Quasi-ideals which are subnear-fields

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1. Introduction

In ring theory the following results are well-known (see [2, Satz 3; Satz 4]):

(A) If a quasi-ideal Q of a ring R is a division subring of R, then Q is a minimal quasi-ideal of R.

(B) A minimal quasi-ideal Q of a ring R is either a zero subring of R or a division subring of R.

In this note we shall extend these results to near-rings and show some applications of extended results.

2. Preliminaries

By a near-ring we mean a non-empty set N in which an addition + and a multiplication • are defined such that

1. \((N, +)\) is a group,
2. \((N, \cdot)\) is a semigroup,
3. \((n+n')n'' = nn'' + n'n'' \ (n, n', n'' \in N)\).

In dealing with general near-ring the neutral element of \((N, +)\) will be denoted by 0.

In this section, \(N\) will denote a near-ring. The set \(N_s\) of all elements \(n\) of \(N\) with \(n0 = 0\) is called the zero-symmetric part of \(N\); \(N\) is called zero-symmetric if \(N = N_s\). An element \(d\) of \(N\) is called distributive if \(d(n+n') = dn + dn'\) for all elements \(n, n'\) of \(N\). The set of all distributive elements of \(N\) will be denoted by \(N_d\).

Let \(A\) and \(B\) be two non-empty subsets of \(N\). We shall define two kinds of products \(AB\) and \(A*B\): \(AB\) denotes the set of all finite sums of
the form

$$\sum a_i b_i \ (a_i \in A, b_i \in B) ;$$

$A*B$ denotes the set of all finite sums of the form

$$\sum (a_i (a_i' + b_i) - a_i a_i') \ (a_i, a_i' \in A, b_i \in B).$$

A subgroup $S$ of $(N, +)$ is called an $N$-subgroup of $N$ if $NS \subseteq S$. A subgroup $M$ of $(N, +)$ is called a subnear-ring of $N$ if $MM \subseteq M$. For instance, $Nn$ is an $N$-subgroup of $N$ for every element $n$ of $N$. The zero-symmetric part $N_0$ of $N$ is a subnear-ring of $N$.

A subgroup $Q$ of $(N, +)$ is called a quasi-ideal of $N$ if $QN \cap NQ \cap N^*Q \subseteq Q$. For instance, every $N$-subgroup of $N$, $dN$ with a distributive element $d$ of $N$ and the zero-symmetric part $N_0$ of $N$ are quasi-ideals of $N$. Clearly $\{0\}$ and $N$ are quasi-ideals of $N$. If $N$ has no quasi-ideals except $\{0\}$ and $N$, we say that $N$ is $Q$-simple.

A near-ring $N$ is called a near-field if it has at least two elements and its non-zero elements form a group with respect to the multiplication defined in $N$.

Let $Z_2$ be the integers modulo 2. Then $(Z_2, +)$ with $0 \cdot 0 = 0 \cdot 1 = 0$, $1 \cdot 0 = 1 \cdot 1 = 1$ is a near-field. As usual, throughout this note, we will exclude those near-fields which are isomorphic to this near-field. So every near-field is zero-symmetric and $Q$-simple (see [1, p. 249 and 3, Theorem 2]).

3. Quasi-ideals which are subnear-fields

A non-zero quasi-ideal $Q$ of a near-ring $N$ is called minimal if $Q$ does not properly contain any non-zero quasi-ideal of $N$.

We have

**Theorem 1.** If a quasi-ideal $Q$ of a near-ring $N$ is a subnear-field of $N$, then $Q$ is a minimal quasi-ideal of the zero-symmetric part $N_0$ of $N$.

**Proof.** Since a near-field is zero-symmetric, the quasi-ideal $Q$ is contained in $N_0$. By [3, Proposition 2], the relation $Q = Q \cap N_0$ implies that $Q$ is a quasi-ideal of $N_0$. 
Let $Q'$ be a quasi-ideal of $N_0$ such that $(0) \neq Q' \subseteq Q$. Then we have

$$Q'Q \cap QQ' \cap Q*Q' \subseteq Q'N_0 \cap N_0Q' \cap N_0*Q' \subseteq Q',$$

which implies that $Q'$ is a quasi-ideal of $Q$. Since a near-field is $Q$-simple, we have $Q' = Q$. Thus $Q'$ is a minimal quasi-ideal of $N_0$.

In Theorem 1, the quasi-ideal $Q$ is a minimal quasi-ideal of $N$, too. In fact, we have

**Proposition 1.** If a quasi-ideal $Q$ of a near-ring $N$ is a minimal quasi-ideal of $N_0$, then $Q$ is a minimal quasi-ideal of $N$.

**Proof.** Let $Q'$ be a quasi-ideal of $N$ such that $(0) \neq Q' \subseteq Q$. Then $Q'$ is contained in $N_0$ and we have

$$Q'N_0 \cap N_0Q' \cap N_0*Q' \subseteq Q'N \cap NQ' \cap N*Q' \subseteq Q',$$

which implies that $Q'$ is a quasi-ideal of $N_0$. Since $Q$ is a minimal quasi-ideal of $N_0$, we have $Q' = Q$. Thus $Q$ is a minimal quasi-ideal of $N$.

4. Minimal quasi-ideals

In view of Theorem 1, we are going to consider those quasi-ideals which are minimal in the zero-symmetric part. We start with

**Proposition 2.** Let $e$ be a distributive idempotent element of a near-ring $N$ and $S$ an $N$-subgroup of $N$. Then $eS$ is a quasi-ideal of $N$ such that $eS = S \cap eN$.

**Proof.** Since $S$ is an $N$-subgroup of $N$, we have $eS \subseteq NS \subseteq S$. On the other hand, we have $eS \subseteq eN$. Hence $eS \subseteq S \cap eN$.

Conversely, any element $a$ of $S \cap eN$ has the form

$$a = s = en \quad (s \in S, \ n \in N),$$

whence $a = en = een = es \in eS$.

Since $S$ and $eN$ are quasi-ideals of $N$, by [3, Proposition 1], the re-
lation $eS=S\cap eN$ implies that $eS$ is a quasi-ideal of $N$.

We now have

**Theorem 2.** Let $E$ be the set of all idempotent elements of a near-ring $N$ and $D$ the set of all elements $d$ of $N$ such that $d(n+n')=dn+dn'$ for all elements $n, n'$ of the zero-symmetric part $N_0$ of $N$.

If a quasi-ideal $Q$ of $N$ is a minimal quasi-ideal of $N_0$, then $Q$ is a subnear-ring of $N$ with $Q\cap E\cap D=\{0\}$ or $Q$ is a subnear-field of $N$.

**Proof.** Since $Q$ is a quasi-ideal of $N_0$, by [3, Corollary to Theorem 1], $Q$ is a subnear-ring of $N_0$. Hence $Q$ is a subnear-ring of $N$.

Suppose that $Q\cap E\cap D\neq\{0\}$. Then there is a non-zero element $e$ in $Q\cap E\cap D$. So the element $e$ is a distributive idempotent element of the subnear-ring $N_e$ of $N$, and $N_e$ is an $N_e$-subgroup of $N_e$. Hence $e(N_e)$ is a quasi-ideal of $N_0$ by Proposition 2.

Since $Q$ is a quasi-ideal of $N_0$, by [3, Proposition 3], we have

$$e(N_e) = N_e \cap eN_e \subseteq N_0 \cap QN_e \subseteq Q.$$  

Moreover, $e(N_e)$ contains the non-zero element $e^4$. So, from the minimality of the quasi-ideal $Q$, it follows that $Q=e(N_e)$.

This implies that $Q$ is a subnear-ring with the identity element $e$. So it remains to be shown that every non-zero element of $Q$ has a left inverse element in $Q$.

Let $n$ be a non-zero element of $Q$. Then we have $Qn=e(N_{n^t})$. Since $N_{n^t}$ is an $N_e$-subgroup of $N_e$, by Proposition 2, $Qn$ is a quasi-ideal of $N_e$ and it contains the non-zero element $en$. Moreover, $Qn$ is contained in $Q$. So, from the minimality of the quasi-ideal $Q$, it follows that $Qn=Q$. Consequently, there exists an element $n'$ in $Q$ such that $n'n=e$.

In case that $N$ is zero-symmetric in Theorem 2, it is evident that $D=N_e$. So we have

**Corollary.** If a quasi-ideal $Q$ of a zero-symmetric near-ring $N$ is minimal, then $Q$ is a subnear-ring of $N$ with $Q\cap N_0 \cap E=\{0\}$ or $Q$ is a subnear-field of $N$.  

5. Applications

Applying Corollary to Theorem 2, we are going to give another proof of the following theorem in [3]:

**Theorem 3.** If a zero-symmetric near-ring $N$ has a cancellable distributive element contained in a minimal quasi-ideal of $N$, then $N$ is a near-field.

**Proof.** Suppose that the zero-symmetric near-ring $N$ has a cancellable distributive element $c$ contained in a minimal quasi-ideal $Q$ of $N$.

Since $c$ is distributive, by [3, Proposition 1], $cN \cap Nc$ is a quasi-ideal of $N$ and it contains the non-zero element $c^2$. Moreover, by [3, Proposition 3], we have $cN \cap Nc \subseteq QN \cap NQ \subseteq Q$. So, from the minimality of the quasi-ideal $Q$, it follows that $Q = cN \cap Nc$.

Since $c^2$ is distributive, similarly we have $Q = c^2N \cap Nc^2$. This implies that the element $c$ has the form

$$c = c^2n = mc^2 \quad (n, m \in N),$$

whence $cn = (mc^2)n = mc$.

Set $e = cn$, then $e$ is contained in $cN \cap Nc = Q$, and $e \neq 0$, since $ce = c \neq 0$. Moreover, we have

$$e^2 = (mc) (cn) = m(c^2n) = me = e.$$

Furthermore, the element $e$ is distributive. In fact, for all elements $n_1, n_2$ of $N$, we have

$$ce(n_1 + n_2) = c(n_1 + n_2) = cn_1 + cn_2 = cen_1 + cen_2,$$

that is, $ce(n_1 + n_2) = c(en_1 + en_2)$. This and the cancellability of the element $c$ imply that $e(n_1 + n_2) = en_1 + en_2$.

Thus the minimal quasi-ideal $Q$ has the non-zero distributive idempotent element $e$. So, by Corollary to Theorem 2, $Q$ is a subnear-field with the identity element $e$.

The element $e$ is the identity element of $N$, too. In fact, multiplying both sides of the equality $ce = c$ by any element $x$ of $N$, we have $xec = xc$, whence $xe = x$. Dually we have $ex = x$ from $ce = c$.

Since $e$ is contained in $Q$, for any element $x$ of $N$, we have

$$x = ex = xe \in QN \cap NQ \subseteq Q,$$
that is, $N=Q$. Thus $N$ is a near-field.

A non-zero $N$-subgroup $S$ of a zero-symmetric near-ring $N$ is called **minimal** if $S$ does not properly contain any non-zero $N$-subgroup of $N$.

We now have the following result which is an extension of [2, Satz 5] to zero-symmetric near-rings:

**Theorem 4.** If a minimal $N$-subgroup $S$ of a zero-symmetric near-ring $N$ has a non-zero distributive idempotent element $e$ of $N$, then $eS$ is a subnear-field of $N$, moreover it is a minimal quasi-ideal of $N$.

**Proof.** By Proposition 2, $eS$ is a quasi-ideal of $N$. Because of Theorem 1 and [3, Corollary to Theorem 1], all we have to prove is that the non-zero elements of $eS$ form a multiplicative subgroup of $N$.

Evidently, $e$ is a left identity element of $eS$. Let $es$ be a non-zero element of $eS$. Then $S(es)$ is a non-zero $N$-subgroup of $N$ contained in $S$. By the minimality of $S$, we have $S(es)=S$. Hence $(eS)(es)=eS$. This implies the existence of a non-zero element $et$ of $eS$ such that $(et)(es)=e$. Thus the non-zero elements of $eS$ form a multiplicative subgroup of $N$.

**References**