Quasi-ideals which are subnear-fields

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https://doi.org/10.15017/1449047

出版情報：九州大学教養部数学雑誌. 15 (1), pp.67-72, 1985-12. 九州大学教養部数学教室
バージョン：
権利関係：

Math. Rep.
XV-1, 1985.

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(Received September 24, 1985)

1. Introduction

In ring theory the following results are well-known (see [**2**, Satz 3; Satz 4]):

(A) If a quasi-ideal Q of a ring R is a division subring of R, then Q is a minimal quasi-ideal of R.

(B) A minimal quasi-ideal Q of a ring R is either a zero subring of R or a division subring of R.

In this note we shall extend these results to near-rings and show some applications of extended results.

2. Preliminaries

By a *near-ring* we mean a non-empty set N in which an addition $+$ and a multiplication $\cdot$ are defined such that

(1) $(N, +)$ is a group,

(2) $(N, \cdot)$ is a semigroup,

(3) $(n+n')n''=nn''+n'n''$ $(n, n', n'' \in N)$.

In dealing with general near-ring the neutral element of $(N, +)$ will be denoted by 0.

In this section, N will denote a near-ring. The set N_0 of all elements n of N with $n0=0$ is called *the zero-symmetric part* of N; N is called *zero-symmetric* if $N=N_0$. An element d of N is called *distributive* if $d(n+n')=dn+dn'$ for all elements n, n' of N. The set of all distributive elements of N will be denoted by N_d.

Let A and B be two non-empty subsets of N. We shall define two kinds of products AB and $A*B$: AB denotes the set of all finite sums of the form

$$\sum a_i b_i \quad (a_i \in A,\ b_i \in B);$$

A^*B denotes the set of all finite sums of the form

$$\sum (a_i(a_i'+b_i)-a_ia_i') \quad (a_i, a_i' \in A,\ b_i \in B).$$

A subgroup S of $(N, +)$ is called an *N-subgroup* of N if $NS \subseteq S$. A subgroup M of $(N, +)$ is called a *subnear-ring* of N if $MM \subseteq M$. For instance, Nn is an N-subgroup of N for every element n of N. The zero-symmetric part N_0 of N is a subnear-ring of N.

A subgroup Q of $(N, +)$ is called a *quasi-ideal* of N if $QN \cap NQ \cap N^*Q \subseteq Q$. For instance, every N-subgroup of N, dN with a distributive element d of N and the zero-symmetric part N_0 of N are quasi-ideals of N. Clearly $\{0\}$ and N are quasi-ideals of N. If N has no quasi-ideals except $\{0\}$ and N, we say that N is *Q-simple*.

A near-ring N is called a *near-field* if it has at least two elements and its non-zero elements form a group with respect to the multiplication defined in N.

Let Z_2 be the integers modulo 2. Then $(Z_2, +)$ with $0 \cdot 0 = 0 \cdot 1 = 0$, $1 \cdot 0 = 1 \cdot 1 = 1$ is a near-field. As usual, throughout this note, we will exclude those near-fields which are isomorphic to this near-field. So every near-field is zero-symmetric and Q-simple (see [**1**, p. 249 and **3**, Theorem 2]).

3. Quasi-ideals which are subnear-fields

A non-zero quasi-ideal Q of a near-ring N is called *minimal* if Q does not properly contain any non-zero quasi-ideal of N.

We have

THEOREM 1. *If a quasi-ideal Q of a near-ring N is a subnear-field of N, then Q is a minimal quasi-ideal of the zero-symmetric part N_0 of N.*

PROOF. Since a near-field is zero-symmetric, the quasi-ideal Q is contained in N_0. By [**3**, Proposition 2], the relation $Q = Q \cap N_0$ implies that Q is a quasi-ideal of N_0.

Let Q' be a quasi-ideal of N_0 such that $\{0\}\neq Q'\subseteq Q$.
Then we have

$$Q'Q\cap QQ'\cap Q^*Q'\subseteq Q'N_0\cap N_0Q'\cap N_0{}^*Q'\subseteq Q',$$

which implies that Q' is a quasi-ideal of Q. Since a near-field is Q-simple, we have $Q'=Q$. Thus Q' is a minimal quasi-ideal of N_0.

In Theorem 1, the quasi-ideal Q is a minimal quasi-ideal of N, too. In fact, we have

Proposition 1. *If a quasi-ideal Q of a near-ring N is a minimal quasi-ideal of N_0, then Q is a minimal quasi-ideal of N.*

Proof. Let Q' be a quasi-ideal of N such that $\{0\}\neq Q'\subseteq Q$. Then Q' is contained in N_0 and we have

$$Q'N_0\cap N_0Q'\cap N_0{}^*Q'\subseteq Q'N\cap NQ'\cap N^*Q'\subseteq Q',$$

which implies that Q' is a quasi-ideal of N_0. Since Q is a minimal quasi-ideal of N_0, we have $Q'=Q$. Thus Q is a minimal quasi-ideal of N.

4. Minimal quasi-ideals

In view of Theorem 1, we are going to consider those quasi-ideals which are minimal in the zero-symmetric part. We start with

Proposition 2. *Let e be a distributive idempotent element of a near-ring N and S an N-subgroup of N. Then eS is a quasi-ideal of N such that $eS=S\cap eN$.*

Proof. Since S is an N-subgroup of N, we have $eS\subseteq NS\subseteq S$. On the other hand, we have $eS\subseteq eN$. Hence $eS\subseteq S\cap eN$.

Conversely, any element a of $S\cap eN$ has the form

$$a=s=en\quad (s\in S,\ n\in N),$$

whence $a=en=een=es\in eS$.

Since S and eN are quasi-ideals of N, by [3, Proposition 1], the relation $eS=S\cap eN$ implies that eS is a quasi-ideal of N.

We now have

Theorem 2. *Let E be the set of all idempotent elements of a near-ring N and D the set of all elements d of N such that $d(n+n')=dn+dn'$ for all elements n, n' of the zero-symmetric part N_0 of N.*

If a quasi-ideal Q of N is a minimal quasi-ideal of N_0, then Q is a subnear-ring of N with $Q\cap E\cap D=\{0\}$ or Q is a subnear-field of N.

Proof. Since Q is a quasi-ideal of N_0, by [3, Corollary to Theorem 1], Q is a subnear-ring of N_0. Hence Q is a subnear-ring of N.

Suppose that $Q\cap E\cap D\neq\{0\}$. Then there is a non-zero element e in $Q\cap E\cap D$. So the element e is a distributive idempotent element of the subnear-ring N_0 of N, and N_0e is an N_0-subgroup of N_0. Hence $e(N_0e)$ is a quasi-ideal of N_0 by Propostion 2.

Since Q is a quasi-ideal of N_0, by [3, Proposition 3], we have

$$e(N_0e)=N_0e\cap eN_0\subseteq N_0Q\cap QN_0\subseteq Q.$$

Moreover, $e(N_0e)$ contains the non-zero element e^3. So, from the minimality of the quasi-ideal Q, it follows that $Q=e(N_0e)$.

This implies that Q is a subnear-ring with the identity element e. So it remains to be shown that every non-zero element of Q has a left inverse element in Q.

Let n be a non-zero element of Q. Then we have $Qn=e(N_0n)$. Since N_0n is an N_0-subgroup of N_0, by Proposition 2, Qn is a quasi-ideal of N_0 and it contains the non-zero element en. Moreover, Qn is contained in Q. So, from the minimality of the quasi-ideal Q, it follows that $Qn=Q$. Consequently, there exists an element n' in Q such that $n'n=e$.

In case that N is zero-symmetric in Theorem 2, it is evident that $D=N_d$. So we have

Corollary. *If a quasi-ideal Q of a zero-symmetric near-ring N is minimal, then Q is a subnear-ring of N with $Q\cap N_d\cap E=\{0\}$ or Q is a subnear-field of N.*

5. Applications

Applying Corollary to Theorem 2, we are going to give an another proof of the following theorem in [3]:

THEOREM 3. *If a zero-symmetric near-ring N has a cancellable distributive element contained in a minimal quasi-ideal of N, then N is a near-field.*

PROOF. Suppose that the zero-symmetric near-ring N has a cancellable distributive element c contained in a minimal quasi-ideal Q of N.

Since c is distributive, by [3, Proposition 1], $cN\cap Nc$ is a quasi-ideal of N and it contains the non-zero element c^2. Moreover, by [3, Proposition 3], we have $cN\cap Nc\subseteq QN\cap NQ\subseteq Q$. So, from the minimality of the quasi-ideal Q, it follows that $Q=cN\cap Nc$.

Since c^2 is distributive, similary we have $Q=c^2N\cap Nc^2$. This implies that the element c has the form

$$c=c^2n=mc^2 \quad (n,\ m\in N),$$

whence $cn=(mc^2)n=mc$.

Set $e=cn$, then e is contained in $cN\cap Nc=Q$, and $e\neq 0$, since $ce=c\neq 0$. Moreover, we have

$$e^2=(mc)(cn)=m(c^2n)=mc=e.$$

Furthermore, the element e is distributive. In fact, for all elements n_1, n_2 of N, we have

$$ce(n_1+n_2)=c(n_1+n_2)=cn_1+cn_2=cen_1+cen_2,$$

that is, $ce(n_1+n_2)=c(en_1+en_2)$. This and the cancellability of the element c imply that $e(n_1+n_2)=en_1+en_2$.

Thus the minimal quasi-ideal Q has the non-zero distributive idempotent element e. So, by Corollary to Theorem 2, Q is a subnear-field with the identity element e.

The element e is the identity element of N, too. In fact, multiplying both sides of the equality $ec=c$ by any element x of N, we have $xec=xc$, whence $xe=x$. Dually we have $ex=x$ from $ce=c$.

Since e is contained in Q, for any element x of N, we have

$$x=ex=xe\in QN\cap NQ\subseteq Q,$$

that is, $N=Q$. Thus N is a near-field.

A non-zero N-subgroup S of a zero-symmetric near-ring N is called *minimal* if S does not properly contain any non-zero N-subgroup of N.

We now have the following result which is an extension of [**2**, Satz 5] to zero-symmetric near-rings:

THEOREM 4. *If a minimal N-subgroup S of a zero-symmetric near-ring N has a non-zero distributive idempotent element e of N, then eS is a subnear-field of N, moreover it is a minimal quasi-ideal of N.*

PROOF. By Proposition 2, eS is a quasi-ideal of N. Because of Theorem 1 and [**3**, Corollary to Theorem 1], all we have to prove is that the non-zero elements of eS form a multiplicative subgroup of N.

Evidently, e is a left identity element of eS. Let es be a non-zero element of eS. Then $S(es)$ is a non-zero N-subgroup of N contained in S. By the minimality of S, we have $S(es)=S$. Hence $(eS)(es)=eS$. This implies the existence of a non-zero element et of eS such that $(et)(es)=e$. Thus the non-zero elements of eS form a multiplicative subgroup of N.

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