Quasi-ideals which are subnear-fields

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Quasi-ideals which are subnear-fields

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1. Introduction

In ring theory the following results are well-known (see [2, Satz 3; Satz 4]):

(A) If a quasi-ideal $Q$ of a ring $R$ is a division subring of $R$, then $Q$ is a minimal quasi-ideal of $R$.

(B) A minimal quasi-ideal $Q$ of a ring $R$ is either a zero subring of $R$ or a division subring of $R$.

In this note we shall extend these results to near-rings and show some applications of extended results.

2. Preliminaries

By a near-ring we mean a non-empty set $N$ in which an addition $+$ and a multiplication $\cdot$ are defined such that

1. $(N, +)$ is a group,
2. $(N, \cdot)$ is a semigroup,
3. $(n+n')n''=nn''+n'n''$ $(n, n', n'' \in N)$.

In dealing with general near-ring the neutral element of $(N, +)$ will be denoted by $0$.

In this section, $N$ will denote a near-ring. The set $N_0$ of all elements $n$ of $N$ with $n0=0$ is called the zero-symmetric part of $N$; $N$ is called zero-symmetric if $N=N_0$. An element $d$ of $N$ is called distributive if $d(n+n')=dn+dn'$ for all elements $n, n'$ of $N$. The set of all distributive elements of $N$ will be denoted by $N_d$.

Let $A$ and $B$ be two non-empty subsets of $N$. We shall define two kinds of products $AB$ and $A*B$: $AB$ denotes the set of all finite sums of
the form
\[ \sum a_i b_i \quad (a_i \in A, b_i \in B) ; \]

\( A*B \) denotes the set of all finite sums of the form
\[ \sum(a_i(a_i + b_i) - a_i a_i) \quad (a_i, a_i' \in A, b_i \in B). \]

A subgroup \( S \) of \((N, +)\) is called an \textit{N-subgroup} of \( N \) if \( NS \subseteq S \). A subgroup \( M \) of \((N, +)\) is called a \textit{subnear-ring} of \( N \) if \( MM \subseteq M \). For instance, \( Nn \) is an \( N \)-subgroup of \( N \) for every element \( n \) of \( N \). The zero-symmetric part \( N_0 \) of \( N \) is a subnear-ring of \( N \).

A subgroup \( Q \) of \((N, +)\) is called a \textit{quasi-ideal} of \( N \) if \( QN \cap NQ \cap N*Q \subseteq Q \). For instance, every \( N \)-subgroup of \( N \), \( dN \) with a distributive element \( d \) of \( N \) and the zero-symmetric part \( N_0 \) of \( N \) are quasi-ideals of \( N \). Clearly \( \{0\} \) and \( N \) are quasi-ideals of \( N \). If \( N \) has no quasi-ideals except \( \{0\} \) and \( N \), we say that \( N \) is \( Q \)-simple.

A near-ring \( N \) is called a \textit{near-field} if it has at least two elements and its non-zero elements form a group with respect to the multiplication defined in \( N \).

Let \( \mathbb{Z}_2 \) be the integers modulo 2. Then \((\mathbb{Z}_2, +)\) with \( 0 \cdot 0 = 0 \cdot 1 = 0 \), \( 1 \cdot 0 = 1 \cdot 1 = 1 \) is a near-field. As usual, throughout this note, we will exclude those near-fields which are isomorphic to this near-field. So every near-field is zero-symmetric and \( Q \)-simple (see [1, p. 249 and 3, Theorem 2]).

3. Quasi-ideals which are subnear-fields

A non-zero quasi-ideal \( Q \) of a near-ring \( N \) is called \textit{minimal} if \( Q \) does not properly contain any non-zero quasi-ideal of \( N \).

We have

\[ \text{THEOREM 1.} \quad \text{If a quasi-ideal } Q \text{ of a near-ring } N \text{ is a subnear-field of } N, \text{ then } Q \text{ is a minimal quasi-ideal of the zero-symmetric part } N_0 \text{ of } N. \]

\[ \text{PROOF.} \quad \text{Since a near-field is zero-symmetric, the quasi-ideal } Q \text{ is contained in } N_0. \text{ By [3, Proposition 2], the relation } Q = Q \cap N_0 \text{ implies that } Q \text{ is a quasi-ideal of } N_0. \]
Let \( Q' \) be a quasi-ideal of \( N_0 \) such that \( \{0\} \neq Q' \subseteq Q \).
Then we have

\[
Q'Q \cap QQ' \cap Q^*Q' \subseteq Q'N_0 \cap N_0Q' \cap N_0^*Q' \subseteq Q',
\]

which implies that \( Q' \) is a quasi-ideal of \( Q \). Since a near-field is \( Q \)-simple, we have \( Q' = Q \). Thus \( Q' \) is a minimal quasi-ideal of \( N_0 \).

In Theorem 1, the quasi-ideal \( Q \) is a minimal quasi-ideal of \( N \), too. In fact, we have

**Proposition 1.** If a quasi-ideal \( Q \) of a near-ring \( N \) is a minimal quasi-ideal of \( N_0 \), then \( Q \) is a minimal quasi-ideal of \( N \).

**Proof.** Let \( Q' \) be a quasi-ideal of \( N \) such that \( \{0\} \neq Q' \subseteq Q \). Then \( Q' \) is contained in \( N_0 \) and we have

\[
Q'N_0 \cap N_0Q' \cap N_0^*Q' \subseteq Q'N_0 \cap N_0Q' \cap N_0^*Q' \subseteq Q',
\]

which implies that \( Q' \) is a quasi-ideal of \( N_0 \). Since \( Q \) is a minimal quasi-ideal of \( N_0 \), we have \( Q' = Q \). Thus \( Q \) is a minimal quasi-ideal of \( N \).

**4. Minimal quasi-ideals**

In view of Theorem 1, we are going to consider those quasi-ideals which are minimal in the zero-symmetric part. We start with

**Proposition 2.** Let \( e \) be a distributive idempotent element of a near-ring \( N \) and \( S \) an \( N \)-subgroup of \( N \). Then \( eS \) is a quasi-ideal of \( N \) such that \( eS = S \cap eN \).

**Proof.** Since \( S \) is an \( N \)-subgroup of \( N \), we have \( eS \subseteq NS \subseteq S \). On the other hand, we have \( eS \subseteq eN \). Hence \( eS \subseteq S \cap eN \).

Conversely, any element \( a \) of \( S \cap eN \) has the form

\[
a = s = en \quad (s \in S, \ n \in N),
\]

whence \( a = en = een = es \in eS \).

Since \( S \) and \( eN \) are quasi-ideals of \( N \), by [3, Proposition 1], the re-
lation \( eS = S \cap eN \) implies that \( eS \) is a quasi-ideal of \( N \).

We now have

**Theorem 2.** Let \( E \) be the set of all idempotent elements of a near-ring \( N \) and \( D \) the set of all elements \( d \) of \( N \) such that \( d(n + n') = dn + dn' \) for all elements \( n, n' \) of the zero-symmetric part \( N_0 \) of \( N \).

If a quasi-ideal \( Q \) of \( N \) is a minimal quasi-ideal of \( N_0 \), then \( Q \) is a subnear-ring of \( N \) with \( Q \cap E \cap D = \{0\} \) or \( Q \) is a subnear-field of \( N \).

**Proof.** Since \( Q \) is a quasi-ideal of \( N_0 \), by [3, Corollary to Theorem 1], \( Q \) is a subnear-ring of \( N_0 \). Hence \( Q \) is a subnear-ring of \( N \).

Suppose that \( Q \cap E \cap D \neq \{0\} \). Then there is a non-zero element \( e \) in \( Q \cap E \cap D \). So the element \( e \) is a distributive idempotent element of the subnear-ring \( N_0 \) of \( N \), and \( N_0 e \) is an \( N_0 \)-subgroup of \( N_0 \). Hence \( e(N_0 e) \) is a quasi-ideal of \( N_0 \) by Proposition 2.

Since \( Q \) is a quasi-ideal of \( N_0 \), by [3, Proposition 3], we have

\[
e(N_0 e) = N_0 e \cap eN_0 \subseteq N_0 Q \cap QN_0 \subseteq Q.
\]

Moreover, \( e(N_0 e) \) contains the non-zero element \( e^4 \). So, from the minimality of the quasi-ideal \( Q \), it follows that \( Q = e(N_0 e) \).

This implies that \( Q \) is a subnear-ring with the identity element \( e \). So it remains to be shown that every non-zero element of \( Q \) has a left inverse element in \( Q \).

Let \( n \) be a non-zero element of \( Q \). Then we have \( Qn = e(N_0 n) \). Since \( N_0 n \) is an \( N_0 \)-subgroup of \( N_0 \), by Proposition 2, \( Qn \) is a quasi-ideal of \( N_0 \) and it contains the non-zero element \( en \). Moreover, \( Qn \) is contained in \( Q \). So, from the minimality of the quasi-ideal \( Q \), it follows that \( Qn = Q \). Consequently, there exists an element \( n' \) in \( Q \) such that \( n'n = e \).

In case that \( N \) is zero-symmetric in Theorem 2, it is evident that \( D = N_0 \). So we have

**Corollary.** If a quasi-ideal \( Q \) of a zero-symmetric near-ring \( N \) is minimal, then \( Q \) is a subnear-ring of \( N \) with \( Q \cap N_0 \cap E = \{0\} \) or \( Q \) is a subnear-field of \( N \).
5. Applications

Applying Corollary to Theorem 2, we are going to give an another proof of the following theorem in [3]:

**Theorem 3.** If a zero-symmetric near-ring $N$ has a cancellable distributive element contained in a minimal quasi-ideal of $N$, then $N$ is a near-field.

**Proof.** Suppose that the zero-symmetric near-ring $N$ has a cancellable distributive element $c$ contained in a minimal quasi-ideal $Q$ of $N$.

Since $c$ is distributive, by [3, Proposition 1], $cN \cap Nc$ is a quasi-ideal of $N$ and it contains the non-zero element $c^2$. Moreover, by [3, Proposition 3], we have $cN \cap Nc \subseteq Q \cap NQ \subseteq Q$. So, from the minimality of the quasi-ideal $Q$, it follows that $Q = cN \cap Nc$.

Since $c^2$ is distributive, similarly we have $Q = c^2N \cap Nc^2$. This implies that the element $c$ has the form

$$c = c^2n = mc^2 \quad (n, m \in N),$$

whence $cn = (mc^2)n = mc$.

Set $e = cn$, then $e$ is contained in $cN \cap Nc = Q$, and $e \neq 0$, since $ce = c \neq 0$. Moreover, we have

$$e^2 = (mc)(cn) = m(c^2n) = mc = e.$$  

Furthermore, the element $e$ is distributive. In fact, for all elements $n_1, n_2$ of $N$, we have

$$ce(n_1 + n_2) = c(n_1 + n_2) = cn_1 + cn_2 = cen_1 + cen_2,$$

that is, $ce(n_1 + n_2) = c(en_1 + en_2)$. This and the cancellability of the element $c$ imply that $e(n_1 + n_2) = en_1 + en_2$.

Thus the minimal quasi-ideal $Q$ has the non-zero distributive idempotent element $e$. So, by Corollary to Theorem 2, $Q$ is a subnear-field with the identity element $e$.

The element $e$ is the identity element of $N$, too. In fact, multiplying both sides of the equality $ce = c$ by any element $x$ of $N$, we have $xce = xc$, whence $xe = x$. Dually we have $ex = x$ from $ce = c$.

Since $e$ is contained in $Q$, for any element $x$ of $N$, we have

$$x = ex = xe \in QN \cap NQ \subseteq Q.$$
that is, \( N = Q \). Thus \( N \) is a near-field.

A non-zero \( N \)-subgroup \( S \) of a zero-symmetric near-ring \( N \) is called minimal if \( S \) does not properly contain any non-zero \( N \)-subgroup of \( N \).

We now have the following result which is an extension of [2, Satz 5] to zero-symmetric near-rings:

**Theorem 4.** If a minimal \( N \)-subgroup \( S \) of a zero-symmetric near-ring \( N \) has a non-zero distributive idempotent element \( e \) of \( N \), then \( eS \) is a subnear-field of \( N \), moreover it is a minimal quasi-ideal of \( N \).

**Proof.** By Proposition 2, \( eS \) is a quasi-ideal of \( N \). Because of Theorem 1 and [3, Corollary to Theorem 1], all we have to prove is that the non-zero elements of \( eS \) form a multiplicative subgroup of \( N \).

Evidently, \( e \) is a left identity element of \( eS \). Let \( es \) be a non-zero element of \( eS \). Then \( S(es) \) is a non-zero \( N \)-subgroup of \( N \) contained in \( S \). By the minimality of \( S \), we have \( S(es) = S \). Hence \( (eS)(es) = eS \). This implies the existence of a non-zero element \( et \) of \( eS \) such that \( (et)(es) = e \). Thus the non-zero elements of \( eS \) form a multiplicative subgroup of \( N \).

**References**