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1. Introduction

In [2], Magill introduced the concept of a laminated near-ring. Let N be an arbitrary near-ring. Each element a in N yields a new near-ring N_a whose additive group coincides with that of N and whose multiplication $*$ is defined by $x * y = xay$ for any two elements x and y in N, where a product in the original near-ring is denoted by juxtaposition. The N_a is called a laminated near-ring of N with respect to the laminator a.

A near-ring is called a Boolean near-ring if its each element is idempotent. The purpose of this note is to characterize those near-rings each of whose laminated near-rings with respect to non-zero laminator is Boolean.

In this note, a near-ring is a right near-ring. For the basic terminology and notation we refer to [3].

2. Necessary condition

It is very easy to characterize those near-rings all of whose laminated near-rings are Boolean.

THEOREM 1. *Let N be a near-ring. Then the following conditions are equivalent :*

(i) *Every laminated near-ring of N is Boolean.*

(ii) *The laminated near-ring of N with respect to zero is Boolean.*

(iii) *N is constant.*

PROOF. (i) $\Rightarrow$ (ii). Evident.

(ii) $\Rightarrow$ (iii). For any element x in N, we have $x0 = x0x = x * x = x$, since the laminated near-ring with respect to zero is Boolean. Thus N is constant.

(iii) $\Rightarrow$ (i). Let a be any element in N. For any element x in N_a, we have $x * x = xax = x$, since x is a constant element in N. Thus N_a is Boolean.

In view of Theorem 1, hereafter, we restrict our attention to those near-rings each of whose laminated near-rings with respect to non-zero laminator is Boolean.

We have the following necessary condition for each laminated near-ring with respect to non-zero laminator to be Boolean.

THEOREM 2. *Let N be a near-ring $\neq \{0\}$. If each laminated near-ring of N with respect to non-zero laminator is Boolean, then N is zero-symmetric or constant.*

PROOF. Suppose that N is not zero-symmetric. Then there exists an element n in N such that $n0 \neq 0$. Since the laminated near-ring with respect to $n0$ is Boolean, we have $x = x(n0)x$ for any element x in N, whence $x = xn0$. Hence $x0 = (xn0)0 = xn0 = x$. Thus N is constant.

In case of constant near-rings, the converse of Theorem 2 holds by Theorem 1. So we will be concerned in case of zero-symmetric near-rings.

3. Some lemmas

We begin with

LEMMA 1. *Let N be a near-ring and a be an element in N. If N is zero-symmetric, then the laminated near-ring N_a of N with respect to a is zero-symmetric.*

PROOF. For any element x in N_a, we have $x * 0 = xa0 = x0 = 0$, since N is zero-symmetric. Thus N_a is zero-symmetric.

The proof of the following lemma is in [**1**].

LEMMA 2. *Let N be a zero-symmetric near-ring. If N has no non-zero nilpotent elements, then $ex = exe$ for each idempotent e and each element x in N.*

LEMMA 3. *Let N be a zero-symmetric near-ring $\neq \{0\}$ and a be a non-zero element in N. Then the laminated near-ring N_a of N with respect to a is Boolean if and only if N is Boolean and a is a right identity of N.*

PROOF. Suppose that N_a is Boolean. Since N_a is zero-symmetric by Lemma 1 and has no non-zero nilpotent elements, by Lemma 2 we have

$$(xa) * (ax) = (xa) * (ax) * (xa)$$

for any element x in N. Noreover, we have

$$(xa) * (ax) = (xa)a(ax) = xax = x$$

and

$$(xa) * (ax) * (xa) = x * (xa) = xaxa = xa.$$

Hence we have $x = xa$ for any element x in N. Thus a is a right identity of N. Therefore we have $x^2 = xax = x$ for any element x in N. Thus N is Boolean.

The converse is evident.

4. Main result

A near-ring is called a trivial integral near-ring if its each non-zero element is a right identity. In other words, N is a trivial integral near-ring if and only if N is a constant near-ring or a zero-symmetric near-ring with the trivial multiplication $xy = x$ if $y \neq 0$ and $x0 = 0$ for any elements x and y in N.

We are now ready to state the main result of this note.

THEOREM 3. *Let N be a near-ring $\neq \{0\}$. Then each laminated near-ring of N with respect to non-zero laminator is Boolean if and only if N is a trivial integral near-ring.*

PROOF. The necessity follows from Theorem 2 and Lemma 3. The sufficiency follows from Theorem 1 and Lemma 3.

COROLLARY 1. *Let N be a near-ring with a non-zero central element. Then each laminated near-ring of N with respect to non-zero laminator is Boolean if and only if N is a two-element field.*

PROOF. Let a be a non-zero central element in N. Suppose that each laminated near-ring of N with respect to non-zero laminator is Boolean. Then, for any non-zero element x in N, we have $a = ax = xa = x$, since a and x are right identities by Theorem 3. Hence $N = \{0, a\}$. Moreover, we have $a0 = 0a = 0$, whence a is an identity of N. So N is a two-element field.

The converse is evident.

COROLLARY 2. *Let N be a near-ring with a non-zero distributive element. Then each laminated near-ring of N with respect to non-zero laminator is Boolean if and only if N is a two-element field.*

PROOF. Let a be a non-zero distributive element in N. Suppose that each laminated near-ring of N with respect to non-zero laminator is Boolean. Since the distributive element a is a zero-symmetric element in N, N is zero-symmetric by Theorem 2 and has no non-zero nilpotent elements by Lemma 3. Hence the distributive idempotent a is central by [1, Lemma 2]. So N is a two-element field by Corollary 1.

A distributively generated near-ring $\neq \{0\}$ has a non-zero distributive element. So the following is an immediate consequence of Corollary 2.

COROLLARY 3. *Let N be a distributively generated near-ring $\neq \{0\}$. Then each laminated near-ring of N with respect to non-zero laminator is Boolean if and only if N is a two-element field.*

References

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