Quasi-ideals which are subnear-fields

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1. Introduction

In ring theory the following results are well-known (see [2, Satz 3; Satz 4]):

(A) If a quasi-ideal $Q$ of a ring $R$ is a division subring of $R$, then $Q$ is a minimal quasi-ideal of $R$.

(B) A minimal quasi-ideal $Q$ of a ring $R$ is either a zero subring of $R$ or a division subring of $R$.

In this note we shall extend these results to near-rings and show some applications of extended results.

2. Preliminaries

By a near-ring we mean a non-empty set $N$ in which an addition $+$ and a multiplication $\cdot$ are defined such that

1. $(N, +)$ is a group,
2. $(N, \cdot)$ is a semigroup,
3. $(n+n')n''=nn''+n'n''$ $(n, n', n'' \in N)$.

In dealing with general near-ring the neutral element of $(N, +)$ will be denoted by $0$.

In this section, $N$ will denote a near-ring. The set $N_0$ of all elements $n$ of $N$ with $n0=0$ is called the zero-symmetric part of $N$; $N$ is called zero-symmetric if $N=N_0$. An element $d$ of $N$ is called distributive if $d(n+n')=dn+dn'$ for all elements $n, n'$ of $N$. The set of all distributive elements of $N$ will be denoted by $N_d$.

Let $A$ and $B$ be two non-empty subsets of $N$. We shall define two kinds of products $AB$ and $A*B$: $AB$ denotes the set of all finite sums of
the form

\[ \sum a_i b_i \ (a_i \in A, b_i \in B) ; \]

\( A*B \) denotes the set of all finite sums of the form

\[ \sum (a_i(a_i'+b_i)-a_i a_i') \ (a_i, a_i' \in A, b_i \in B). \]

A subgroup \( S \) of \((N, +)\) is called an \( N\)-subgroup of \( N \) if \( NS \subseteq S \). A subgroup \( M \) of \((N, +)\) is called a subnear-ring of \( N \) if \( MM \subseteq M \). For instance, \( Nn \) is an \( N\)-subgroup of \( N \) for every element \( n \) of \( N \). The zero-symmetric part \( N_0 \) of \( N \) is a subnear-ring of \( N \).

A subgroup \( Q \) of \((N, +)\) is called a quasi-ideal of \( N \) if \( QN \cap NQ \subseteq N^*Q \subseteq Q \). For instance, every \( N\)-subgroup of \( N \), \( dN \) with a distributive element \( d \) of \( N \) and the zero-symmetric part \( N_0 \) of \( N \) are quasi-ideals of \( N \). Clearly \( \{0\} \) and \( N \) are quasi-ideals of \( N \). If \( N \) has no quasi-ideals except \( \{0\} \) and \( N \), we say that \( N \) is \( Q\)-simple.

A near-ring \( N \) is called a near-field if it has at least two elements and its non-zero elements form a group with respect to the multiplication defined in \( N \).

Let \( \mathbb{Z}_2 \) be the integers modulo 2. Then \((\mathbb{Z}_2, +)\) with \( 0 \cdot 0 = 0 \cdot 1 = 0 \), \( 1 \cdot 0 = 1 \cdot 1 = 1 \) is a near-field. As usual, throughout this note, we will exclude those near-fields which are isomorphic to this near-field. So every near-field is zero-symmetric and \( Q\)-simple (see [1, p. 249 and 3, Theorem 2]).

3. Quasi-ideals which are subnear-fields

A non-zero quasi-ideal \( Q \) of a near-ring \( N \) is called minimal if \( Q \) does not properly contain any non-zero quasi-ideal of \( N \).

We have

**Theorem 1.** If a quasi-ideal \( Q \) of a near-ring \( N \) is a subnear-field of \( N \), then \( Q \) is a minimal quasi-ideal of the zero-symmetric part \( N_0 \) of \( N \).

**Proof.** Since a near-field is zero-symmetric, the quasi-ideal \( Q \) is contained in \( N_0 \). By [3, Proposition 2], the relation \( Q = Q \cap N_0 \) implies that \( Q \) is a quasi-ideal of \( N_0 \).
Let $Q'$ be a quasi-ideal of $N_o$ such that $(0) \neq Q' \subseteq Q$.

Then we have

$$Q'Q \cap QQ' \cap Q^*Q' \subseteq Q'N_o \cap N_oQ' \cap Q'N_o^*Q' \subseteq Q',$$

which implies that $Q'$ is a quasi-ideal of $Q$. Since a near-field is $Q$-simple, we have $Q' = Q$. Thus $Q'$ is a minimal quasi-ideal of $N_o$.

In Theorem 1, the quasi-ideal $Q$ is a minimal quasi-ideal of $N$, too. In fact, we have

**Proposition 1.** If a quasi-ideal $Q$ of a near-ring $N$ is a minimal quasi-ideal of $N_o$, then $Q$ is a minimal quasi-ideal of $N$.

**Proof.** Let $Q'$ be a quasi-ideal of $N$ such that $(0) \neq Q' \subseteq Q$. Then $Q'$ is contained in $N_o$ and we have

$$Q'N_o \cap N_oQ' \cap Q'N_o^*Q' \subseteq Q'N \cap Q^*Q' \subseteq Q',$$

which implies that $Q'$ is a quasi-ideal of $N_o$. Since $Q$ is a minimal quasi-ideal of $N_o$, we have $Q' = Q$. Thus $Q$ is a minimal quasi-ideal of $N$.

### 4. Minimal quasi-ideals

In view of Theorem 1, we are going to consider those quasi-ideals which are minimal in the zero-symmetric part. We start with

**Proposition 2.** Let $e$ be a distributive idempotent element of a near-ring $N$ and $S$ an $N$-subgroup of $N$. Then $eS$ is a quasi-ideal of $N$ such that $eS = S\cap eN$.

**Proof.** Since $S$ is an $N$-subgroup of $N$, we have $eS \subseteq NS \subseteq S$. On the other hand, we have $eS \subseteq eN$. Hence $eS \subseteq S\cap eN$.

Conversely, any element $a$ of $S \cap eN$ has the form

$$a = s = en \ (s \in S, \ n \in N),$$

whence $a = en = een = es \in eS$.

Since $S$ and $eN$ are quasi-ideals of $N$, by [3, Proposition 1], the re-
lation $eS=S\cap eN$ implies that $eS$ is a quasi-ideal of $N$.

We now have

**Theorem 2.** Let $E$ be the set of all idempotent elements of a near-ring $N$ and $D$ the set of all elements $d$ of $N$ such that $d(n+n')=dn+dn'$ for all elements $n$, $n'$ of the zero-symmetric part $N_0$ of $N$.

If a quasi-ideal $Q$ of $N$ is a minimal quasi-ideal of $N_0$, then $Q$ is a subnear-ring of $N$ with $Q\cap E\cap D=\{0\}$ or $Q$ is a subnear-field of $N$.

**Proof.** Since $Q$ is a quasi-ideal of $N_0$, by [3, Corollary to Theorem 1], $Q$ is a subnear-ring of $N_0$. Hence $Q$ is a subnear-ring of $N$.

Suppose that $Q\cap E\cap D\neq \{0\}$. Then there is a non-zero element $e$ in $Q\cap E\cap D$. So the element $e$ is a distributive idempotent element of the subnear-ring $N_0$ of $N$, and $N_0e$ is an $N_0$-subgroup of $N_0$. Hence $e(N_0e)$ is a quasi-ideal of $N_0$ by Proposition 2.

Since $Q$ is a quasi-ideal of $N_0$, by [3, Proposition 3], we have

$$e(N_0e)=N_0e\cap eN_0\subseteq N_0Q\cap QN_0\subseteq Q.$$ 

Moreover, $e(N_0e)$ contains the non-zero element $e$. So, from the minimality of the quasi-ideal $Q$, it follows that $Q=e(N_0e)$.

This implies that $Q$ is a subnear-ring with the identity element $e$. So it remains to be shown that every non-zero element of $Q$ has a left inverse element in $Q$.

Let $n$ be a non-zero element of $Q$. Then we have $Qn=e(N_0n)$. Since $N_0n$ is an $N_0$-subgroup of $N_0$, by Proposition 2, $Qn$ is a quasi-ideal of $N_0$ and it contains the non-zero element $en$. Moreover, $Qn$ is contained in $Q$. So, from the minimality of the quasi-ideal $Q$, it follows that $Qn=Q$. Consequently, there exists an element $n'$ in $Q$ such that $nn'=e$.

In case that $N$ is zero-symmetric in Theorem 2, it is evident that $D=N_0$. So we have

**Corollary.** If a quasi-ideal $Q$ of a zero-symmetric near-ring $N$ is minimal, then $Q$ is a subnear-ring of $N$ with $Q\cap N_0\cap E=\{0\}$ or $Q$ is a subnear-field of $N$. 
5. Applications

Applying Corollary to Theorem 2, we are going to give an another proof of the following theorem in [3]:

**Theorem 3.** If a zero-symmetric near-ring $N$ has a cancellable distributive element contained in a minimal quasi-ideal of $N$, then $N$ is a near-field.

**Proof.** Suppose that the zero-symmetric near-ring $N$ has a cancellable distributive element $c$ contained in a minimal quasi-ideal $Q$ of $N$.

Since $c$ is distributive, by [3, Proposition 1], $cN \cap Nc$ is a quasi-ideal of $N$ and it contains the non-zero element $c^2$. Moreover, by [3, Proposition 3], we have $cN \cap Nc \subseteq QN \cap NQ \subseteq Q$. So, from the minimality of the quasi-ideal $Q$, it follows that $Q = cN \cap Nc$.

Since $c^2$ is distributive, similarly we have $Q = c^2N \cap Nc^2$. This implies that the element $c$ has the form

$$c = c^2n = mc^2 (n, m \in N),$$

whence $cn = (mc^2)n = mc$.

Set $e = cn$, then $e$ is contained in $cN \cap Nc = Q$, and $e \neq 0$, since $ce = c \neq 0$. Moreover, we have

$$e^2 = (mc)(cn) = m(c^2n) = me = e.$$

Furthermore, the element $e$ is distributive. In fact, for all elements $n_1, n_2$ of $N$, we have

$$ce(n_1 + n_2) = c(n_1 + n_2) = cn_1 + cn_2 = cnen_1 + cnen_2,$$

that is, $ce(n_1 + n_2) = c(en_1 + en_2)$. This and the cancellability of the element $c$ imply that $e(n_1 + n_2) = en_1 + en_2$.

Thus the minimal quasi-ideal $Q$ has the non-zero distributive idempotent element $e$. So, by Corollary to Theorem 2, $Q$ is a subnear-field with the identity element $e$.

The element $e$ is the identity element of $N$, too. In fact, multiplying both sides of the equality $ce = c$ by any element $x$ of $N$, we have $xe = xc$, whence $xe = x$. Dually we have $ex = x$ from $ce = c$.

Since $e$ is contained in $Q$, for any element $x$ of $N$, we have

$$x = ex = xe \in QN \cap NQ \subseteq Q,$$
that is, \( N = Q \). Thus \( N \) is a near-field.

A non-zero \( N \)-subgroup \( S \) of a zero-symmetric near-ring \( N \) is called minimal if \( S \) does not properly contain any non-zero \( N \)-subgroup of \( N \).

We now have the following result which is an extension of [2, Satz 5] to zero-symmetric near-rings:

**Theorem 4.** If a minimal \( N \)-subgroup \( S \) of a zero-symmetric near-ring \( N \) has a non-zero distributive idempotent element \( e \) of \( N \), then \( eS \) is a subnear-field of \( N \), moreover it is a minimal quasi-ideal of \( N \).

**Proof.** By Proposition 2, \( eS \) is a quasi-ideal of \( N \). Because of Theorem 1 and [3, Corollary to Theorem 1], all we have to prove is that the non-zero elements of \( eS \) form a multiplicative subgroup of \( N \).

Evidently, \( e \) is a left identity element of \( eS \). Let \( es \) be a non-zero element of \( eS \). Then \( S(es) \) is a non-zero \( N \)-subgroup of \( N \) contained in \( S \). By the minimality of \( S \), we have \( S(es) = S \). Hence \( (eS)(es) = eS \). This implies the existence of a non-zero element \( et \) of \( eS \) such that \( (et)(es) = e \). Thus the non-zero elements of \( eS \) form a multiplicative subgroup of \( N \).

**References**