Quasi-ideals which are subnear-fields

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1. Introduction

In ring theory the following results are well-known (see [2, Satz 3; Satz 4]):

(A) If a quasi-ideal $Q$ of a ring $R$ is a division subring of $R$, then $Q$ is a minimal quasi-ideal of $R$.

(B) A minimal quasi-ideal $Q$ of a ring $R$ is either a zero subring of $R$ or a division subring of $R$.

In this note we shall extend these results to near-rings and show some applications of extended results.

2. Preliminaries

By a near-ring we mean a non-empty set $N$ in which an addition $+$ and a multiplication $\cdot$ are defined such that

1. $(N, +)$ is a group,
2. $(N, \cdot)$ is a semigroup,
3. $(n+n')n''=nn''+n'n''$ $(n, n', n'' \in N)$.

In dealing with general near-ring the neutral element of $(N, +)$ will be denoted by 0.

In this section, $N$ will denote a near-ring. The set $N_0$ of all elements $n$ of $N$ with $n0 = 0$ is called the zero-symmetric part of $N$; $N$ is called zero-symmetric if $N=N_0$. An element $d$ of $N$ is called distributive if $d(n+n')=dn+dn'$ for all elements $n, n'$ of $N$. The set of all distributive elements of $N$ will be denoted by $N_d$.

Let $A$ and $B$ be two non-empty subsets of $N$. We shall define two kinds of products $AB$ and $A*B$: $AB$ denotes the set of all finite sums of
the form
\[ \sum a_i b_i \ (a_i \in A, b_i \in B); \]

\(A^*B\) denotes the set of all finite sums of the form
\[ \sum(a_i (a'_i + b_i) - a_i a'_i) \ (a_i, a'_i \in A, b_i \in B). \]

A subgroup \(S\) of \((N, +)\) is called an \(N\)-subgroup of \(N\) if \(NS \subseteq S\). A subgroup \(M\) of \((N, +)\) is called a subnear-ring of \(N\) if \(MM \subseteq M\). For instance, \(Nn\) is an \(N\)-subgroup of \(N\) for every element \(n\) of \(N\). The zero-symmetric part \(N_0\) of \(N\) is a subnear-ring of \(N\).

A subgroup \(Q\) of \((N, +)\) is called a quasi-ideal of \(N\) if \(QN \cap NQ \cap N^*Q \subseteq Q\). For instance, every \(N\)-subgroup of \(N\), \(dN\) with a distributive element \(d\) of \(N\) and the zero-symmetric part \(N_0\) of \(N\) are quasi-ideals of \(N\). Clearly \(\{0\}\) and \(N\) are quasi-ideals of \(N\). If \(N\) has no quasi-ideals except \(\{0\}\) and \(N\), we say that \(N\) is \(Q\)-simple.

A near-ring \(N\) is called a near-field if it has at least two elements and its non-zero elements form a group with respect to the multiplication defined in \(N\).

Let \(Z_2\) be the integers modulo 2. Then \((Z_2, +)\) with \(0 \cdot 0 = 0 \cdot 1 = 0, 1 \cdot 0 = 1 \cdot 1 = 1\) is a near-field. As usual, throughout this note, we will exclude those near-fields which are isomorphic to this near-field. So every near-field is zero-symmetric and \(Q\)-simple (see \([1, p. 249 and 3, Theorem 2]\)).

### 3. Quasi-ideals which are subnear-fields

A non-zero quasi-ideal \(Q\) of a near-ring \(N\) is called minimal if \(Q\) does not properly contain any non-zero quasi-ideal of \(N\).

We have

**Theorem 1.** If a quasi-ideal \(Q\) of a near-ring \(N\) is a subnear-field of \(N\), then \(Q\) is a minimal quasi-ideal of the zero-symmetric part \(N_0\) of \(N\).

**Proof.** Since a near-field is zero-symmetric, the quasi-ideal \(Q\) is contained in \(N_0\). By \([3, Proposition 2]\), the relation \(Q = Q \cap N_0\) implies that \(Q\) is a quasi-ideal of \(N_0\).
Let $Q'$ be a quasi-ideal of $N_0$ such that $(0) \not= Q' \subseteq Q$. Then we have
\[ Q'Q \cap QQ' \cap Q^*Q' \subseteq Q'N_0 \cap Q^*Q' \subseteq Q', \]
which implies that $Q'$ is a quasi-ideal of $Q$. Since a near-field is $Q$-simple, we have $Q' = Q$. Thus $Q'$ is a minimal quasi-ideal of $N_0$.

In Theorem 1, the quasi-ideal $Q$ is a minimal quasi-ideal of $N$, too. In fact, we have

**Proposition 1.** If a quasi-ideal $Q$ of a near-ring $N$ is a minimal quasi-ideal of $N_0$, then $Q$ is a minimal quasi-ideal of $N$.

**Proof.** Let $Q'$ be a quasi-ideal of $N$ such that $(0) \not= Q' \subseteq Q$. Then $Q'$ is contained in $N_0$ and we have
\[ Q'N_0 \cap N_0Q' \cap N_0^*Q' \subseteq Q'N \cap NQ' \cap N^*Q' \subseteq Q', \]
which implies that $Q'$ is a quasi-ideal of $N_0$. Since $Q$ is a minimal quasi-ideal of $N_0$, we have $Q' = Q$. Thus $Q$ is a minimal quasi-ideal of $N$.

4. Minimal quasi-ideals

In view of Theorem 1, we are going to consider those quasi-ideals which are minimal in the zero-symmetric part. We start with

**Proposition 2.** Let $e$ be a distributive idempotent element of a near-ring $N$ and $S$ an $N$-subgroup of $N$. Then $eS$ is a quasi-ideal of $N$ such that $eS = S \cap eN$.

**Proof.** Since $S$ is an $N$-subgroup of $N$, we have $eS \subseteq NS \subseteq S$. On the other hand, we have $eS \subseteq eN$. Hence $eS \subseteq S \cap eN$.

Conversely, any element $a$ of $S \cap eN$ has the form
\[ a = s = en \quad (s \in S, \, n \in N), \]
whence $a = en = een = es \subseteq eS$.

Since $S$ and $eN$ are quasi-ideals of $N$, by [3, Proposition 1], the re-
lation $eS=S\cap eN$ implies that $eS$ is a quasi-ideal of $N$.

We now have

**Theorem 2.** Let $E$ be the set of all idempotent elements of a near-ring $N$ and $D$ the set of all elements $d$ of $N$ such that $d(n+n')=dn+dn'$ for all elements $n$, $n'$ of the zero-symmetric part $N_0$ of $N$.

If a quasi-ideal $Q$ of $N$ is a minimal quasi-ideal of $N_0$, then $Q$ is a subnear-ring of $N$ with $Q\cap E \cap D=\{0\}$ or $Q$ is a subnear-field of $N$.

**Proof.** Since $Q$ is a quasi-ideal of $N_0$, by [3, Corollary to Theorem 1], $Q$ is a subnear-ring of $N_0$. Hence $Q$ is a subnear-ring of $N$.

Suppose that $Q\cap E \cap D=\{0\}$. Then there is a non-zero element $e$ in $Q\cap E \cap D$. So the element $e$ is a distributive idempotent element of the subnear-ring $N_0$ of $N$, and $N_0e$ is an $N_0$-subgroup of $N_0$. Hence $e(N_0e)$ is a quasi-ideal of $N_0$ by Proposition 2.

Since $Q$ is a quasi-ideal of $N_0$, by [3, Proposition 3], we have

\[ e(N_0e)=N_0e\cap eN_0 \subseteq N_0Q \cap QN_0 \subseteq Q. \]

Moreover, $e(N_0e)$ contains the non-zero element $e^2$. So, from the minimality of the quasi-ideal $Q$, it follows that $Q=e(N_0e)$.

This implies that $Q$ is a subnear-ring with the identity element $e$. So it remains to be shown that every non-zero element of $Q$ has a left inverse element in $Q$.

Let $n$ be a non-zero element of $Q$. Then we have $Qn=e(N_0n)$. Since $N_0n$ is an $N_0$-subgroup of $N_0$, by Proposition 2, $Qn$ is a quasi-ideal of $N_0$ and it contains the non-zero element $en$. Moreover, $Qn$ is contained in $Q$. So, from the minimality of the quasi-ideal $Q$, it follows that $Qn=Q$. Consequently, there exists an element $n'$ in $Q$ such that $n'n=e$.

In case that $N$ is zero-symmetric in Theorem 2, it is evident that $D=N_0$. So we have

**Corollary.** If a quasi-ideal $Q$ of a zero-symmetric near-ring $N$ is minimal, then $Q$ is a subnear-ring of $N$ with $Q\cap N_0 \cap E=\{0\}$ or $Q$ is a subnear-field of $N$. 
5. Applications

Applying Corollary to Theorem 2, we are going to give an another
proof of the following theorem in [3]:

**Theorem 3.** If a zero-symmetric near-ring \( N \) has a cancellable distributive
element contained in a minimal quasi-ideal of \( N \), then \( N \) is a near-field.

**Proof.** Suppose that the zero-symmetric near-ring \( N \) has a cancel-
lable distributive element \( c \) contained in a minimal quasi-ideal \( Q \) of \( N \).

Since \( c \) is distributive, by [3, Proposition 1], \( cN \cap Nc \) is a quasi-ideal
of \( N \) and it contains the non-zero element \( c^2 \). Moreover, by [3, Propo-
sition 3], we have \( cN \cap Nc \subseteqQN \cap NQ \subseteq Q \). So, from the minimality of the
quasi-ideal \( Q \), it follows that \( Q=cN \cap Nc \).

Since \( c^2 \) is distributive, similarly we have \( Q=c^2N \cap Nc^2 \). This implies
that the element \( c \) has the form

\[
c=c^2n=mc^2 \quad (n, m \in N),
\]

whence \( cm=(mc)Nn=mc \).

Set \( e=cn \), then \( e \) is contained in \( cN \cap Nc \), and \( e \neq 0 \), since \( ce=c \neq 0 \). Moreover, we have

\[
e^2=(mc)(cn)=m(c^2n)=mc=e.
\]

Furthermore, the element \( e \) is distributive. In fact, for all elements
\( n_1, n_2 \) of \( N \), we have

\[
ce(n_1+n_2)=c(n_1+n_2)=cn_1+cn_2=cen_1+cen_2,
\]

that is, \( ce(n_1+n_2)=c(en_1+en_2) \). This and the cancellability of the
element \( c \) imply that \( e(n_1+n_2)=en_1+en_2 \).

Thus the minimal quasi-ideal \( Q \) has the non-zero distributive idem-
potent element \( e \). So, by Corollary to Theorem 2, \( Q \) is a subnear-field
with the identity element \( e \).

The element \( e \) is the identity element of \( N \), too. In fact, multiplying both sides of the equality \( ec=c \) by any element \( x \) of \( N \), we have

\[
xec=x, \text{ whence } xe=x.
\]

Dually we have \( ex=x \) from \( ce=c \).

Since \( e \) is contained in \( Q \), for any element \( x \) of \( N \), we have

\[
x=ex=xe \in QN \cap NQ \subseteq Q.
\]
that is, \( N = Q \). Thus \( N \) is a near-field.

A non-zero \( N \)-subgroup \( S \) of a zero-symmetric near-ring \( N \) is called \textit{minimal} if \( S \) does not properly contain any non-zero \( N \)-subgroup of \( N \).

We now have the following result which is an extension of \([2, \text{Satz 5}]\) to zero-symmetric near-rings:

\textbf{THEOREM 4.} If a minimal \( N \)-subgroup \( S \) of a zero-symmetric near-ring \( N \) has a non-zero distributive idempotent element \( e \) of \( N \), then \( eS \) is a subnear-field of \( N \), moreover it is a minimal quasi-ideal of \( N \).

\textbf{Proof.} By Proposition 2, \( eS \) is a quasi-ideal of \( N \). Because of Theorem 1 and \([3, \text{Corollary to Theorem 1}]\), all we have to prove is that the non-zero elements of \( eS \) form a multiplicative subgroup of \( N \).

Evidently, \( e \) is a left identity element of \( eS \). Let \( es \) be a non-zero element of \( eS \). Then \( S(es) \) is a non-zero \( N \)-subgroup of \( N \) contained in \( S \). By the minimality of \( S \), we have \( S(es) = S \). Hence \((eS)(es) = eS \). This implies the existence of a non-zero element \( et \) of \( eS \) such that \((et)(es) = e \). Thus the non-zero elements of \( eS \) form a multiplicative subgroup of \( N \).

\textbf{References}


