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https://doi.org/10.15017/1449052

出版情報：九州大学教養部数学雑誌. 16 (1), pp.7-15, 1987-12. 九州大学教養部数学教室
バージョン：

Math. Rep.
16-1, 1987.

Existence of nonnegative solutions of some semilinear elliptic equations

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(Received September 18,1987)

1. Introduction

In this note we consider the existence of nonnegative solutions of the semilinear elliptic boundary value problem :

$$(1)\quad \begin{cases} -\Delta u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u \geqq 0, \ u \not\equiv 0 & \text{in } \Omega, \\ u \in C^2(\overline{\Omega}). \end{cases}$$

Here Ω is a bounded domain in R^N $(N \geqq 3)$ with smooth boundary $\partial\Omega$ and $f(x, u) : \overline{\Omega} \times [0, +\infty) \longrightarrow R$.

We assume the following conditions on the nonlinear term $f(x, u)$:

(F_1) f is locally Hölder continuous in $\overline{\Omega} \times R^+$, where $R^+ = [0, +\infty)$,

(F_2) $a_1(x)t^p + a_2(x) \leqq f(x, t) \leqq b_1(x)t^p + b_2(x)$ $\quad (x \in \overline{\Omega},\ t \geqq 0)$,

where

$0 < p < (N+2)/(N-2)$ and $p \neq 1$, and furthermore

$a_\ell(x)$ and $b_\ell(x)$ are continuous functions in $\overline{\Omega}$ $(\ell = 1, 2)$ which may change sign in Ω,and there is a sphere $B_r(x_0)$ $(\subset \Omega)$ with centre x_0 and radius r such that

$a_1(x) > 0$ and $a_2(x) \geqq 0$ in $\overline{B_r(x_0)}$.

In most earlier works ([1, 2, 4, 5, 9]), various authors have considered problem (1) of the case that $f(x, u)$ depends only on u or satisfies some nonlinear conditions at $u=0$ and $u=+\infty$.

For example, the standard assumptions on f are as follows;

(F_3) $|f(x, t)| \leqq a_1 + a_2 t^p$ $(x \in \overline{\Omega},\ t \geqq 0)$, $f(x, 0) \geqq 0$ $(x \in \overline{\Omega})$, where a_1 and a_2 denote positive constant, and $1 < p < (N+2)/(N-2)$.

(F_4) $\lim_{t \to \infty} (F(x, t)/t) < \lambda_1$, $(> \lambda_1)$ uniformly in $x \in \overline{\Omega}$,

and

(F_5) $\lim_{t \to +0} (f(x, t)/t) > \lambda_1$ $(< \lambda_1)$ uniformly in $x \in \overline{\Omega}$.

Here λ_1 is the first eigenvalue of $(-\Delta)$ with Dirichlet boundary condition.

It is obvious that $f(x, u)$ with (F_2) does not always satisfy (F_3), (F_4) and (F_5).

Now we shall consider the special case of (1) :

$$
(2) \qquad \begin{cases} -\Delta u = m(x)\, u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u \geqq 0, \quad u \not\equiv 0 & \text{in } \Omega, \end{cases}
$$

Here p satisfies the following condition :

(P) $0 < p < (N+2)/(N-2)$, $p \neq 1$,

and

$$
(M) \qquad \begin{cases} m(x) \in C^{\alpha}(\overline{\Omega}) \qquad (\alpha \in (0, 1)), \\ m(x) \text{ may change sign in } \Omega \text{ and } \max\{m(x) \mid x \in \overline{\Omega}\} > 0. \end{cases}
$$

Under the conditions (P) and (M), $f(x, u) = m(x) u^p$ satisfies (F_1) and (F_2), but does not satisfy (F_4) and (F_5).

In [3], Brown and Lin proved the existence of positive eigenfunction of problem :

$$
(3) \qquad -\Delta u = \lambda m(x) u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,
$$

where $\lambda \in R$ and $m(x)$ satisfies condition (M). Recently Hess and Kato [8] have generalized this problem to a more general case of uniformly elliptic operator of second order.

At first in §2, we shall study the problem (2) under the conditions (P) and (M), and then in §3 the problem (1) under the conditions $(F_1) - (F_2)$ with $0<p<1$. Roughly speaking, our results are as follows :

THEOREM 1. *Under the conditions* (P) *and* (M), *there exists a solution of the problem* (2) *and if* $0 \leqq m(x)$ *in* Ω, *then the solution is positive.*

COROLLARY. *If* $0 \leqq m(x)$ *in* Ω *and* $0 < p < 1$, *then the problem* (2) *has at most one positive solution.*

THEOREM 2. *Under the conditions* $(F_1)-(F_2)$ *with* $0< p <1$, *there is a solution of the problem* (1).

We shall prove theorem 1 by the standard variational argument and theorem 2 by usual sub-super solution method.

Finally, we state the following definitions :

DEFINITION 1. A function $u\in H_0^1(\Omega)$ is said to be a *weak solution* of the problem (1), provided that

$$\int_\Omega \{\nabla u \cdot \nabla v - f(x,u)v\}dx = 0 \quad \forall v \in H_0^1(\Omega).$$

DEFINITION 2. A function $\varphi\in H^1(\Omega)$ is said to be a *weak subsolution* of the problem (1), provided that

$$\int_\Omega \{\nabla \varphi \cdot \nabla v - f(x,\varphi)v\}dx \leqq 0 \quad \forall v \in H_0^1(\Omega), \quad v \geqq 0,$$

$$\varphi \leqq 0 \quad \text{on} \quad \partial\Omega,$$

$$f(x, \varphi) \in L^2(\Omega).$$

For a *weak supersolution* $\Psi\in H^1(\Omega)$ the inquality signs in the above definition are reversed.

2. The case $f(x, u) = m(x)u^p$

We shall prove Theorem 1. The proof below is due to the standard argument (See e. g. [2]) , and the proof is included for completenes.

Suppose $m(x_0) > 0$ at some $x_0 \in \Omega$.

Put $V \equiv \{v \in H_0^1(\Omega) : \int_\Omega m(x) v^{p+1}dx > 0, \ v \geqq 0 \ \text{ in } \Omega\}$.

It is obvious that V is a nonempty subset of $H_0^1(\Omega)$.

Consider the following minimization problem :

$$\text{(4)} \qquad \operatorname*{Inf}_{v\in V} \left\{ \frac{\int_\Omega |\nabla v|^2 dx}{\left[\int_\Omega m(x) v^{p+1}dx\right]^{2/p+1}} \right\}.$$

(i) The case $0 < p < 1$.

If $v \in V$, then we have, by Hölder's inequality,

(5) $$0 < \left[\int_\Omega m(x)\, v^{p+1}dx\right]^{\frac{2}{p+1}} \leqq c\left[\int_\Omega v^2dx\right] \leqq cS^2\int_\Omega |\nabla v|^2dx,$$

where S is the Sobolev's constant.

(ii) The case $1 < p < (N+2)/(N-2)$.

Since $1 < p+1 < 2N/(N-2)$, we have for any $v \in V$

(6) $$0 < \left[\int_\Omega m\, v^{p+1}dx\right]^{\frac{2}{p+1}} \leqq c\left[\int_\Omega v^{p+1}dx\right]^{\frac{2}{p+1}} \leqq cS^2\int_\Omega |\nabla v|^2dx,$$

where S is the Sobolev's constant.

Then we have

(7) $$0 < \frac{1}{c\,S^2} < \frac{\left[\int_\Omega |\nabla v|^2dx\right]}{\left[\int_\Omega m\, v^{p+1}dx\right]^{2/(p+1)}} \quad (\forall\, v \in V)$$

From (7) it follows that there exists a positive number d such that

(8) $$d = \inf_{v \in V} \frac{\left[\int_\Omega |\nabla v|^2dx\right]}{\left[\int_\Omega m\, v^{p+1}dx\right]^{2/(p+1)}}$$

Now we can select a subsequence $\{v_n\}$ in V such that

(9) $$\begin{cases} \int_\Omega m\, v_n^{p+1}dx = 1, \\ \lim_{n\to\infty}\int_\Omega |\nabla v_n|^2dx = d \end{cases}$$

Since $\{v_n\}$ is weakly compact in $H_0^1(\Omega)$, then we can choose a subsequence of $\{v_n\}$, which for simplicity we also denote by $\{v_n\}$, and $v_0 \in H_0^1(\Omega)$ such that

(10) $$\begin{cases} v_n \longrightarrow v_0 \quad \text{weakly in } H_0^1(\Omega), \\ v_n \longrightarrow \quad \text{strongly in } L^q(\Omega), \end{cases}$$

where $1 < q < 2N/(N-2)$.

Then we have

$$\|v_0\|_{H_0^1}^2 < \underline{\lim}_{n\to\infty} \|v_n\|_{H_0^1}^2 = d.$$

On the other hand,

$$\left| \int_\Omega m(x)\ (v_n^{p+1} - v_0^{p+1})\ dx \right|$$

$$\leqq c < \int_\Omega \left| v_n - v_0 \right|^{p+1} dx)^{1/(p+1)} \left(\int_\Omega (\left| v_n \right|^{p+1} + \left| v_0 \right|^{p+1}) dx \right)^{p/p+1}$$

and by Sobolev's Imbedding Theorem we obtain

$$\lim_{n \to \infty} \int_\Omega m(x)\ (v_n^{p+1} - v_0^{p+1}) dx = 0.$$

Therefore we conclude that

$$(11) \quad \begin{cases} \int_\Omega m(x)\ v_0^{p+1} dx = 1, \\ \int_\Omega |\nabla v_0|^2\, dx = d. \end{cases}$$

Hence v_0 satisfies

$$(12) \quad \begin{cases} \int_\Omega \nabla v_0 \cdot \nabla v\ dx = d \int_\Omega m(x)\, v_0^p v\ dx \quad (\forall v \in H_0^1(\Omega)) \\ v_0 \in H_0^1(\Omega), \\ v_0 \geqq 0 \quad \text{in } \Omega. \end{cases}$$

Set $u = d^{\frac{1}{p-1}} v_0$. Then u is a nonnegative weak solution of the problem (2). Therefore u satisfies

$$(13) \quad \begin{cases} \int_\Omega \nabla u \cdot \nabla v\ dx = \int_\Omega m(x)\ u^p v\ dx \quad (\forall v \in H_0^1(\Omega)), \\ u \geqq 0 \ \text{ in } \Omega, \\ u \in H_0^1(\Omega), \\ \int_\Omega |\nabla u|^2 dx = d^{\frac{p+1}{p-1}} < \infty. \end{cases}$$

Now by virtue of the standard argument, we can show that

$$m(x)\, u^p(x) \in L^q(\Omega) \qquad (\forall q > N/2),$$

and then the above weak solution u belongs to $C^1(\overline{\Omega})$.

Therefore $m(x)\ u^p(x) \in C^\beta(\overline{\Omega})$,

where
$$\beta = \begin{cases} \min\ (\alpha,\ p) & \text{if } p < 1, \\ \alpha & \text{if } p > 1. \end{cases}$$

Hence $u \in C^{2,\beta}(\overline{\Omega})$.

Furthemore if $m(x) \geqq 0$ in Ω, then by the maximum principle the solution u is positive in Ω.

Therefore we can conculude the following :

THEOREM 1. *Suppose that the conditions* (P) *and* (M) *are satisfied. Then, the problem* (2) *has a solution, and if* $0 \leqq m(x)$ *in* Ω, *then the solution u is positive.*

COROLLARY. *If* $0 \leqq m(x)$ *in* Ω *and* $0 < p < 1$, *then the problem* (2) *has at most one positive solution.*

PROOF. We follow Spruck's argument (Theorem 1. 1 in [10]).

Let u, v be positive solutions of the problem (2).

Set $w_1 = u^{1-p}$ and $w_2 = v^{1-p}$. Then w_k satisfies

$$\Delta w_k = -(1-p)m(x) - p/(1-p)\ \{|\nabla w_k|^2/w_k\}\ (k=1,\ 2).$$

From this it follows that

$$\Delta(w_1 - w_2) = -p/(1-p)\ \{|\nabla w_1|^2/w_1 - |\nabla w_2|^2/w_2\}$$
$$= \sum_{j=1}^{N} b_j(x)\ (w_1 - w_2)_{x_j} + c(x)\ (w_1 - w_2) \text{ in } \Omega,$$

where $c(x) = p/(1-p)\ \{|\nabla w_2|^2/(w_1 w_2)\} \geqq 0$. $w_1 - w_2 = 0$ on $\partial\Omega$.
By the maximum principle (e. g. [6]), we have $w_1 \equiv w_2$ in $\overline{\Omega}$.

3. The general case f(x, u)

We assume the conditions (F_1) and (F_2) are satisfied.

At first we give some examples of $f(x,\ u)$ satisfying (F_1) and (F_2).

EXAMPLE. $f(x,\ u) = a_1(x)\ u^p + a_2(x)$.

$f(x,\ u) = a_1(x)\ u^p\ (e^{-u} + 1) + a_2(x)$,

In this section we assume that $0 < p < 1$.

Now we shall construct a subsolution φ and a supersolution Ψ of the problem (1).

LEMMA 1. *There exists a weak subsolution* φ *of the problem* (1).

PROOF.

By the assumption (F_2), there exist a sphere $B_r(x_0)$ $(\subset \Omega)$ with centre x_0 and radius r and a number $\delta > 0$ such that

$$a_1(x) > \delta \text{ and } a_2(x) \geqq 0 \quad \text{in } B_r(x_0).$$

Consider the eigenvalue problem :

$$(14)\quad \begin{cases} -\Delta z = \mu z & \text{in } B_r(x_0), \\ z = 0 & \text{on } \partial B_r(x_0). \end{cases}$$

Let μ_0 be the first eigenvalue of (14) and z the corresponding positive eigenfunction satisfying $\max\{z(x) \mid x \in B_r(x_0)\} = 1$.
Set

$$\varphi = \begin{cases} s\,z & x \in B_r(x_0) \\ 0 & x \in \Omega - B_r(x_0), \end{cases}$$

where constant $s \in R^+$ is not greater than $(\delta/\mu_0)^{\frac{1}{1-p}}$.
In $B_r(x_0)$, φ satisfies

$$\begin{aligned} \Delta\varphi + f(x, \varphi) &\geqq \Delta(sz) + a_1(x)(sz)^p + a_2(x) \\ &\geqq -s\mu_0 z + \delta(sz)^p \\ &\geqq (sz)^p\{\delta - s^{1-p}\mu_0\} > 0. \end{aligned}$$

Then $\varphi \in H_0^1(\Omega)$ is a weak subsolution of the problem (1).

LEMMA 2. *There exists a weak supersoluition* Ψ *of the problem* (1).

PROOF. Let D be a bounded domain with smooth boundary ∂D such that $D \supset \overline{\Omega}$.

Consider the following eigenvalue problem :

$$(15) \quad \begin{cases} -\Delta w = \lambda w & \text{in } D, \\ w = 0 & \text{on } \partial D. \end{cases}$$

Let λ_0 be the first eigenvalue of (15) and w_0 the corresponding positive eigenfunction satisfying $\max\{w_0(x) \mid x \in D\} = 1$.

Put $h = \min\{w_0(x) \mid x \in \overline{\Omega}\}\ (>0)$ and $\Psi = t\,w_0$.

Here the constant t is not less than $(2/(\lambda_0 h))^{\frac{1}{1-p}}$.

Without loss of generality, we assume $|b_1|_\infty + |b_2|_\infty \leqq 1$.

Then in Ω, Ψ satisfies

$$\begin{aligned} \Delta\Psi + f(x, \Psi) &\leqq -\lambda_0 t w_0 + b_1(x)(tw_0)^p + b_2(x) \\ &\leqq -\lambda_0 t w_0 + (tw_0)^p + 1 \\ &\leqq t^p\{1 + (1/t)^p - \lambda_0 h t^{1-p}\} \leqq 0. \end{aligned}$$

Then $\Psi \in C^2(\overline{\Omega})$ is a weak supersolution of the problem (1).

Therefore if we choose t such that

$$t\,h > s,$$

then

$$0 \leqq \varphi(x) \leqq \Psi(x) \quad \text{in } \overline{\Omega}.$$

Now using Hess's result (Theorem 1 in [7]), we can conclude the following :

LEMMA 3. *Under the above conditions, the problem* (1) *has a nonnegative weak* *solution* $u \in H_0^1(\Omega)$ *with* $\varphi \leqq u \leqq \Psi$ *in* Ω.

For any weak solution u of Lemma 3, the following inequality holds :

$$\int_\Omega |f(x, u(x))|^q dx < \infty \qquad (\forall q > 1).$$

By the standard argument, we know that the weak solution u belongs to $W^{2,q}(\Omega)$ $(\forall q > N/2)$ and then $u \in C^1(\overline{\Omega})$.
Using the condition (F_1), we know $f(x, u(x)) \in C^\alpha(\overline{\Omega})$.
Therefore we can conclude that

$$u \in C^{2,\alpha}(\overline{\Omega}).$$

Now we have proved the following :
THEOREM 2. *Under the condition* (F_1) *and* (F_2) *with* $0 < p < 1$, *the problem* (1) *has a nonnegative classical solution.*
Finally, we make the following :
REMARK. Under the conditions (F_1) and (F_2) with $p>1$, we can not state any result of the problem (1). In the next note we will mention some results about this.

Acknowledgement. The authors wish to express their sincere thanks to Professor Akira Ono for his valuable comments.

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