\infty} w_i$ and $\prod_{i=-\infty}^{\infty} w'_i$ be $FG(\Omega)$-cycles such that $\lambda(w_i) = \lambda(w'_i)$ and $w_i \neq w'_i$ for $i \in \mathbf{Z}$, and $\prod_{i=k}^{m} z_i$ and $\prod_{i=k}^{m} z'_i$ be $FG(\Omega)$-admissible sequences such that $\lambda(z_i) = \lambda(z_i)$ for $i = k, k+1, \cdots, m$, $t(w_{k-1}) = i(z_k)$, $t(w'_{k-1}) = i(z'_k)$, $z_m \neq z'_m$ and $t(z_m) = t(z'_m)$. For any cylinder set $\prod_{i=p}^{q} z_i$ of $\Sigma(FG(\Omega))$ $(p < q)$ there exists an $FG(\Omega)$-admissible sequence $\prod_{i=r}^{p-1} z_i$ $(r < p)$ with $i(z_r) = t(z_m) = t(z'_m)$ from the irreducibility of $FG(\Omega)$. Let $\prod_{i=q+1}^{\infty} z_i$ be a $FG(\Omega)$-admissible sequence with $t(z_q) = i(z_{q+1})$. Then $\prod_{i=-\infty}^{k-1} w_i \prod_{i=k}^{m} z_i \prod_{i=r}^{\infty} z_i$ and $\prod_{i=-\infty}^{k-1} w' \prod_{i=k}^{m} z'_i \prod_{i=r}^{\infty} z_i$ are different elements in the cylinder set $\prod_{i=p}^{q} z_i$ with the same Λ-image, and hence (1) does not hold. We complete the proof.

For a subshift Ω of $S^{\mathbf{Z}}$ and a positive integer valued function f defined on S we define

$$\Omega_f = \{ \prod_{i=-\infty}^{\infty} \prod_{h=1}^{f(x_i)} (x_i , h) \mid \prod_{i=-\infty}^{\infty} x_i \in \Omega \}.$$

Then Ω_f is a subshift.

For a labeled graph $\mathsf{G} = (W, V, i, t, S, \lambda)$ and a positive integer valued function f on the label set S, we define a labeled graph $\mathsf{G}_f = (W_f , V_f , i_f , t_f , S_f , \lambda_f)$ by the following way :

$$W_f = \{ (w, h) \mid h = 1, 2, \cdots, f(\lambda(w)), \ w \in W \}.$$

We consider an equivalence relation in W_f by $(w, h) \sim (w', h')$ if either $i(w) = i(w')$ and $h = h' = 1$ or $w = w'$ and $h = h'$.

V_f is the set of all equivalence classes $[(w, h)]$ for (w, h) in W_f.

$i_f(w, h) = [(w, h)]$,

$$t_f(w, h) = \begin{cases} [(w, h+1)] & \text{if } h = 1, 2, \cdots, f(\lambda(w)) - 1, \\ [(w', 1)] & \text{if } h = f(\lambda(w)), \end{cases}$$

where w' is an arc with $i(w') = t(w)$.

$\lambda_f(w, h) = (\lambda(w), h) \in S \times \mathbf{N}$.

$S_f = \lambda_f(W_f)$.

It is easy to see that $\Omega(\mathsf{G}_f) = (\Omega(\mathsf{G}))_f$ and that if G is a Fischer graph so is G_f.

LEMMA 2. *Let* G *be a Fischer graph, f be a positive integer valued function on the label set of* G *and n be a positive integer.*

(1) G *has a cycle of type* n *if and only if* G_f *has a cycle of type* n.

(2) G *has twin cycles of type* n *if and only if* G_f *has twin cycles of type* n.

(3) G *has a cycle with a handle if and only if* G_f *has a cycle with a handle,*

(4) G *has fraternal cycles with a handle if and only if* G_f *has fraternal cycles with a handle.*

PROOF. Note that a G_f-cycle has a form $\prod_{i=-\infty}^{\infty} \prod_{h=1}^{f(\lambda(w_i))} (w_i , h)$ for a G-cycle $\prod_{i=-\infty}^{\infty} w_i$ and that for a G-admissible finite sequence $\prod_{i=1}^{n} w_i$ and a positive integer k there exists a finite sequence α with $\prod_{i=1}^{n} \lambda(w_i) = \alpha^k$ if and only if there exists a finite sequence $\widetilde{\alpha}$ with $\prod_{i=1}^{n}\prod_{h=1}^{f(\lambda(w_i))}(\lambda(w_i),\ h)=\widetilde{\alpha}^k$. Then Lemma 2 is a direct conclusion from the definitions.

LEMMA 3. *Let Ω be a transitive subshift of S^Z, f be a positive integer valued function defined on S and φ be a map from* S *into a finite set. Then*
(1) *Ω is a sofic system if and only if Ω_f is a sofic system.*
(2) *Ω is a sofic system if and only if Ω^φ is a sofic system.*
(3) *Ω is a sofic system if and only if ${}^\varphi\Omega$ is a sofic system.*

PROOF. (1) If Ω is a sofic system there is a labeled graph G with $\Omega=\Omega(\mathsf{G})$. Since $\Omega_f=\Omega(\mathsf{G}_f)$, Ω_f is a sofic system. Conversely, if Ω_f is a transitive sofic system there exists a Fischer graph G' with $\Omega_f=\Omega(\mathsf{G}')$. A terminal G'-vertex of an arc with label $(a,\ h)$ $(h\neq f(a))$ is an initial G'-vertex of an arc with label $(a,\ \mathrm{h}+1)$ and is not a terminal G'-vertex nor an initial G'-vertex of any other G'-arc. We construct a labeled graph G by canceling all these vertexes from the labeled graph G' and replacing each chain of G'-arcs with label $\prod_{h=1}^{f(a)}(a,\ h)$ by an arc with label a. Then we have $\Omega=\Omega(\mathsf{G})$ and hence Ω is a sofic system. (2) and (3) follow from the fact that Ω is conjugate to Ω^φ and ${}^\varphi\Omega$.

PROOF of THEOREM 4, THEOREM 5 and THEOREM 6.

Combining arguments of Parry and Sullivan [7] with a result of Nasu [6] we obtain the same subshift from mutually flow equivalent subshifts by finite steps of three kinds of deformations of subshifts. Two of them are the same ones as mentioned in the proof of Theorem 1 and the third one is a deformation of subshift from Ω to Ω_f, which corresponds to a deformation of labeled graph from G to G_f. Then Theorem 4 follows from Lemma 3, Theorem 5 follows from Lemma 1 and Lemma 2, and Theorem 6 follows from Theorem 2, Theorem 3, Theorem 4 and Theorem 5.

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