

On fixed point sets of differentiable involutions

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On fixed point sets of differentiable involutions

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1. Introduction

Let (M^n, T) be a closed smooth manifold dimension n with a differentiable involution, and let F^m be the union of the m -dimensional components of the fixed point with the normal bundle ν_m . The aim of this paper is to give some relations among the Whitney numbers of $\{[F^m, \nu_m]\}$, which are used to investigate involutions fixing a projective space and a point. Let $S^{<k>}(\xi)$ be the characteristic class of an n -dimensional vector bundle ξ which is given by replacing the i -th elementary symmetric polynomial of α_i by the i -th Whitney class $W_i(\xi)$ in the symmetric polynomial

$$\sum_{i_1+i_2+\dots+i_n=k} \alpha_1^{i_1} \alpha_2^{i_2} \dots \alpha_n^{i_n}$$

The main theorem will be:

THEOREM 1.1. *Let $[F^m] \in H_m(F^m; \mathbb{Z}_2)$ be the fundamental class of F^m . Then*

$$(1) \sum_{0 \leq m \leq n} \langle S^{<m>}(\nu_m), [F^m] \rangle = 0$$

$$(2) \sum_{0 \leq m \leq n} \langle m S^{<m>}(\nu_m) + \sum_{1 \leq j \leq \beta} \binom{\beta}{j} S^{<m-j>}(\nu_m) S_j(\nu_m) + S^{<m-\beta>}(\nu_m) S_\beta(\tau(F_m)), [F^m] \rangle = 0$$

where $S_j(\)$ indicates the characteristic class corresponding to the symmetric polynomial $\sum \alpha_i^j$, and $\beta \leq n-1$.

The paper is organized as follows. In § 2 we get some relations among the characteristic classes which will be used to calculate the Wu classes of the real projective space bundle $P(\xi)$ associated with a smooth vector bundle ξ in §3, and to calculate the Gysin homomorphism of the classifying map $f: P(\xi) \rightarrow RP^N$ for the canonical line bundle

over $P(\xi)$ in § 4. Making use of the results obtained in § 2, the Boardman homomorphism and the Quillen theorem instead of the results of Kosniowski and Stong [8], we prove Theorem 1.1 in § 4. In § 5 we show that an involution fixing a $(2m+1)$ -dimensional real projective space RP^{2m+1} and a point is bordant to the Z_2 -manifold (RP^{2m+2}, T) with

$$T[x_0, x_1, \dots, x_{2m+2}] = [-x_0, x_1, \dots, x_{2m+2}]$$

and we investigate the dimension of a Z_2 -manifold fixing RP^{2m} and a point.

2. The structure of the cohomology ring of a projective space bundle

Let ξ be a differentiable vector bundle of dimension $k+1$ over an n -dimensional smooth manifold M^n . Denote by $P(\xi)$ the projective space bundle. The cohomology ring $H^*(P(\xi); Z_2)$ is a $H^*(M^n; Z_2)$ -module with

$$ax = \pi^*(a) \cup x,$$

where $\pi: P(\xi) \rightarrow M^n$ is the projection. Let η_ξ be the canonical line bundle over $P(\xi)$ and let c be the first Whitney class of η_ξ . The Leray-Hirsch theorem implies that $H^*(P(\xi); Z_2)$ is a free $H^*(M^n; Z_2)$ -module with a basis $1, c, c^2, \dots, c^k$, and

$$(2.1) \quad c^{k+1} = W_1(\xi)c^k + W_2(\xi)c^{k-1} + \dots + W_k(\xi)c + W_{k+1}(\xi)$$

where $W_i(\xi)$ denotes the i -th Whitney class of ξ .

Applying the splitting principle to a $(k+1)$ -dimensional vector bundle ξ , we formally write

$$W(\xi) = \prod_{j=1}^{k+1} (1 + \alpha_j)$$

Denote by $S^{<i>}(\xi)$ the characteristic class corresponding to the symmetric polynomial

$$\sum_{i_1 + i_2 + \dots + i_{k+1} = i} \alpha_1^{i_1} \alpha_2^{i_2} \dots \alpha_{k+1}^{i_{k+1}}$$

Then we have the following

PROPOSITION 2.2. *Let*

$$(2.3) \quad c^{k+t} = f_{i,1}c^k + f_{i,2}c^{k-1} + \dots + f_{i,k}c + f_{i,k+1}.$$

Then

- (1) $f_{i,j} = W_j(\xi)$
- (2) $f_{i,1} = S^{<t>}(\xi)$
- (4) $f_{i,j} = S^{<t-1>}(\xi) W_j(\xi) + S^{<t-2>}(\xi) W_{j+1}(\xi) + \dots + S^{<t+j-k-2>}(\xi) W_{k+1}(\xi)$

PROOF. (1) is an immediate result of (2.1). It follows from (2.1) and (2.3) that

$$(2.4) \quad \begin{cases} f_{i+1,1} = f_{i,1} W_1(\xi) + f_{i,2} \\ f_{i+1,2} = f_{i,1} W_2(\xi) + f_{i,3} \\ \dots\dots\dots \\ f_{i+1,k} = f_{i,1} W_k(\xi) + f_{i,k+1} \\ f_{i+1,k+1} = f_{i,1} W_{k+1}(\xi). \end{cases}$$

Let $f_j = f_{j,1}$ for brevity. Then we have

$$(2.5) \quad f_{i+1} = f_i W_1(\xi) + f_{i-1} W_2(\xi) + \dots + f_{i-k+1} W_k(\xi) + f_{i-k} W_{k+1}(\xi)$$

We comprehend $W_j(\xi)$ to be the j -th elementary symmetric polynomial $\mathfrak{S}_j(\alpha)$ of α_j , and so we have

$$(2.6) \quad f_{i+1} = f_i \mathfrak{S}_1(\alpha) + f_{i-1} \mathfrak{S}_2(\alpha) + \dots + f_{i-k+1} \mathfrak{S}_k(\alpha) + f_{i-k} \mathfrak{S}_{k+1}(\alpha)$$

Let $f_j^{(i)} = f_j^{(i-1)} + \alpha_i f_{j-1}^{(i-1)}$ where $f = f_j^{(1)} + \alpha_1 f_{j-1}$. Proceeding inductively to substitute $f_j^{(i)}$ for $f_j^{(i-1)}$ in (2.6), we finally obtain

$$f_{i+1}^{(k)} = f_i^{(k)} \alpha_{k+1}$$

and

$$f_{i+1}^{(k)} = \alpha_{k+1}^{i+1}.$$

We now suppose as the inductive hypothesis that

$$f_j^{(1)} = \sum_{i_2 + \dots + i_{k+1} = j} \alpha_2^{i_2} \dots \alpha_{k+1}^{i_{k+1}}$$

Since $f_j^{(1)} = f_j + \alpha_1 f_{j-1}$, it follows that

$$f_j = f_j^{(1)} + \alpha_1 f_{j-1}^{(1)} + \alpha_1^2 f_{j-2}^{(1)} + \dots + \alpha_1^{t-2} f_2^{(1)} + \alpha_1^{t-1} f_1^{(1)} + \alpha_1^t$$

and we have $f_j = S^{<t>}(\xi)$. (3) is an immediate result of (2) and (2.4).

3. On the Wu classes of the projective space bundle associated with a real vector bundle.

According to Adams [1], the KO -group $\widetilde{KO}(RP^n)$ of the n -dimensional projective space RP^n is the cyclic group $Z/2^f$ with a generator γ_n -1 , where γ_n is the canonical line bundle over RP^n and f_n is a number of a set $\{s \mid 0 < s \leq n, s \equiv 1, 2, 4 \text{ modulo } 8\}$. Let ξ be a real vector bundle of dimension $k+1$ over RP^n with $\xi \sim_s u \gamma_n$. It means that there exist trivial bundles θ and θ' such that $\xi \oplus \theta = u \gamma_n \oplus \theta'$. Applying the splitting principle, we have

$$(3.1) \quad S^{< i >}(\xi) = \sum_{i_1 + i_2 + \dots + i_{k+1} = i} \alpha_1^{i_1} \alpha_2^{i_2} \dots \alpha_{k+1}^{i_{k+1}}$$

$$= \sum_{j_1 + j_2 + \dots + j_u = i} x^{j_1 + j_2 + \dots + j_u} = \binom{-u}{i} x^i$$

where $x = W_1(\gamma_n)$ is a generator of $H^1(RP^n; Z_2)$. The i -th Whitney class $W_i(\xi)$ of ξ is $\binom{u}{i} x^i$. Using Proposition 2.2, we obtain

PROPOSITION 3.2.

$$\begin{aligned} c^{k+i} &= \binom{-u}{i} c^k x^i + \sum_{s=1}^k \binom{-u}{i-s} \binom{u}{s+1} c^{k-1} x^{i+1} + \sum_{s=1}^{k+1} \binom{-u}{i-2s} \binom{u}{s+2} c^{k-2} x^{i+2} \\ &+ \sum_{1 \leq s \leq k-j+2} \binom{-u}{i-s} \binom{u}{j+s-1} c^{k-j+1} x^{i+j-1} + \dots + \binom{-u}{i-1} \binom{u}{k+1} x^{i+k} \end{aligned}$$

where c is the first Whitney class of the canonical line bundle γ_ε of the projective space bundle associated with ξ .

The tangent bundle of the projective space bundle $P(\xi)$ associated with a vector bundle ξ over RP^n is stably equivalent to

$$\gamma_\varepsilon \otimes \pi^! \xi \oplus (n+1) \gamma_n$$

where $\pi: P(\xi) \rightarrow RP^n$ is the projection. The Wu class $v_i(M)$ of a manifold M of dimension m is defined by $\langle Sq^i x, [M] \rangle = \langle x v_i(M), [M] \rangle$, where $[M]$ indicates the fundamental class of M . We use Proposition 3.2 to have the following (cf. [5]).

PROPOSITION 3.3. *Let*

$$v_i(P(\xi)) = \lambda_0 x^i + \lambda_1 x^{i-1} c + \lambda_2 x^{i-2} c^2 + \dots + \lambda_k x^{i-k} c^k.$$

If $\xi \sim_s u \gamma_n$ then

$$\lambda_j = \sum_{a+b=t} \binom{n-t+j}{a} \binom{k-j}{b} \binom{-u}{b-j}.$$

4. A proof of Theorem 1.1.

Let $\xi \rightarrow F$ be a smooth real vector bundle of dimension $k+1$ over an m -dimensional closed smooth manifold F , and let $f: P(\xi) \rightarrow RP^N$ be the classifying map for the canonical line bundle η_ξ . We now investigate the Gysin homomorphism

$$f_1: H^i(P(\xi); Z_2) \xrightarrow{D} H_{m+k-i}(P(\xi); Z_2) \xrightarrow{f_*} H_{m+k-i}(RP^N; Z_2) \xrightarrow{D^{-1}} H^{N-m-k+i}(RP^N; Z_2)$$

where D is the Poincare duality isomorphism.

PROPOSITION 4.1. *Let $a \in H^p(F; Z_2)$ and $c = W_1(\eta_\xi)$. Then*

$$f_1(ac^q) = \langle S^{\langle m-p \rangle}(\xi) a, [F] \rangle \bar{x}^{N-m-k+p+q}$$

where $\bar{x}$ indicates the generator of $H^1(RP^N; Z_2)$.

PROOF. Take the dual class $(\bar{x}^{N-m-k+p+q})_* \in H_{N-m-k+p+q}(RP^N)$ which equals to $\bar{x}^{m+k-p-q} \cap [RP^N]$. Let $f_1(ac^q) = \lambda \bar{x}^{N-m-k+p+q}$. Since $f^*(\bar{x}) = c$, Proposition 2.2 implies that

$$\begin{aligned} \lambda &= \langle \bar{x}^{m+k-p-q}, f_*(ac^q \cap [P(\xi)]) \rangle \\ &= \langle ac^{m+k-p}, [P(\xi)] \rangle \\ &= \langle a S^{\langle m-p \rangle}(\xi) c^k, [P(\xi)] \rangle \\ &= \langle a S^{\langle m-p \rangle}(\xi), [F] \rangle \end{aligned}$$

Q. E. D.

We now recall the Boardman map (cf. [6])

$$\beta: \mathfrak{N}^*(X) \rightarrow H^*(X; Z_2) [[t_1, t_2, \dots]]$$

which is a multiplicative natural transformation satisfying

(1) for the cobordism first characteristic class $W_1^N(\gamma)$ of a line bundle γ

$$\beta(W_1(\gamma)) = W_1(\gamma) + \{W_1(\gamma)\}^2 t_1 + \dots + \{W_1(\gamma)\}^{i+1} t_i + \dots$$

(2) for finite CW complex X , β is injective.

We next recall the Conner-Floyd characteristic class (cf. [2], [6])

$$c_i: Vect(X) \rightarrow H^*(X; Z_2) [[t_1, t_2, \dots]]$$

which is a map assigning a real vector bundle over X to a formal power series of t_i with the coefficient in $H^*(X; Z_2)$ such that

- (1) $c_i(f_! \xi) = f^* c_i(\xi)$
- (2) $c_i(\xi \oplus \eta) = c_i(\xi) c_i(\eta)$
- (3) for a line bundle η
 $c_i(\eta) = 1 + W_1(\eta) t_1 + \{W_1(\eta)\}^2 t_2 + \dots$

Denoting by $D_N: \mathfrak{R}^*(N) \rightarrow \mathfrak{R}_*(N)$ the Atiyah Thom Poincare duality isomorphism. Then we have the Umkehrung homomorphism $f_!$ for a map $f: M \rightarrow N$ between closed manifolds M and N :

$$f!: \mathfrak{R}^*(M) \xrightarrow{D_M} \mathfrak{R}_*(M) \xrightarrow{f_*} \mathfrak{R}_*(N) \xrightarrow{D_N^{-1}} \mathfrak{R}^*(N)$$

which satisfies $D_N f_!(1) = [M \rightarrow N] \in \mathfrak{R}_*(N)$. For the bordism group $\mathfrak{R}_n^{Z_2}$ of Z_2 -manifolds of dimension n , the bordism group $\sum_{s+t=n} \mathfrak{R}_s(BO(t))$ of vector bundles and the bordism group $\mathfrak{R}_{n-1}(BO(1))$ of free Z_2 -manifolds, there exists an exact sequence (cf. [7])

$$(4.2) \quad 0 \rightarrow \mathfrak{R}_n^{Z_2} \xrightarrow{f_*} \sum_{s+t=n} \mathfrak{R}_s(BO(t)) \xrightarrow{g} \mathfrak{R}_{n-1}(BO(1)) \rightarrow 0$$

where $j^*[M, T] = \sum_i [F_i, \nu_i]$, F_i the fixed point component of (M, T) and ν_i the normal bundle of F_i , and $\partial[M, \xi] = [P(\xi) \xrightarrow{f} RP^N \subset BO(1)]$, f the classifying map for the canonical line bundle. We now remark that

$$(4.3) \quad \beta(f_!(1)) = f_! c_i(\nu_f),$$

where ν_f is the virtual normal bundle of $f: M \rightarrow N$ (cf. [9]).

A PROOF OF THEOREM 1.1. The exact sequence (4.2) implies that $\sum \partial[F^m, \nu_m] = 0$. Let $f_m: P(\nu_m) \rightarrow RP^N$ be the classifying map for the canonical line bundle η_ε . (4.3) implies $\sum f_m! (c_i(\nu_{f_m})) = 0$. We now compute

$$\begin{aligned} f_m! c_i(\nu_{f_m}) &= f_m! \left\{ \frac{f_m^* \{c_i(\gamma_N)\}^{N+1}}{c_i(\gamma_{\nu_m}) \otimes \pi^! \nu_m c_i(\pi^! \tau(F^m))} \right\} \\ &= c_i(\gamma_N)^{N+1} f_m! \left\{ \frac{1}{c_i(\gamma_{\nu_m}) \otimes \pi^! \gamma_m c_i(\pi^! \tau(F^m))} \right\} \end{aligned}$$

Denote by $\mathfrak{D}$ an ideal generated by $\{t_1, t_2, \dots, t_{\beta-1}, t_\beta^2, t_{\beta+1}, \dots\}$. By virtue of the splitting principle, we have

$$c_i(\gamma_{\nu_m} \otimes \pi^! \nu_m) = 1 + \{(n-m) c^\beta + \sum_{1 \leq j \leq m} \binom{\beta}{j} S_j(\pi^! \nu_m) c^{\beta-j}\} t_\beta \mod \mathfrak{D}$$

and

$$c_i(\pi^! \tau(F)) = 1 + s_\beta(\pi^! \tau(F)) t_\beta \pmod{\mathfrak{D}}.$$

Since $c_i(\gamma_N)$ is an invertible element, by making use of Proposition 4.1 we complete the proof.

5. Involutions fixing real projective spaces

Suppose that RP^m is embedding in M^n with the normal bundle ν which is stably equivalent to $u\gamma_m$. Then

LEMMA 5.1.

- (1) $S_j(\nu) = ux^j$ in $H^*(RP^m; Z_2)$
- (2) $S_j(\tau(P(\nu))) = u(c+x)^j + (n+m+u)c^j + (m+1)x^j$ in $H^*(P(\nu); Z_2)$ where x is the generator of $H^1(RP^m; Z_2)$, and $c = W_1(\gamma_\nu)$.

PROOF. $\xi \sim_s \xi'$ implies that $S_j(\xi) = S_j(\xi')$ and (1) follows. $\tau(P(\nu)) \sim_s \gamma \otimes \pi^! \nu \oplus (m+1) \pi^! \gamma_m$ and

$$S_j(\tau(P(\nu))) = S_j(\gamma_\nu \otimes \pi^! \nu) + (m+1)x^j.$$

Let $\nu \oplus \theta = u\gamma_m \oplus \theta'$, with trivial bundles θ and θ' . Then

$$S_j(\gamma_\nu \otimes \pi^! \nu) + \dim S_j(\gamma_\nu) = u S_j(\gamma_\nu \otimes \pi^! \gamma_m) + \dim \theta' S_j(\gamma_\nu)$$

and $S_j(\gamma_\nu \otimes \pi^! \nu) = u(c+x)^j + (n+m+u)c^j$. Q. E. D.

Let (M, T) be a closed Z_2 -manifold of dimension n fixing the disjoint union $\sum_{1 \leq i \leq s} RP^{m_i}$ of real projective spaces, and let ν_i be the normal bundle of RP^{m_i} which is stably equivalent to $u_i \gamma_{m_i}$, where u_i is a non-negative integer. Then it follows from Theorem 1.1 and Lemma 5.1 that

PROPOSITION 5.2.

- (1) $\sum_{1 \leq i \leq s} \binom{-u_i}{m_i} = 0 \pmod{2}$
- (2) $\sum_{1 \leq i \leq s} \{m_i \binom{-u_i}{m_i} + \sum_{1 \leq j \leq \beta} \binom{\beta}{j} \binom{-u_i}{m_i - j} u_i + \binom{-u_i}{m_i - \beta} (m_i + 1)\} = 0 \pmod{2}$
- (3) if $\beta = 2^t$ then

$$\sum_{1 \leq i \leq s} \{m_i \binom{-u_i}{m_i} + (u_i + m_i + 1) \binom{-u_i}{m_i - \beta}\} = 0 \pmod{2}$$

We then have the following

COROLLARY 5.3. *Suppose that a close Z_2 -manifold (M, T) fixing $RP^n + \{a \text{ point}\}$ has the normal bundle ν of RP^n with $\nu \sim_s u\gamma_m$. Then*

- (1) *if m is odd, then u is odd.*
- (2) *if $m=2^i$, then u is odd.*

PROOF. (1) is the immediate result of Proposition 5.2 (1). Applying Proposition 5.2 (3) to $\beta=2^i$, we have

$$m \binom{-u}{m} + \binom{-u}{0} (u+m+1) \equiv 0 \pmod{2}.$$

and $u+1 \equiv 0$.

Q. E. D.

We also obtain

COROLLARY 5.4. *There is no involution fixing $2k$ copies of RP^n and a point such that the normal bundles of the projective spaces are stably equivalent each other.*

THEOREM 5.5. *A Z_2 -manifold fixing a projective space RP^{2n+1} of dimension $2n+1$ and a point is bordant to a Z_2 -manifold RP^{2n+2} with the action $\tilde{T} [x_0, x_1, \dots, x_{2n+2}] = [-x_0, x_1, \dots, x_{2n+2}]$.*

PROOF. Suppose that (M, T) is a Z_2 -manifold whose fixed point set is $RP^{2n+1} + \{a \text{ point}\}$ and the normal bundle ν of RP^{2n+1} is stably equivalent to $u\gamma_{2n+1}$, where ν is of dimension $k+1$. By Corollary 5.3 (1) u is odd. Conner and Floyd proved that Euler characteristic numbers modulo 2 of M and the fixed point set coincide. We use this fact to prove that $\chi(M) \equiv 1$, where $\chi(\)$ denotes the Euler characteristic modulo 2. Suppose that k is odd, then the dimension of M is odd and $\chi(M) \equiv 0$. This is a contradiction. Therefore k is even. Generalized Whitney numbers $\langle W_\omega(N)g^*(y), [N] \rangle$ for a singular manifold $(N \rightarrow Y)$ determines the bordism class $[N \rightarrow Y]$ in $\mathfrak{N}_*(Y)$ (cf. [7]). Since k is even and u is odd, the first Whitney class of $P(\nu)$ is $c+x$. Let us compare the following characteristic numbers of $[P(\nu) \xrightarrow{f} RP^N]$ and $[RP^{2n+1} \subset RP^N]$, where f is the classifying map for the line bundle γ_ν :

$$\langle \{W_1(P) - f^*(\tilde{x})\}^{2n+k+1}, [P(\nu)] \rangle = 0 \quad \text{if } k > 0$$

and

$$\langle \{W_1(RP^{n+k}) - i^*(\tilde{x})\}^{2n+k+1}, [RP^{2n+k+1}] \rangle = 1.$$

Therefore $k=0$ and ν is equivalent to γ_{2n+1} . Then we have

$$j^*([M, T]) = [RP^{2n+1}, \gamma_{2n+1}] + [\{a \text{ point}\}, \theta],$$

where θ is the trivial bundle. Let $(RP^{2n+2}, \tilde{T})$ be a Z_2 -manifold with a Z_2 -action

$$\tilde{T} [x_0, x_1, \dots, x_{2n+2}] = [-x_0, x_1, \dots, x_{2n+2}]$$

Then

$$j^*([RP^{2n+2}, \tilde{T}]) = j^*([M, T])$$

and the exact sequence (4.2) deduces that (M, T) is bordant to $(RP^{2n+2}, \tilde{T})$.

Q. E. D.

THEOREM 5.6. *Suppose that an involution (M^n, T) fixes $RP^{2m} + \{a \text{ point}\}$, $m > 0$. Let ν be the normal bundle of RP^{2m} with $\nu \sim_s u\gamma_{2m}$. Then u is odd and $n \leq 4m+1$.*

PROOF. We firstly assume that u is even. Then Lemma 5.1 implies that $S_j(\tau(P(\nu))) = n c^j + x^j$. We compare generalized Whitney numbers of $[P(\nu) \xrightarrow{f} RP^N]$, f the classifying map of γ_ν , and $[RP^{n-1} \xrightarrow{i} RP^N]$. If $n > 2m$, then

$$\langle c^{n-2m-1} \{S_{2m}(\tau(P(\nu))) - n f^*(\tilde{x}^{2m})\}, [P(\nu)] \rangle = 1$$

and

$$\langle x^{n-2m-1} \{S_{2m}(\tau(RP^{n-1})) - n i^*(\tilde{x}^{2m})\}, [RP^{n-1}] \rangle = 0$$

where $x = W_1(\gamma_{n-1})$, $\tilde{x} = W_1(\gamma_N)$ and $c = W_1(\gamma_\nu)$. This is a contradiction. Then if u is even, $\dim M = 2m$. This means that some component of M with the involution has a fixed point set consisting of a point. Since there is no involution fixing a point except the involution on a zero dimensional manifold, if m is positive, then u is odd. We next assume that u is odd. Then Lemma 5.1 implies that

$$S_j(\tau(P(\nu))) = (c+x)^j + (n+1)c^j + x^j.$$

If $n > 4m+1$, then

$$\begin{aligned} A &= c^{n-4m-2} \{S_{4m+1}(\tau(P(\nu))) + (n+1)f^*(\tilde{x})^{4m+1} + [S_{2m+1}(\tau(P(\nu)))] \\ &\quad + (n+1)f^*(\tilde{x})^{2m+1}\} [S_{2m}(\tau(P(\nu))) + (n+1)f^*(\tilde{x})^{2m}] \\ &= c^{n-4m-2} (c+x)^{2m+1} x^{2m} = c^{n-2m-1} x^{2m} \end{aligned}$$

and

$$B = x^{n-4m-2} \{ S_{4m+1}(\tau(RP^{n-1})) + (n+1)i^*(\bar{x})^{4m+1} + [S_{2m+1}(\tau(RP^{n-1})) + (n+1)i^*(\bar{x})^{2m+1}] [S_{2m}(\tau(RP^{n-1})) + (n+1)i^*(\bar{x})^{2m}] \} = 0$$

Therefore $\langle A, [P(\nu)] \rangle = 1$ and $\langle B, [RP^{n-1}] \rangle = 0$. This is a contradiction. Then the theorem follows.

The homogenous space $SU(n)/SO(n)$ is diffeomorphic to a manifold $X_n = \{P \in SU(n) \mid {}^tP = P\}$. Let Z_2 act on X_n by $P \rightarrow P^{-1}$. Denote by $F(X_n, Z_2)$ the fixed point set. Then we have

PROPOSITION 5.7. $F(X_n, Z_2)$ is a disjoint union of the Grassmann manifolds $\{G_{2k}(R^n); k=0, 1, 2, \dots\}$.

PROOF. Each element P of $F(X_n, Z_2)$ belongs to $SO(n)$ and ${}^tP = P$. Let $F_{2k}(X_n, Z_2)$ consist of elements of $F(X_n, Z_2)$ whose trace is $n - 4k$. Letting each P of $F_{2k}(X_n, Z_2)$ correspond to the subspace $\{x \mid Px = -x\}$ in R^n , we see that $F_{2k}(X_n, Z_2)$ is diffeomorphic to the Grassmann manifold $G_{2k}(R^n)$.

Q. E. D.

Hence we obtain the 3-dimensional projective space RP^3 with the involution $[x_0, x_1, x_2, x_3] \rightarrow [-x_0, x_1, x_2, x_3]$ whose fixed point set is $RP^3 + \{a \text{ point}\}$ and the 5-dimensional manifold X_3 with the involution $P \rightarrow P^{-1}$ whose fixed point set is $RP^2 + \{a \text{ point}\}$.

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