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加藤, 久子
九州大学教養部数学教室

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Regularity of weak solutions of the Navier-Stokes equations

Hisako KATO

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§ 1. Introduction

We consider the initial-boundary value problem for the Navier-Stokes equations in three space dimension;

$$(1.1)\qquad \begin{cases} u_t-\Delta u+u\cdot\nabla u=f-\nabla p, & \text{in } D\times(0,T), \\ \operatorname{div} u=0, & \text{in } D\times(0,T), \\ u|_{\partial D}=0, & \text{in } (0,T), \\ u(x,0)=a(x), & \text{in } D. \end{cases}$$

Here, the function $u=(u^1(x,t), u^2(x,t), u^3(x,t))$ represents the velocity field of the fluid and $p=p(x,t)$ is the pressure. The function $a=(a^1(x), a^2(x), a^3(x))$ is a given initial velocity and $f=(f^1(x,t), f^2(x,t), f^3(x,t))$ is a given external force. Let D be a bounded domain in R^3 with smooth boundary ∂D. In this paper, let $f=0$ for simplicity.

E. Hopf [8] proved in 1951 the existence of a global (in time) weak solution (more generally, for an arbitrary domain D in R^n). Various researches have been made by many mathematicians, but no one has yet succeeded in proving or disproving the uniqueness of the weak solutions. Recently, many results have been established with respect to the regularity of weak solutions; see Scheffer [10] [11] [12], Caffarelli, Kohn and Nirenberg [1], Giga [4] [6], Sohr and Von Wahl [14] etc.

Let A be Stokes operator. In this paper we discuss the Hausdorff dimension of the set E of possible time singularity of weak solutions belonging to $L_2((0,T); D(A^{\gamma}))$ and we shall show that if $\gamma=\frac{3}{4}-\varepsilon$, then 2ε-dimensional Hausdorff measure of the set E is zero, and moreover if $\gamma=\frac{3}{4}$, the set E is empty.

§ 2. weak solution and strong solution

Let H_0^m = closure of $C_0^\infty(D)$ in H^m,

$C_{0,\sigma}^\infty = \{\varphi \in C_0^\infty(D);\ \operatorname{div}\varphi = 0\}$,

H_σ = closure of $C_{0,\sigma}^\infty$ in L_2,

$H_{0,\sigma}^1$ = closure of $C_{0,\sigma}^\infty$ in H^1,

and operator P be the orthogonal projection from L_2 onto H_σ. By Stokes operator A we denote the Friedrichs extension of the symmetric operator $-P\Delta$ in H_σ with domain $D(A) = H^2 \cap H_0^1 \cap H_\sigma$. It is well known that $D(A^{1/2}) = H_{0,\sigma}^1$

We give the definition of weak solutions of (1.1).

DEFINITION 2.1 (see Giga [6], Masuda [9])
Let the initial function a be in H_σ. A measurable function v on $D \times (0, T)$ is called a weak solution of the problem (1.1) if

(i) $v \in L_\infty((0,T); H_\sigma) \cap L_2((0,T); H_{0,\sigma}^1)$,

(ii) v is weakly continuous from $[0, T)$ to H_σ,

(iii) $\int_0^T \left\{ -(v, \varphi_t) + (\nabla v, \nabla \varphi) + (v \cdot \nabla v, \varphi) \right\} dt = (a, \varphi(0))$,

for all $\varphi \in C^\infty(D \times [0, T))$, $\operatorname{div}\varphi = 0$ such that φ vanishes outside a compact set of $D \times [0, T)$,

and moreover the energy inequalities hold for v ;

(iv) $\|v(t)\|^2 + 2\int_0^t \|\nabla v\|^2\, dt \leqq \|a\|^2$ for $t \in [0, T)$,

(iv)′ $\|v(t)\|^2 + 2\int_s^t \|\nabla v\|^2\, dt \leqq \|v(s)\|^2$ for almost all $s \geqq 0$, including $s = 0$ and all $t > s$.

E. Hopf proved the existence of a weak solution in $D \times (0, \infty)$ [8]. To our purpose, the following three propositions are useful.

PROPOSITION 2.1 (uniqueness of weak solutions; see Serrin [13], Temam [15], Masuda [9]).

Let u and v be weak solutions of (1.1) in $D\times (0, T)$ taking the same initial value $a\in H_\sigma$ and satisfying the energy inequality (iv) *in definition 2.1. Of the two functions u and v let u belong also to the space $L_q((0, T); L_p(D))$ for some p and q such that*

$$\frac{3}{p}+\frac{2}{q}\leqq 1, p>3.$$

Then it holds $u=v$ *on* $[0, T)$.

From the problem (1.1), the following initial value problem is introduced;

$$(2.1)\qquad \begin{cases} \dfrac{du}{dt}+Au=Fu+Pf(t), \text{ in } (0,T), \\ u(0)=a, \end{cases}$$

where $Fu=-P(\mathrm{u}\cdot\nabla)u$.
Let $Pf\equiv 0$. This equation is considered in the integral form;

$$(2.2)\qquad u(t)=e^{-tA}a+\int_0^t \mathrm{e}^{-(t-s)A}Fu(s)\,ds, \qquad \text{in } (0, T).$$

PROPOSITION 2.2 (existence and uniqueness of strong solution, see Fujita and Kato [3], Giga [5])
Let $\frac{1}{4}\leqq\gamma<1$, and a be in $D(A^\gamma)$. Then there is a unique strong solution u of (2.2) such that

(i) $u\in C([0, T); D(A^\gamma))$, $\mathrm{u}(0)=a$,
(ii) $u\in C((0, T); D(A^\alpha))$, and $\|A^\alpha u(t)\|=o(t^{\gamma-\alpha})$ *as* $t\rightarrow+0$ *for all* α, $\gamma<\alpha<1$, *where T is a positive number depending on a.*

Moreover $u\in C^\infty(\overline{D}\times(0, T))$.

PROPOSITION 2.3 (estimate of maximal interval, see Giga [4], [5])
Let $\sigma>\frac{1}{4}$ and $[0, T_*)$ be the maximal interval on which u given in proposition 2.2 exists. If a is in $D(A^\sigma)$, then the estimate

$$(2.3)\qquad T_*^{\sigma-1/4}\geqq\frac{C}{\|A^\sigma a\|}$$

holds with a constant C independent of a.

With respect to the proof of proposition 2.3, we note that it holds

$$\begin{cases} t^{\alpha - 1/4} \| A^{\alpha} e^{-tA} a \| \to 0 \ (t \to +0), \\ t^{\alpha - 1/4} \| A^{\alpha} e^{-tA} a \| = t^{\alpha - 1/4} \| A^{\alpha - \sigma} e^{-tA} A^{\sigma} a \| \leqq t^{\sigma - 1/4} \| A^{\sigma} a \|, \end{cases}$$

for $\frac{1}{4} < \sigma < \alpha < 1$, and it follows (2.3) from these relations.

§ 3. regularity of weak solutions

To discuss the regularity of weak solutions we shall show the following lemma, and then we recall the concept of the Hausdorff dimension.

LEMMA 3. 1
Let a be in $D(A^{\sigma})$ *for* $\sigma > \frac{1}{4}$. *Then, any weak solution* v *of the problem (1.1) agrees with the strong solution* u *given in proposition 2.2 on the maximal interval* $[0, T_*)$, *where the strong solution exists.*

PROOF. The strong solution u is a weak solution. By imbedding theorem (see Henry [7]); $D(A^{\alpha}) \subset L_q(D)$ for $\frac{1}{q} > \frac{3-4\alpha}{6}$, there exists p such that $\|u\|_{L_p(D)} \leqq c \|u\|_{D(A^{\sigma})}$, $p > 3$.

In fact, since for $3-4\sigma > 0$ it holds $\frac{6}{3-4\sigma} > 3$, we may choose p such that $\frac{6}{3-4\sigma} > p > 3$ and for $3-4\sigma \leqq 0$, we choose an arbitrary number p such that $p > 3$. Consequently, by considering $u \in C([0, T]; D(A^{\sigma}))$ for any $T < T_*$, $u \in L_q((0, T); L_p(D))$ for $q = \frac{2p}{p-3}$. By proposition 2.1 we have $v = u$ on $[0, T_*)$. The proof of lemma 3.1 is complete.

DEFINITION 3.1 (see Federer [2])
Let X be a separable metric space and let $p > 0$. The p-dimensional Hausdorff measure of a subset Y of X is

$$(3.1) \quad \mu_p(Y) = \lim_{\varepsilon \to +0} \mu_{p,\varepsilon}(Y) = \sup_{\varepsilon > 0} \mu_{p,\varepsilon}(Y),$$

where

$$(3.2) \quad \mu_{p,\varepsilon}(Y) = \inf \left\{ \sum_{n=1}^{\infty} (\operatorname{diam} U_n)^p ;\ U_n \in O_{\varepsilon},\ Y \subset \bigcup_{n=1}^{\infty} U_n \right\},$$

and $O_{\varepsilon} = \left\{ U \text{ is ball; diameter of } U \leqq \varepsilon \right\}$.

The Hausdorff dimension of a subset Y of X is the number;
$\sup\{p>0;\ \mu_p(Y)=+\infty\}$.
It is well known that

(i) $\mu_{p,\varepsilon_1}(Y)\geqq\mu_{p,\varepsilon_2}(Y)$ for $0<\varepsilon_1<\varepsilon_2$,

(ii) $\mu_p(\bigcup_{n=1}^{\infty}Y_n)\leqq\sum_{n=1}^{\infty}\mu_p(Y_n)$,

(iii) if $\mu_p(Y)<+\infty$, then $\mu_q(Y)=0$ for all $q>p$,

(iv) $\sup\{p>0;\ \mu_p(Y)=+\infty\}$
$=\inf\{p>0;\ \mu_p(Y)=0\}$
$=\inf\{p>0;\ \mu_p(Y)<+\infty\}$

(v) let λ_n be n-dimensional Lebesgue measure and μ_n be n-dimensional Hausdorff measure over R^n, then $\lambda_n(Y)=\mu_n(Y)$ for measurable set Y.

Now, we shall show the theorem with respect to the regularity of weak solutions.

THEOREM 3. 1.
Let v be a weak solution of the problem (1. 1) *on* $(0,\infty)$. *Suppose that v belong also to the space $L_2((0,T);D(A^\gamma))$ for some $\gamma\geqq\frac{1}{2}$ and $0<T<\infty$. Then, if $\gamma=\frac{3}{4}-\varepsilon$ $(0<\varepsilon\leqq\frac{1}{4})$ there is a closed set E of Lebesgue measure zero in $(0,T)$ whose 2ε-dimensional Hausdorff measure $\mu_{2\varepsilon}(E)$ is zero and such that v is in $C^\infty(\bar{D}\times((0,T)-E))$. Moreover if $\gamma\geqq\frac{3}{4}$, the above possible singularity set E is empty and v is in $C^\infty(\bar{D}\times(0,T))$.*

PROOF. From the definition of weak solusions there exists a set E_1 of Lebesgue measure zero such that v satisfies the energy inequality;

$\|v(t)\|^2+2\int_s^t\|\nabla v\|^2\,dt\leqq\|v(s)\|^2$ for any $s\in(0,\infty)-E_1$ *and* for all $t>s$.
On the other hand let $E_2=\{t\in(0,T);\ v(t)\notin D(A^\gamma)\}$ then the set E_2 has Lebesgue measure zero by the assumption. We put $E_0=(E_1\cap(0,T))\cup E_2$. *For any* $t\in(0,T)-E_0$ there exists the strong solution u with the initial value $v(t)$ such that $u=v$ on $[t,t+T_*(t))$ by lemma 3. 1.

Therefore v is smooth in $\overline{D}\times(t, t+T_*(t))$. Let

$E=(0, T)-\bigcup_{t\in(0,T)-E_0}\left\{(t, t+T_*(t))\cap(0, T)\right\}$. E is a closed set in $(0, T)$

and $v\in C^\infty(\overline{D}\times((0, T)-E))$. Since $\bigcup_{t\in(0,T)-E_0}(t, t+T_*(t))$ is open set

in R^1, we may put $\bigcup_{t\in(0,T)-E_0}(t, t+T_*(t))=\bigcup_{n=1}^{\infty}(r_n, s_n)$,

where (r_n, s_n) are mutually disjoint intervals and $\bigcup_{n=1}^{\infty}(r_n, s_n)$ represents the union of countable intervals at most. Since it holds $E_0\subset E$ and $E-E_0\subset\{r_n;\ n=1, 2, 3\cdots\}$, $E-E_0$ has Lebesgue measure zero. Thus E has Lebesgue measure zero.For any $t\in(r_n, s_n)\cap(0, T)$ it follows

$$t+T_*(t)\leqq s_n,$$

and therefore

$$(3.\ 3)\quad (s_n-t)^{\gamma-1/4}\geqq T_*(t)^{\gamma-1/4}\geqq\frac{C}{\|A^\gamma v(t)\|}$$

by proposition 2. 3. Let $\gamma=\frac{3}{4}-\varepsilon$. In the case of satisfying $s_n\leqq T$ for all n, we have

$$C\sum_{n=1}^{\infty}\int_{r_n}^{s_n}\frac{dt}{(s_n-t)^{1-2\varepsilon}}\leqq\sum_{n=1}^{\infty}\int_{r_n}^{s_n}\|A^\gamma v(t)\|^2\,dt\leqq\int_0^T\|A^\gamma v(t)\|^2\,dt<+\infty.$$

Therefore, it follows

$$(3.\ 4)\quad \sum_{n=1}^{\infty}(s_n-r_n)^{2\varepsilon}<+\infty.$$

If there exist the numbers n_0 satisfying $s_{n_0}>T$, then such a number n_0 is only one, since $(r_{n_0}, s_{n_0})\cap(0, T)=(r_{n_0}, T)$ and $(r_n, s_n)\cap(0, T)$ are mutually disjoint. Hence analogously in the above it holds

$\sum_{n=1,\, n\neq n_0}^{\infty}(s_n-r_n)^{2\varepsilon}<+\infty$. Therefore if s_{n_0} is finite, $\sum_{n=1}^{\infty}(s_n-r_n)^{2\varepsilon}<+\infty$,

and if $s_{n_0}=+\infty$, by considering the interval $(0, r_{n_0})$ instead of $(0, T)$, the relation (3. 4) is valid.

Let $\gamma = \frac{3}{4}$. If $s_n \leqq T$, then it follows from (3. 3)

$$C\int_{r_n}^{s_n} \frac{dt}{(s_n - t)} \leqq \int_{r_n}^{s_n} \| A^{3/4} v\ (t) \|^2\ dt \leqq \int_0^T \| A^{3/4} v\ (t) \|^2\ dt < +\infty.$$

On the other hand, the left-hand side is infinite. It means that $s_n > T$ for all n and that since such a number n is only one, we may denote

$$\bigcup_{n=1}^{\infty} (r_n, s_n) \cap (0, T) = (r_{n_0}, T).$$

By considering that E has Lebesgue measure zero, we have $(r_{n_0}, T) = (0, T)$. Consequently, the time singularity set E is empty.

Lastly, from (3. 4) we reduce to 2ε-dimensional Hausdorff measure $\mu_{2\varepsilon}(E) = 0$ by Scheffer's argument. (see [10])
By (3. 4), for any $\sigma > 0$ there exists a number N such that

$$\text{(3. 5)} \quad \sum_{i=N+1}^{\infty} (s_i - r_i) \leqq \sum_{i=N+1}^{\infty} (s_i - r_i)^{2\varepsilon} \leqq \sigma.$$

We can represent

$$[0, T] - \bigcup_{i=1}^{N} (r_i, s_i) = K_1 \cup K_2 \cup \cdots \cup K_m,$$

where K_i are mutually disjoint closed intervals, and it holds $E \subset K_1 \cup K_2 \cup \cdots \cup K_m$. Since (r_i, s_i) $(i = N+1, N+2, \cdots\cdots)$ is included in one and only one K_j, we denote I_j the set of i's satisfying $K_j \supset (r_i, s_i)$.

It holds

$$\text{(3. 6)} \quad K_j = \bigcup_{i \in I_j} (r_i, s_i) \cup (K_j \cap E),$$

and

$$\text{(3. 7)} \quad \operatorname{diam} K_j = \sum_{i \in I_j} (s_i - r_i).$$

The equality (3. 7) is showed as follows. For any $x \in K_j \cap E$, the distance $\mathrm{d}\,(x, \bigcup_{i \in I_j} (r_i, s_i))$ is zero, and thus the $K_j \cap E$ is included in the cluster set of

$\bigcup_{i \in I_j} (r_i,\ s_i)$, where we used that the set E has Lebesgue measure zero. Therefore it follows

$$(3.\ 8)\qquad \sum_{j=1}^{m} (\mathrm{diam}\ K_j)^{2\varepsilon} = \sum_{j=1}^{m} \left\{ \sum_{i \in I_j} (s_i - r_i) \right\}^{2\varepsilon}$$

$$\leqq \sum_{j=1}^{m} \sum_{i \in I_j} (s_i - r_i)^{2\varepsilon} = \sum_{n=N+1}^{\infty} (s_n - r_n)^{2\varepsilon} \leqq \sigma .$$

By (3. 5) and (3. 7) it holds

$$(3.\ 9)\qquad \mathrm{diam}\ K_j \leqq \sigma .$$

Consequently, by the definition (3. 2), and the inequality (3. 8), $\mu_{2\varepsilon,\ \sigma}(E) \leqq \sigma$. Let $\sigma \to 0$, then it means that 2ε-dimensional Hausdorff measure of the set E is zero. This completes the proof of theorem 3. 1.

REMARK. It is well known that the weak solution constructed by E. Hopf belongs to $L_2((0,\ \infty);\ D(A^{1/2}))$ and $\mu_{1/2}(E) = 0$. We note that the singularity set E is empty for $\gamma = \frac{3}{4}$. More generally, Giga [6] proved regularity criteria by considering $L_q((0,\ T);\ L_p(D))$ instead of $L_2((0,\ T);\ D(A^{\gamma}))$. But it seems that our result for $\gamma = \frac{3}{4}$ is not involved in [6].

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