Some Degenerate Nonlinear Parabolic Equations

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Some Degenerate Nonlinear Parabolic Equations

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0. Introduction

In this paper we consider the existence, uniqueness and the behaviour of solutions for the following initial-boundary value problem of the degenerate nonlinear parabolic equation:

$$
\text{(I)}\quad
\begin{cases}
u_t=\Delta(A(u))+\sum_{k=1}^{N}\dfrac{\partial B_k(u)}{\partial x_k}+F(u) & \text{in } Q=\Omega\times R^+ \qquad (0.1)\\
u(x,t)=0 & \text{on } \Gamma=\partial\Omega\times R^+\\
u(x,0)=u_0(x) & \text{in } \Omega,
\end{cases}
$$

where Ω is a bounded domain in R^N $(N>1)$ with a smooth boundary $\partial\Omega$, Δ denotes the N-dimensional Laplace operator, $A(u)$ is a function of class $C^1(R)$ with $A(0)=0$, $a(u)\equiv A'(u)>0$ $(u\neq 0)$ and $a(0)=0$, and $B_k(u)$, $F(u)$ and $u_0(x)$ satisfy some hypotheses which will be described in § 1.

The "porous media equation $u_t=\Delta(u^m)$ $(m>1)$" which has been investigated by various authors (Cf. Ch. 5 in [6] and [14] for a survey) is the special case of (0.1). Gilding [7] has proved the existence, uniqueness for non-negative solutions of Problem (I) in the case $N=1$ and $F(u)=0$. Kalashnikov [8] and Kershner [9] have proved the existence, uniqueness and localization for non-negative solutions of the Cauchy problem for (0.1) in the case $N=1$, $A(u)=u^m$, $B_1(u)=u^n$ and $F(u)=-\lambda u^p$, where $m>1$, $n>1$, $p>0$ and $\lambda>0$ are constants. Nakao [11, 12] has proved the existence and decay estimates for solutions of Problem (I) in the case $N>1$. In [5], Ebihara and Nanbu verified the existence of global classical solutions of Problem (I) in the special case.

At first, we shall prove the existence and uniqueness for the solution of Problem (I) which is more general than the case $u_t=\Delta(u^m)+\sum_{k=1}^{N}\dfrac{\partial(u^{n_k})}{\partial x_k}-\lambda u^p$, where $m>1$, $n_k\geqq 1$, $p>1$ and $\lambda>0$ are constants. For $N=1$ and

$F(u)=0$, Gilding [7] proved the uniqueness under the condition $(B_1'(u))^2 = 0(a(u))$ as $u \to 0+$. His proof relies on the continuity of solution. For $N>1$, the continuity of solutions of Problem (I) is not known, and so we shall give a different proof from Gilding's one, which is suggested by the argument in [2].

Secondly, we shall investigate the behaviour of solutions of Problem (II):

$$
\text{(II)} \quad \begin{cases} u_t = \Delta(|u|^{m-1}u) + \sum_{k=1}^{N} \dfrac{\partial B_k(u)}{\partial x_k} + F(u) & \text{in } Q \qquad (0.2) \\ u(x,t) = 0 & \text{on } \Gamma \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}
$$

where $m>1$ is a constant, $B_k(u) \in C^1(R)$ with $B_k'(0)=0$, $F(s) \in C^1(R)$ with $F(0)=0$ and $F(s) \times (sign\ s) \leqq 0$, and $u_0 \in L^\infty(\Omega)$. We shall derive a decay estimate for solutions of Problem (II) as the following:

$$
|u(x,t)| \leqq C^*(a_1 t + a_2)^{-\gamma} \qquad \text{on } Q \qquad (0.3)
$$

where $\gamma = 1/(m-1)$, and C^*, a_1 and a_2 are positive constants which only depend on the data m, N, u_0 and Ω. For the case $F(s)=B_k(s)=0 (k=1, \ldots, N)$, Aronson and Peletier [3] and Alikakos [1] have proved the decay estimate of type (0.3), independently. The estimate (0.3) will be derived by Alikakos's technique. Recently, Nakao [11], [12] has proved the decay estimate of the norm $\|B_k(u(t))\|_2^2 + \|\nabla(|u|^{m-1}u)(t)\|_2^2$ for solutions of Problem (II).

Finally, we shall investigate another behaviour for non-negative solutions of Problem (III):

$$
\text{(III)} \quad \begin{cases} u_t = \Delta(u^m) + \sum_{k=1}^{N} b_k^0 \dfrac{\partial (u^{n_k})}{\partial x_k} - \lambda u^p & \text{in } Q \qquad (0.4) \\ u(x,t) = 0 & \text{on } \Gamma \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}
$$

where $m>1$, $n_k \geqq 1$, $p \geqq 1$, b_k^0 and $\lambda \geqq 0$ are constants, and $u_0(x)$ is a given non-negative function of class $L^\infty(\Omega)$.

In [3] Aronson and Peletier proved the positivity property and the large time behaviour for solutions of the initial-boundary value problem for the porous media equation $u_t = \Delta(u^m)$. Afterwards, Bertsch, Nanbu and Peletier [4] have generalized these results in [3] to the equation $u_t = \Delta(u^m) - \lambda u^p$ $(\lambda > 0)$. We shall generalize these results in ([3], [4]) to the equation

(0.4) with $b_k^0 \neq 0$. The proof of the positivity property for solutions is based on the construction of an appropriate subsolution. We shall prove that if $n_k \geqq m$ $(k=1, \ldots, N)$ and $p \geqq m$, then the positivity property holds, that is, for any initial function $u_0(x)$ there is a finite time $T>0$ satisfying $P(t) \equiv \{x \in \Omega: u(x, t)>0\} = \Omega$ for every $t \geqq T$, and if $n_k = m$ $(k=1, \ldots, N)$, the large time behaviour of (6.5)–(6.7) in §6 holds.

1. Rreliminaries

Here are the hypotheses to be made on the data of Problem (I).

H.1. $A(u) \in C^1(R)$, $A(0)=0$, $a(u) \equiv A'(u)>0$ $(u \neq 0)$ and $a(0)=0$.

H.2. $B_k(u) \in C^1(R)$, $B_k(0)=b_k(0)=0$, where $b_k(s) \equiv dB_k(s)/ds$.

H.3. Given any $M>0$ there exists a constant $\mu_0(M)$ such that $b_k^2(s) \leqq \mu_0(M)\, a(s)$ holds for $s \in [-M, M]$ $(k=1, \ldots, N)$.

H.4. $F(s): R \to R$ is locally Lipshitz continuous and $F(0)=0$.

H.5. $u_0(x) \in L^\infty(\Omega)$.

H.6. Ω is a bounded, arcwise connected open set of R^N $(N>1)$ with the smooth boundary $\partial\Omega$.

We introduce the notation in this paper.

Let $R=(-\infty, -\infty)$, $R^+=(0, +\infty)$, $Q=\Omega \times R^+$, $Q_T=\Omega \times (0, T]$, $\Gamma=\partial\Omega \times R^+$, $\Gamma_T=\partial\Omega \times [0, T]$, $u_{x_k}=\partial u/\partial x_k$, $\nabla\phi=(\phi_{x_1}, \ldots, \phi_{x_N})$, $\|u(t)\|_p = \{\int_\Omega |u(x,t)|^p dx\}^{1/p}$ and $\|u\|_\infty = \sup_{x \in \Omega} |u(x)|$.

DEFINITION 1. *A solution u* of Problem (I) *on* $[0, T]$ is a function u with the properties

(i) $u \in C([0, T]: L^1(\Omega)) \cap L^\infty(Q_T)$,

(ii)
$$\int_\Omega u(t)\phi(t)\, dx - \iint_{Q_t} \{u\phi_t + A(u)\Delta\phi - \sum_{k=1}^N B_k(u)\phi_{x_k}\}\, dx d\tau = \int_\Omega u_0\phi(0)\, dx + \iint_{Q_t} F(u)\phi\, dx d\tau \tag{1.1}$$

for any $t \in [0, T]$ and every $\phi \in C^2(\bar{Q}_T)$ such that $\phi=0$ on $\partial\Omega \times [0, T]$.

A solution u of Problem (I) *on* $[0, \infty)$ means a solution on each $[0, T]$ and *a subsolution* (*supersolution*) is defined by (i), and (ii) with equality replaced by $\leqq (\geqq)$ and adding $\varphi \geqq 0$ in Q_T.

We state two lemmata. The first one will be used in proof of uniqueness theorem. The second one will be used in §6.

LEMMA 1. *Let $A(u)$ and $B_k(u)$ satisfy hypotheses* (H. 1)-(H. 3). *Then, given $M>0$ there exists a positive constant $\mu_0=\mu_0(M)$ such that*

$$\{B_k(s_1)-B_k(s_2)\}^2\leqq\mu_0(A(s_1)-A(s_2))(s_1-s_2) \tag{1.2}$$

for all $s_1,\ s_2\in[-M,M]$ ($k=1,\ \dots,\ N$).

PROOF.
$$\begin{aligned}\{B_k(s_1)-B_k(s_2)\}^2&=(s_1-s_2)^2\{\int_0^1 b_k(s_1\tau+s_2(1-\tau))d\tau\}^2\\ &\leqq(s_1-s_2)^2\int_0^1 b_k^2(s_1\tau+s_2(1-\tau))d\tau\\ &\leqq(s_1-s_2)^2\mu_0\int_0^1 a(s_1\tau+s_2(1-\tau))d\tau \quad \text{(by H. 3))}\\ &=\mu_0(s_1-s_2)[A(s_1)-A(s_2)].\end{aligned}$$

Let $d:\bar{\Omega}\to[0,+\infty)$ be given by
$$d(x)=\min\{|x-z|:z\in\partial\Omega\}.$$
By $d(x)$ we define the set $\Omega_s=\{x\in\Omega:\ 0<d(x)\leqq s\}$ and $\Sigma_s=\bar{\Omega}-\Omega_s=\{x\in\Omega: d(x)\geqq s\}$. The following property of $d(x)$ is well known.

LEMMA 2 (Serrin [15]). *Let Ω satisfy* (H. 6). *There exists a constant $\sigma\in R^+$ such that for every $x\in\Omega_\sigma$ there is a unique $z(x)\in\partial\Omega$ which satisfies*

$$d(x)=|x-z(x)|.$$

Moreover, $d(x)\in C^2(\Omega_\sigma)$.

A proof of Lemma 2 can be found in [15].

2. Uniqueness for Problem (I)

In this section we shall verify the uniqueness of solutions of Problem (I). We start with the following problem:

$$\text{(IV)}\quad\begin{cases} u_t=\Delta A(u)+\sum_{k=1}^N b_k(u)\dfrac{\partial u}{\partial x_k}+g(x,t) & \text{in } Q \qquad (2.1)\\ u(x,t)=0 & \text{on } \Gamma\\ u(x,0)=u_0(x) & \text{in } \Omega\end{cases}$$

where $A(u)$, $b_k(u)\equiv dB_k(u)/du$ and u_0 satisfy the above (H. 1) – (H. 6) and

$g(x,t)\in L^1(Q_T)$ for any $T>0$.

DEFINITION 2. *A solution u* of Problem (IV) *on* $[0,T]$ is a function u with the properties

(i) $u\in C([0,T]:L^1(\Omega))\cap L^\infty(Q_T)$,

(ii)
$$\int_\Omega u(t)\phi(t)dx-\iint_{Q_t}\{u\phi_t+A(u)\Delta\phi-\sum_{k=1}^N B_k(u)\phi_{x_k}\}dxd\tau$$
$$=\int_\Omega u_0\phi(0)dx+\iint_{Q_t}g\phi dxd\tau \tag{2.2}$$

for any $t\in[0,T]$ and every $\phi\in C^2(\bar{Q}_T)$ such that $\phi=0$ on $\partial\Omega\times[0,T]$. *A subsolution* (*supersolution*) of Problem (IV) is defined by (i), and (ii) with equality replaced by $\leqq(\geqq)$ and adding $\varphi\geqq 0$ in Q_T.

PROPOSITION 1. *Let $\underline{u}$ be a subsolution of Problem* (IV) *on* $[0,T]$ *with data $\underline{g}$, $\underline{u}_0$ and let $\bar{u}$ be a supersolution of problem* (IV) *on* $[0,T]$ *with data $\bar{g}$, $\bar{u}_0$. Then for each $\tau\in[0,T]$ and for every $\lambda\geqq\mu_0$, where μ_0 is the constant in Lemma* 1 *determined by* $M=\max(\|\underline{u}\|_{\infty(Q_T)}, \|\bar{u}\|_{\infty(Q_T)})$, *we have*

$$e^{\lambda\tau}\int_\Omega(\underline{u}(\tau)-\bar{u}(\tau))^+dx\leqq\int_\Omega(\underline{u}_0-\bar{u}_0)^+dx$$
$$+\iint_{Q_\tau}(\underline{g}-\bar{g}+\lambda(\underline{u}-\bar{u}))^+\cdot e^{\lambda s}dxds, \tag{2.3}$$

where $r^+=\max\{r,0\}$.

PROOF. Set

$$\bar{a}(x,t)=\begin{cases}\dfrac{A(\underline{u})-A(\bar{u})}{\underline{u}-\bar{u}} & \text{for } \underline{u}\neq\bar{u}\\ 0 & \text{for } \underline{u}=\bar{u},\end{cases}$$

$$\bar{b}_k(x,t)=\begin{cases}\dfrac{B_k(\underline{u})-B_k(\bar{u})}{\underline{u}-\bar{u}} & \text{for } \underline{u}\neq\bar{u}\\ 0 & \text{for } \underline{u}=\bar{u},\end{cases}$$

and $M=\max(\|\underline{u}\|_{\infty(Q_T)}, \|\bar{u}\|_{\infty(Q_T)})$.

By virtue of (H.1) and (H.2) $\bar{a}(x,t)$ and $\bar{b}_k(x,t)$ are of class $L^\infty(Q_T)$ and $\bar{a}(x,t)\geqq 0$ in $\bar{Q}_T$. By Lemma 1 there exists a constant $\mu_0=\mu_0(M)$ such that

$$\{\bar{b}_k(x,t)\}^2\leqq\mu_0\bar{a}(x,t) \qquad \text{in } \bar{Q}_T. \tag{2.4}$$

Now by the standard mollification of functions, we can construct sequences of smooth functions $\{a_n\}$, $\{b_{k,n}\}$ such that

(i) $a_n, b_{k,n} \in C^\infty(\bar{Q}_T)$,

(ii) $(1/n) \leqq a_n \leqq \|\bar{a}\|_{\infty(Q_T)} + (1/n)$ in $\bar{Q}_T$

(iii) $(a_n - \bar{a})/\sqrt{a_n} \to 0$ as $n \to \infty$ in $L^2(Q_T)$,

(iv) $\|b_{k,n}\|_\infty \leqq \|\bar{b}_k\|_\infty + (1/n)$ in $\bar{Q}_T$,

$b_{k,n} - \bar{b}_k \to 0$ as $n \to \infty$ in $L^2(Q_T)$

and (v) $\{b_{k,n}\}^2 \leqq \mu_0 a_n$ on $\bar{Q}_T (k=1, \ldots, N)$.

Fix $\tau \in [0, T]$.

Let $\chi \in C_0^\infty(\Omega)$ be such that $0 \leqq \chi \leqq 1$, and ϕ_n the solution of the following problem:

$$
\text{(V)} \quad \begin{cases} \phi_{nt} + a_n \Delta\phi_n + \sum_{k=1}^{N} b_{k,n} \dfrac{\partial \phi_n}{\partial x_k} = \lambda \phi_n & \text{in } Q_\tau \qquad (2.5) \\ \phi_n(x, t) = 0 & \text{on } \partial\Omega \times (0, \tau) \\ \phi_n(x, \tau) = \chi(x) & \text{in } \Omega, \end{cases}
$$

where $\lambda \geqq \mu_0$.

By the well known results, the Problem (V) has a unique classical solution $\phi_n \in C^2(\bar{Q}\tau)$ which satisfies the following properties:

$$
0 \leqq \phi_n(x, t) \leqq \exp(\lambda(t - \tau)) \quad \text{in } \bar{Q}_\tau, \tag{2.6}
$$

and
$$
\int_\Omega |\nabla \phi_n(x, t)|^2 dx + \int_t^\tau \int_\Omega a_n |\Delta \phi_n|^2 dx dt + \lambda \int_t^\tau \int_\Omega |\nabla \phi_n|^2 dx dt \leqq \int_\Omega |\nabla \chi|^2 dx \equiv C_0 \tag{2.7}
$$

for every $t \in [0, \tau]$ and each $\lambda \geqq \mu_0$.

(2.6) is an immediate consequence of maximum principle. Let's prove (2.7). Multiplying (2.5) by $\Delta\phi_n$ and integrating over $\Omega \times (t, \tau)$, then we find

$$
-\int_t^\tau \frac{\partial}{\partial s} \{ \int_\Omega (\frac{1}{2}) |\nabla \phi_n(x, s)|^2 dx \} ds + \int_t^\tau \int_\Omega a_n |\Delta \phi_n|^2 dx ds
$$

$$
+ \lambda \int_t^\tau \int_\Omega |\nabla \phi_n|^2 \, dx ds = -\int_t^\tau \int_\Omega \Delta \phi_n \sum_{k=1}^{N} b_{k,n} \frac{\partial \phi_n}{\partial x_k} dx ds
$$

$$
\leqq \frac{1}{2} \int_t^\tau \int_\Omega a_n |\Delta \phi_n|^2 dx ds + \frac{1}{2} \int_t^\tau \int_\Omega \sum_{k=1}^{N} \frac{b_{k,n}^2}{a_n} (\frac{\partial \phi_n}{\partial x_k})^2 dx ds
$$

$$\leqq\frac{1}{2}\int_t^\tau a_n|\Delta\phi_n|^2dxds+\frac{1}{2}\mu_0\int_t^\tau\int_\Omega|\nabla\phi_n|^2dxds.$$

By $2\lambda-\mu_0\geqq\lambda$, we obtain (2.7).

Now set $t=\tau$ and $\phi=\phi_n$ in (2.2). By (2.5)-(2.7), we have

$$\int_\Omega(\underline{u}(\tau)-\bar{u}(\tau))\chi(x)dx-\iint_{Q_\tau}\{\underline{u}-\bar{u})(\bar{a}-a_n)\Delta\phi_n$$

$$+(\underline{u}-\bar{u})\sum_{k=1}^N(\bar{b}_k-b_{k,n})(\phi_n)_{x_k}\}dxdt$$

$$\leqq\int_\Omega(\underline{u}_0-\bar{u}_0)\phi_n(0)\,dx+\iint_{Q_\tau}(\underline{g}-\bar{g}+\lambda(\underline{u}-\bar{u}))\phi_ndxdt$$

$$\leqq e^{-\lambda\tau}\int_\Omega(\underline{u}_0-\bar{u}_0)^+dx+\iint_{Q_\tau}e^{\lambda(t-\tau)}(\underline{g}-\bar{g}+\lambda(\underline{u}-\bar{u}))^+dxdt.$$

However $\iint_{Q_\tau}|a-a_n||\Delta\phi_n|\,dxdt\leqq\iint_{Q_\tau}\frac{|\bar{a}-a_n|}{\sqrt{a_n}}\sqrt{a_n}\,|\Delta\phi_n|\,dxdt$

$$\leqq C_0\|\frac{\bar{a}-a_n}{\sqrt{a_n}}\|_{L^2(Q_T)}\to 0\quad\text{as } n\to\infty,$$

and $\iint_{Q_\tau}\sum_{k=1}^N|\bar{b}_k-b_{k,n}||(\phi_n)_{x_k}|\,dxdt\leqq\|\nabla\phi_n\|_2\sum_{k=1}^N\|\bar{b}_k-b_{k,n}\|_{L^2(Q_T)}$

$$\leqq C_0/(\mu_0)\sum_{k=1}^N\|\bar{b}_k-b_{k,n}\|_{L^2(Q_T)}\to 0\qquad\text{as } n\to\infty.$$

Thus we have

$$e^{\lambda\tau}\int_\Omega(\underline{u}(\tau)-\bar{u}(\tau))\chi(x)dx$$

$$\leqq\int_\Omega(\underline{u}_0-\bar{u}_0)^+\,dx+\iint_{Q_\tau}e^{\lambda t}(\underline{g}-\bar{g}+\lambda(\underline{u}-\bar{u}))^+dxdt$$

which holds for any $\chi\in C_0^\infty(\Omega)$ with $0\leqq\chi\leqq1$. Hence this inequality continues to hold for $\chi=\text{sign}\{(\underline{u}(\tau)-\bar{u}(\tau))^+\}$, and it completes the proof.

Q. E. D.

THEOREM 1. *Let* (H. 1)-(H. 6) *be satisfied.*

(i) (Uniqueness). *Let* $\underline{u}$ *and* $\bar{u}$ *be solutions of Problem* (I) *on* $[0, T]$ *with data* $\underline{u}_0$ *and* $\bar{u}_0$, *respectively. Let* K *be a Lipschitz constant for* $F(u)$ *on* $[-M, M]$, *where* $M=\max(\|\underline{u}\|_{L^\infty(Q_T)}, \|\bar{u}\|_{L^\infty(Q_T)})$. *Then for each* $t\in[0, T]$ *we have*

$$\|\underline{u}(t)-\bar{u}(t)\|_{L^1(\Omega)} \leqq \exp(K_0 t)\|\underline{u}_0-\bar{u}_0\|_{L^1(\Omega)}. \tag{2.8}$$

Here $K_0 \equiv \max(K, \mu_0)$, *where* $\mu_0=\mu_0(M)$ *is the constant determined in* Lemma 1.

(ii) (Comparison principle). *Let* $\bar{u}$ *be a supersolution of Problem* (I) *on* $[0, T]$ *with data* $\bar{u}_0$ *and let* $\underline{u}$ *be a subsolution of Problem* (I) *on* $[0, T]$ *with data* $\underline{u}_0$. *If* $\underline{u}_0 \leqq \bar{u}_0$, *then*

$$\underline{u}(x, t) \leqq \bar{u}(x, t) \quad a.e. \text{ in } Q_T. \tag{2.9}$$

PROOF. Set $\underline{g}=F(\underline{u})$ and $\bar{g}=F(\bar{u})$ in Proposition 1. Let $\lambda=\max(K, \mu_0) \equiv K_0$ and $h(t)=\exp(K_0 t)\int_\Omega (\underline{u}-\bar{u})^+ dx$. By the relation $(F(\underline{u})-F(\bar{u})+K_0(\underline{u}-\bar{u}))^+ \leqq 2K_0(\underline{u}-\bar{u})^+$, (2.3) gives us $h(t) \leqq h(0)+2K_0\int_0^t h(s)ds$ for $t \in [0, T]$. By Gronwall's Inequality, we have $h(t) \leqq h(0) \exp(2K_0 t)$ which implies (2.8).

(ii) follows from (i).

3. Existence of solution of problem (I)

We shall verify the existence of solutions of Problem (I). Hereafter, instead of (H. 4), we set the following hypothesis:

H. 4. A. $F(u) \in C^1(R)$, $F(0)=0$, and $F(u) \times (\text{sign} u) \leqq 0$.

First we assume $u_0 \in C_0^\infty(\Omega)$. Set $M \equiv \sup|u_0|$.

Let $0<\varepsilon<1$ and let us consider the following regularized problem $(\mathrm{I})_\varepsilon$ under the hypotheses (H. 1), (H. 2), (H. 4. A) and (H. 6):

$$(\mathrm{I})_\varepsilon \quad \begin{cases} u_t=\Delta(\varepsilon u+A(u))+\sum_{k=1}^N \dfrac{\partial B_k(u)}{\partial x_k}+F(u) & \text{in } Q \qquad (3.1) \\ u(x, t)=0 & \text{on } \Gamma \\ u(x, 0)=u_0(x) & \text{in } \Omega. \end{cases}$$

By the well known result (Ch. V of [10]), the problem $(\mathrm{I})_\varepsilon$ has a unique classical solution u_ε with $|u_\varepsilon(x, t)| \leqq M$ on $\bar{Q}$. Set $A_\varepsilon(s)=\varepsilon s+A(s)$ and $A'_\varepsilon(s)=\varepsilon+a(s)$. Through this proof we denote u_ε by u. Multiply (3.1) by $\int_0^u b_n^2(s)ds$ and integrate over Ω. This gives

$$\frac{d}{dt}\{\int_\Omega[\int_0^u(\int_0^s b_n^2(\tau)d\tau)ds]dx\}+\int_\Omega \nabla A_\varepsilon(u)\cdot\nabla(\int_0^u b_n^2(s)ds)dx$$

$$=\sum_{k=1}^N\int_\Omega\frac{\partial B_k(u)}{\partial x_k}(\int_0^u b_n^2(s)ds)dx+\int_\Omega F(u)\,(\int_0^u b_n^2(s)ds)dx$$

$$\leqq\sum_{k=1}^N\int_\Omega\frac{\partial}{\partial x_k}[\int_0^u b_k(s)\,(\int_0^s b_0^2(\tau)d\tau)ds]dx+0 \qquad (\text{by (H. 4. A)})$$

$$=0 \quad (n=1,\ \dots,\ N).$$

Hence

$$\int_\Omega \nabla A_\varepsilon(u)\cdot\nabla(\int_0^u b_n^2(s)ds)dx\leqq-\frac{d}{dt}\{\int_\Omega[\int_0^u(\int_0^s b_n^2(\tau)d\tau)ds]dx\} \qquad (n=1,\ \dots,\ N). \tag{3.2}$$

Multiplying (3.1) by $(A_\varepsilon(u))_t$ and integrating over Ω, we find that

$$\int_\Omega A'_\varepsilon(u)u_t^2dx+\frac{d}{dt}(\frac{1}{2}\int_\Omega|\nabla A_\varepsilon(u(t))|^2dx)$$

$$=\sum_{k=1}^N\int_\Omega\frac{\partial B_k(u)}{\partial x_k}A'_\varepsilon(u)u_tdx+\frac{d}{dt}\,(\int_\Omega[\int_0^u F(s)A'_\varepsilon(s)ds]dx).$$

Since

$$\sum_{k=1}^N\int_\Omega\frac{\partial B_k(u)}{\partial x_k}A'_\varepsilon(u)u_tdx\leqq\frac{1}{2}\int_\Omega A'_\varepsilon(u)u_t^2dx+\frac{1}{2}\sum_{k=1}^N\int_\Omega b_k^2(u)u_{x_k}^2A'_\varepsilon(u)dx$$

and

$$\frac{1}{2}\sum_{k=1}^N\int_\Omega b_k^2(u)u_{x_k}^2A'_\varepsilon(u)dx$$

$$\leqq\frac{1}{2}\sum_{k=1}^N\int_\Omega\nabla(\int_0^u b_k^2(s)ds)\cdot\nabla A_\varepsilon(u)dx$$

$$\leqq-\frac{d}{dt}\{\frac{1}{2}\sum_{k=1}^N\int_\Omega[\int_0^u(\int_0^s b_k^2(\tau)d\tau)ds]dx\} \quad (\text{by (3.2)}),$$

we have for each $t\in R^+$

$$\int_\Omega A'_\varepsilon(u)u_t^2dx+\frac{d}{dt}\{\int_\Omega|\nabla A_\varepsilon(u(t))|^2dx$$

$$-\int_\Omega[\int_0^u 2F(s)A'_\varepsilon(s)ds]dx+\sum_{k=1}^N\int_\Omega[\int_0^u(\int_0^s b_n^2(\tau)d\tau)ds]dx\}$$

$$\leqq 0. \tag{3.3}$$

Using the condition (H. 4. A), from (3.3) it follows that

$$\int_0^T\int_\Omega A'_\varepsilon(u_\varepsilon)[(u_\varepsilon)_t]^2dxdt+\sup_{0\leq t\leq T}\int_\Omega|\nabla A_\varepsilon(u_\varepsilon(x,t))|^2dx$$

$$\leqq\int_\Omega|\nabla A_\varepsilon(u_0)|^2dx-\int_\Omega\{\int_0^{u_0}[2F(s)A'_\varepsilon(s)-\sum_{n=1}^N(\int_0^s b_n^2(\tau)d\tau)]ds\}dx$$

$$\leqq C_0(u_0)\equiv C_0, \tag{3.4}$$

where C_0 depends only u_0, but not on ε or T.
Set $v_\varepsilon=A(u_\varepsilon)$. Then the estimate (3.4) implies

$$|v_\varepsilon(x,t)|\leqq A_\varepsilon(M)\leqq M+A(M) \qquad \text{in } Q_T, \tag{3.5}$$

$$\|\nabla v_\varepsilon(t)\|_{L^2(\Omega)}\leqq C_0 \qquad \text{on } [0,T], \tag{3.6}$$

$$\|(v_\varepsilon)_t\|^2_{L^2(0,T:L^2(\Omega))}=\iint_{Q_T}[A'(u_\varepsilon)(u_\varepsilon)_t]^2dxdt$$

$$\leqq\sup_{\|u\|_\infty\leq M}A'_\varepsilon(u)\iint_{Q_T}A'_\varepsilon(u_\varepsilon)[(u)_t]^2dxdt$$

$$\leqq(\mathrm{M}+\sup_{|s|\leq M}a(s))C_0. \tag{3.7}$$

From (3.5), (3.6) and (3.7), if follows that $\{v_\varepsilon\}_{0<\varepsilon<1}$ is equicontinuous from $[0,T]$ into $L^2(\Omega)$ with value in a bounded subset of $H^1(\Omega)$. Hence by Arzela-Asocoli's Theorem there exists a function $v\in C([0,T]:L^2(\Omega))$ and $\varepsilon_n\downarrow 0$ such that

$$v_{\varepsilon_n}\to v \qquad \text{in } C([0,T]:L^2(\Omega)).$$

Then $u_{\varepsilon_n}\to A^{-1}(v)=u$ and $A(u_{\varepsilon_n})\to A(u)$ in $C([0,T]:L^2(\Omega))$.

Taking the limit as $\varepsilon_n\to 0$ in the variational equation

$$\int_\Omega u_{\varepsilon_n}(T)\phi(T)dx-\int_\Omega u_0\phi(0)dx$$

$$=\iint_{Q_T}u_{\varepsilon_n}\phi_t+A_{\varepsilon_n}(u_{\varepsilon_n})\Delta\phi-\sum_{k=1}^N B_k(u_{\varepsilon_n})\phi_{x_k}+F(u_{\varepsilon_n})\phi\}dxdt$$

which is valid for all $\phi\in C^2(\bar{Q}_T)$ with $\phi=0$ on $\partial\Omega\times[0,T]$, we have that u is a solution of Problem (I) on $[0,T]$ for each $T>0$. By the standard argument, we can easily remove the restriction $u_0\in C_0^\infty(\Omega)$, but we omit it here.

Now we can conclude the following:

THEOREM 2. *Let hypotheses* (H.1), (H.2), (H.4.A), (H.5) *and* (H.6) *be satisfied. Then Problem* (I) *has a solution on* $[0,\infty)$. *In addition* *if we assume* (H. 3), *Problem* (I) *has a unique solution.*

4. Behaviour of solutions of problem (II)

In § 3 we have considered the regularized problem $(\mathrm{I})_\varepsilon$. Now we shall consider the following regularized problem $(\mathrm{II})_\varepsilon$:

$$(\mathrm{II})_\varepsilon \quad \begin{cases} u_t = \Delta(|u|^{m-1}u) + \varepsilon\Delta u + \sum_{k=1}^{N} \frac{\partial B_k(u)}{\partial x_k} + F(u) & \text{in } Q \quad (4.1) \\ u(x,t) = 0 & \text{on } \Gamma \\ u(x,0) = u_0 & \text{in } \Omega, \end{cases}$$

where $0<\varepsilon<1$ and $u_0(x) \in C_0^\infty(\Omega)$.
Let (H. 2) and (H. 4. A) be sasisfied.

On the basis of theorem in Ch. V of [10], we know that Problem $(\mathrm{II})_\varepsilon$ has a unique classical solution $u_\varepsilon(x,t) \in C^{2;1}_{x;t}(\bar{Q})$. By Alikakos's argument, we can obtain the following:

PROPOSITION 2. *Let $u_\varepsilon(x,t)$ be the solution of Problem* $(\mathrm{II})_\varepsilon$. *Then we have*

(i) $|u_\varepsilon(x,t)| \leqq M \equiv \sup_\Omega |u_0|$ in $\bar{Q}$ (4. 2)

(ii) $\|u_\varepsilon(t)\|_{m+1} \leqq \{(m-1)k_m t + \|u_0\|_{m+1}^{-(m-1)}\}^{-\gamma}$ $t \in [0,\infty)$ (4. 3)

where k_m is a constant which depends on m and Sobolev *constant S_0, and $\gamma = 1/(m-1)$,*
and

(iii) *there exist positive constants C^*, a_1 and a_2 which depend only on the data u_0 and Ω such that*

$$|u_\varepsilon(x,t)| \leqq C^*(a_1 t + a_2)^{-\gamma} \quad \textit{in } \bar{Q}. \tag{4. 4}$$

Here $a_2 = \|u_0\|_{m+1}^{-(m-1)}$ and $a_1 = (m-1)S_0^{-2}(mes\Omega)^{1-m/m+1}$, where S_0 is Sobolev *constant such that $\|v\|_2 \leqq S_0 \|\nabla v\|_2$ for $v \in \mathring{W}_2^1(\Omega)$.*

PROOF. For brevity, we denote u_ε by u.
(i) is an immediate consequence of the maximum principle. To get the estimates (ii) and (iii), we shall use Alikatos's argument in the proof of Th. 3.1 of [1].

Multiply (4. 1) by $|u|^{\lambda-1}u$ $(\lambda \geqq m)$ and integrate over Ω.

Since $\int_\Omega b_k(u)u_{x_k}|u|^{\lambda-1}udx=\int_\Omega(\int_0^u b_k(s)|s|^{\lambda-1}sds)_{x_k}dx=0$

holds, we have

$$\frac{d}{dt}\left(\frac{1}{\lambda+1}\int_\Omega|u|^{\lambda-1}u^2dx\right)$$

$$\leqq -\varepsilon\int_\Omega \nabla u\nabla(|u|^{\lambda-1}u)dx-\frac{4m\lambda}{(\lambda+m)^2}\int_\Omega|\nabla(|u|^{(\lambda+m-2)/2}u)|^2dx$$

$$+\int_\Omega F(u)|u|^{\lambda-1}udx.$$

By (H. 4. A) and estimate (i), we know that $\int_\Omega F(u)u|u|^{\lambda-1}dx\leqq 0$ and thus we obtain

$$\frac{d}{dt}\left(\frac{1}{\lambda+1}\int_\Omega|u|^{\lambda-1}u^2dx\right)\leqq-\frac{4m\lambda}{(\lambda+m)^2}\int_\Omega|\nabla(|u|^{(\lambda+m-2)/2}u)|^2dx.$$

Set $v=|u|^{(\lambda+m-2)/2}u$ and $\rho_\lambda(t)=\int_\Omega|v(x,t)|^{2(\lambda+1)/(\lambda+m)}dx$. Since $v(t)\in \mathring{W}_2^1(\Omega)$, using Sobolev's and Hölder's inequality, we get

$$\rho_\lambda(t)\leqq S_0^{2(\lambda+1)/(\lambda+m)}\left(\int_\Omega|\nabla(|u|^{\frac{\lambda+m-2}{2}}u|^2dx\right)^{\frac{\lambda+1}{\lambda+m}}(mes\Omega)^{\frac{m-1}{\lambda+m}}.$$

Then we have

$$\frac{d}{dt}\left(\frac{1}{\lambda+1}\int_\Omega|u|^{\lambda+1}dx\right)\leqq\frac{-4m\lambda}{(\lambda+m)^2}S_0^{-2}(mes\Omega)^{\frac{1-m}{\lambda+1}}\rho_\lambda(t)^{\frac{\lambda+m}{\lambda+1}}$$

$$=-\frac{4m\lambda}{(\lambda+m)^2}S_0^{-2}(mes\Omega)^{\frac{1-m}{\lambda+1}}\|u(t)\|_{\lambda+1}^{\lambda+m}.$$

Set $\mu_{\lambda+1}(t)=\int_\Omega|u(x,t)|^{\lambda+1}dx$.
Thus for each $\lambda\geqq m$ and $t>0$, we have

$$\frac{d\mu_{\lambda+1}(t)}{dt}\leqq-(\lambda+1)\frac{4m\lambda}{(\lambda+m)^2}S_0^{-2}(mes\Omega)^{-\frac{m-1}{\lambda+1}}(\mu_{\lambda+1}(t))^{\frac{\lambda+m}{\lambda+1}}$$

which implies

$$\|u(t)\|_{m+1}\leqq\{(m^2-1)S_0^{-2}(mes\Omega)^{\frac{1-m}{m+1}}t+\|u_0\|_{m+1}^{m-1}\}^{-\gamma}$$

$$\equiv\{(m^2-1)k_mt+\|u_0\|_{m+1}^{m-1}\}^{-\gamma}.$$

Set $v(x,\tau)=u(x,t)\ (a_1t+a_2)^\gamma\equiv u(x,t)\theta(t)^\gamma$ and $\tau=\log\theta(t)$, where $a_1=(m-1)k_m$, $a_2=\|u_0\|_{m+1}^{m-1}$ and $\gamma=1/(m-1)$.
Then v satisfies the following problem:

$$(\mathrm{VI})\quad \begin{cases} a_1 v_\tau = e^\tau \varepsilon \Delta v + \Delta(|v|^{m-1}v) + F(ve^{-\gamma\tau})e^{m\gamma\tau} + a_1\gamma v \\ \qquad + \sum_{k=1}^{N} b_k(ve^{-\gamma\tau})\, e^\tau v_{x_k} & \text{in } \Omega\times(\tau_0, +\infty) \\ v(x,\tau)=0 & \text{on } \partial\Omega\times(\tau_0, +\infty) \\ v(x,\tau_0)=u_0(x) & \text{in } \Omega, \end{cases} \tag{4.5}$$

where $\tau_0=\log a_2$.
Set $\lambda=m+2^k-2$ $(k=1, 2, \dots)$ and $|v|^{\lambda-1}v=|v|^{m-1}v^{2^k-1}$.
We note that $|v(\tau)|^{\lambda-1}v(\tau)\in \mathring{W}^1_2(\Omega)$.
Multiply (4.5) by $|v|^{\lambda-1}v$ and integrate over Ω. This gives that for $\lambda=m+2^k-2$ $(k=1, 2, \dots)$

$$\frac{d}{d\tau}\Big(\frac{1}{\lambda+1}\int_\Omega |v|^{\lambda+1}dx\Big)\leqq -(a_1)^{-1}\int_\Omega \nabla(|v|^{m-1}v)\nabla|v|^{\lambda-1}v)\,dx + \frac{1}{m-1}\int_\Omega |v|^{\lambda+1}dx. \tag{4.6}$$

This (4.6) just corresponds to (3.20) in Alikakos[1]. By Alikakos's result in [1], there exists a constant C_3, which depends only on the data, such that

$$\sup_{\tau\geq\tau_0}[\sup_{x\in\Omega}|v(x,\tau)|]\leqq C_3\max\{1, \sup_{\tau\geq\tau_0}\|v(\tau)\|_m, \|u_0\|_\infty\}. \tag{4.7}$$

However by (4.3), we know

$$\|v(\tau)\|_m^m\leqq\theta(t)^{m/m-1}\|u(t)\|^m_{m+1}(mes\Omega)^{1/(m+1)}\leqq C_4,$$

where C_4 depends only on u_0, S_0 and Ω, and is independent of t and ε.
Set $C^*=C_3\max\{1, C_4, \|u_0\|_\infty\}$. Then we have (4.4)

Now we may conclud:

THEOREM 3. *Let u be the solution of Problem* (II). *Then*

(i) $|u(x,t)|\leqq\|u_0\|_\infty$ a.e. *in* $\bar{Q}$

(ii) $\|u(t)\|_{m+1}\leqq((m-1)k_m t+\|u_0\|^{-\frac{1}{\gamma}}_{m+1})^{-\gamma}$ on $[0,\infty)$

(iii) $|u(x,\mathrm{t})|\leqq C^*(a_1t+a_2)^{-\gamma}$ a.e. *in* $\bar{Q}$,

where k_m, a_1, a_2 and C^* are constants stated in Proposition 2.

5. Positivity property of non-negative solutions of problem (II)

We consider the positivity property of non-negative solutions of Problem

(II) for non-negative initial functions $u_0 \in L^\infty(\Omega)$ and $F(u)=0$.

By Theovem 2, we know that problem (II) has a non-negative solution on $[0,\infty)$. We suppose that

H. 7. $b_k(0)=0$ and

there exist constants $b_0(\in[0,\infty))$ and $s_0(\in(0,\infty))$ such that for $s\in(0,s_0)$

$$|b_k(s)|\leqq b_0 s^{m-1}, \quad (k=1, \ldots, N). \tag{5.1}$$

Consider the function

$$z(x,t;\rho,\tau)=\{[\frac{m-1}{4m}(t+\tau)^{-1}(\rho^2-|x|^2\log^{-1}(t+\tau))]_+\} \tag{5.2}$$

where $\tau>1$, $0<\rho<1$, $[f]_+=\max(f,0)$, $|x|^2=\sum_{k=1}^N x_k^2$ and $\gamma=1/(m-1)$.

If τ is large enough such that

$$(\frac{m-1}{4m}\,\frac{1}{\tau})^\gamma<s_0 \tag{5.3}$$

and

$$1-\frac{N(m-1)}{2}\log^{-1}\tau-b_0\frac{N(m-1)}{2m}(\log\tau)^{-1/2}>0, \tag{5.4}$$

then in the set $D_{\rho,\tau}=\{(x,t):z(x,t;\rho,\tau)>0\}$ we have

$$\begin{aligned} Lz &= \Delta z^m+\sum_{k=1}^N b_k(z)z_{x_k}-z_t \\ &\geqq \gamma z(t+\tau)^{-1}\{1-\frac{N(m-1)}{2}\log^{-1}(t+\tau)-b_0\frac{N(m-1)}{2m}(\log(t+\tau))^{-1/2}\} \\ &\geqq \gamma z(t+\tau)^{-1}\{1-\frac{N(m-1)}{2}\log^{-1}\tau-b_0\frac{N(m-1)}{2m}(\log\tau)^{-1/2}\}\geqq 0, \end{aligned}$$

where we use the relation $x_k\leqq|x|\leqq\rho(\log(t+\tau))^{1/2}<(\log(t+\tau))^{1/2}$. Furthermore we note that on the lateral boundary of $D_{\rho,\tau}$ $\nabla z^m=0$ holds, and the support of $z(x,t;\rho,\tau)$ grows with time t.

Now, by the same argument as in Proposition 4 of [3] and using (ii) of Theorem 1 in § 2, we can prove the followings:

LEMMA 3. *Let u be a solution of Problem* (II) *in the case* $F(u)=0$. *Suppose that* (H. 7) *and*

$$u(x,\theta)\geqq z(x-y,0:\rho,\tau)$$

for some $y\in\partial\Omega$ *and some* $\theta\in[0,\infty)$, *where* $\rho\sqrt{\log\tau}<\frac{d(y)}{2}$.

Then for any $\rho^*\in(0,\rho/2]$ *and* x^* *such that*

$$|x^*-y|=\rho^*\sqrt{\log\tau},$$

we have

$$u(x,t^*+\theta)\geqq z(x-x^*,\ 0:\rho-\rho^*,\tau^*)\quad(x\in\Omega), \tag{5.5}$$

where $\tau^*=t^*+\tau=\tau^{[\rho/(\rho-\rho^*)]^2}$.

THEOREM 4 (Cf. *Proposition* 4 in [3]) *Let* u *be the solution of Problem* (II) *with* $F(u)=0$. *Under* (H. 7), *there exists a time* $T>0$ *such that*

$$P(t)\equiv\{x\in\Omega:u(x,t)>0\}=\Omega \text{ for every } t\geq T.$$

COROLLARY. (Cf. Lemma 7 in [4]). *If we replace* $F(u)=0$ *by* $F(u)=-\lambda u^p(\lambda>0,\ p\geqq m)$ *in Theorem* 4, *then the positivity property continues to hold.*

On the other hand, Kalashnikov's example in [8] gives us the following remark.

REMARK. Let $b_k(u)=u^{n_k-1}$ $(k=1,\ \dots,\ N)$.
If min $(n_1,\ n_2,\ \dots,\ n_N)<m$, then the positivity property fails to hold, i. e. for certain initial function u_0 the support of $u(x,t)$ remains in a subset of Ω for all $t\geqq 0$.

6. The large time behaviour of solutions of problem (III) in the case $n_k=m$ $(k=1,\ \dots,\ N)$.

We consider the Aronson-Peletier's estimate for solutions of Problem (III) in the case $n_k=m$ $(k=1,\ \dots,\ N)$. For simplity, we study the case $\lambda=0$. The case $\lambda>0$ will be considered by the same way in [4], and then we omit it here.

In the following we use the argument in [3].

Now we consider the following initial-value problem $(P)_\delta$:

$$(P)_\delta\qquad\begin{cases}(g^m)''+\dfrac{N-1+\delta}{\eta}(g^m)'+\alpha g+\beta\eta g'=0 & (6.1)\\ g(0)=c,\ \ g'(0)=0\end{cases}$$

where α and β are positive numbers, related by the equation

$$\alpha(m-1)+2\beta=1,$$

$\delta\equiv\sum_{k=1}^{N}|b_k^0|(>0)$ and c is an arbitrary positive number.

In [3], Aronson and Peletier discussed in detail the problem $(P)_0$ in the case $\delta=0$. But if we consider the equation (0.4), we must consider the estimate of $(\delta/\eta)(g^m)'$ derived from the term $\sum_{k=1}^{N} b_k^0(u^m)_{x_k}$. However we can also cosider the problem $(P)_\delta$ in the same way as the case $\delta=0$. For convenience we shall state it's outline. By the standard argument it can be shown that there is a unique solution $g\in C^2$ $([0,\eta_0))$ for some small η_0 and this solution can be continued as long as it is positive, and as long as g is positive we have

$$(g^m)'(\eta)=-\frac{\alpha-\beta(N+\delta)}{\eta^{N-1+\delta}}\int_0^\eta\zeta^{N-1+\delta}g(\zeta)d\zeta-\beta g(\eta)\eta$$

$$\leqq-\beta\eta g(\eta)<0,$$

provided that $\alpha\geqq\beta(N+\delta)$. This implies that

$$g(\eta)\leqq(c^{m-1}-\frac{m-1}{2m}\beta\eta^2)^r$$

and hence $a\equiv\sup\{\eta>0:g>0 \text{ on } [0,\eta)\}<+\infty$.

Set $\alpha=2\beta(N+\delta)$. Then g^m is decreasing in $\eta\in[0,a)$ and

$$0>\kappa\equiv(g^m)'(a)>-\beta ac.$$

Let us denote the solution of Problem $(P)_\delta$ by $g(\eta:c)$ and the corresponding value of a and k by $a(c)$ and $k(c)$. In [3], Aronson and Peletier proved the properties of $a(c)$ and $k(c)$ such that

$$g(\eta;c)=cg(c^{-(m-1)/2}\eta;1),\ a(c)=c^{(m-1)/2}a(1) \text{ and } k(c)=c^{(m-1)/2}k(1).$$

Now let us consider the two parameters family of function

$$v(x,t:c,\tau)=(t+\tau)^{-\alpha}[g(\eta;c)]_+ \tag{6.2}$$

where $g(\eta;c)$ is the solution of Problem $(P)_\delta$, $\eta=|x|(t+\tau)^{-\beta}$, and α and β are positive numbers related by

$$\alpha(m-1)+2\beta=1 \text{ and } \alpha=2\beta(N+\delta).$$

Let $e:\Omega\to R$ be defined by

$$\begin{cases} -\Delta e+\sum_{k=1}^{N} b_k^0 e_{x_k}=1 & \text{in } \Omega \\ e=0 & \text{on } \partial\Omega. \end{cases}$$

By the well known results, $e\in C^{2+\alpha}(\bar{\Omega})$ for $\alpha\in(0,1)$ and $e>0$ in Ω and $\frac{\partial e}{\partial\nu}<0$ on $\partial\Omega$.

Let K be the set $K=\{\phi\in L^1(\Omega):\phi\geqq ke$ a.e. in Ω for some $k\in R^+\}$.

PROPOSITION 3. *Let u be a solution of Problem* (III) *in the case* $\lambda=0$ *and* $n_k=m(k=1, \ldots, N)$.
Then there exists a $T^*\in[0,\infty)$ *which depends only on the data such that*

$$u^m(x, T^*)\in K.$$

PROOF. Let σ be Serrin's number of Lemma 2.
Put $\sigma_0=\min(1,\sigma)$ and fix $s\in(0,\sigma_0)$. Let $y\in\partial\Sigma_s=\{x\in\Omega:d(x)=s\}$.
By Theorem 4, there exists a time $T>0$ such that

$$u(x,t)>0 \qquad \text{for all } x\in\Omega \text{ and } t\geqq T.$$

Without loss of generality, we may assume $T=0$.
Set $\mu=\min\{u(x,0): x\in\bar{\Sigma}_{s/2}\}$.

Let $\bar{v}(x,t)=v(x-y,t;c,\tau)=(t+\tau)^{-\alpha}[g(|x-y|(t+\tau)^{-\beta};c)]_+ \qquad (6.3)$

where the parameters c and τ are chosen such that

support of $\bar{v}(x,0)=\bar{B}_{s/2}(y)$,

the positive numbers α and β are determined by the relation of $\alpha(m-1)+2\beta=1$ and $\alpha=2\beta(N+\delta)$, and $g(\eta;c)$ is the solution of Problem $(\mathrm{P})_\delta$.
Now we can find the parameters c and τ such that

$$c=\mu\tau^\alpha \text{ and } \tau=\{s/[2a(1)\mu^{(m-1)/2}]\}^2.$$

After some computation, we have

$$\begin{aligned} L\bar{v}&=\Delta(\bar{v}^m)+\sum_{k=1}^{N} b_k^0(\bar{v}^m)_{x_k}-\bar{v}_t \\ &=(t+\tau)^{-\alpha m-2\beta}\{(g^m)''+\frac{N-1}{\eta}(g^m)'+\sum_{k=1}^{N} b_k^0(x_k-y_k)\frac{1}{\eta}(g^m)'\} \\ &\quad+(t+\tau)^{-\alpha-1}\{\alpha g+\beta\eta g'\} \\ &\geqq(t+\tau)^{-\alpha-1}\{(g^m)''+\frac{N-1+\delta}{\eta}(g^m)'+\alpha g+\beta\eta g'\} \end{aligned}$$

$$=0 \qquad \text{in } \bar{Q}\cap\{(x,t):|x-y|\leqq a(c)(t+\tau)^{\beta}\}$$

as long as $a(c)(t+\tau)^{\beta}\leqq s<1$, because of $|x_k-y_k|\leqq|x-y|\leqq s<1$.
The support of $\bar{v}(x,t)$ expands as t increases and first touch when $(T^*+\tau)^{\beta}a(c)=s$, that is, $T^*=(2^{1/\beta}-1)\tau$.
By (ii) of Theorem 1 in §2, we have

$$u(x,t)\geqq\bar{v}(x,t) \qquad \text{in } H^*\equiv\Omega\times[0,T^*].$$

By virtue of $(g^m)'(a(c);c)\equiv\kappa<0$ and the argument in [3], we can obtain the conclusion (Cf. Proposition 5 in [3]).

Now, let $f_\lambda^*(x)$ be the unique solution of the problem

$$(\mathrm{VII})_\lambda \quad \begin{cases} \Delta(f^m)+\sum_{k=1}^{N}b_k^0\frac{\partial(f^m)}{\partial x_k}+\gamma f-\lambda f^m=0 & \text{in } \Omega \\ f=0 & \text{on } \partial\Omega \end{cases}$$

where $\gamma=1/(m-1)$ and $\lambda\geqq0$.

We remark that the existence and uniqueness of the positive solution of Problem $(\mathrm{VII})_\lambda$ $(\lambda\geqq0)$ can be proved by the same way as the proof of the case $b_k^0=\lambda=0$ $(k=1,\dots,N)$ in [3] (Cf. Proposition 1 in [3]).

Combining our Proposition 3 and Theorem 3 in [3], we can conclude the following result.

THEOREM 5. *Let u be the solution of Problem* (III) *with $n_k=m$ $(k=1,\dots,N)$ and $\lambda=0$.*

There exists a constant $C\in R^+$ which depends only on the data such that

$$|(1+t)^{\gamma}u(x,t)-f_0^*(x)|\leqq Cf_0^*(x)(1+t)^{-1} \qquad \text{in } \bar{Q} \tag{6.4}$$

where $f_0^*(x)$ is the solution of Problem $(\mathrm{VII})_0$.

Finally, combining our Proposition 3 and Theorems 12-16 in [4], we can prove the following asymptotic behaviour of solutions.

THEOREM 6. *Let u be the solution of Problem* (III) *with $n_k=m(k=1,\dots,N)$, $\lambda>0$ and $\beta=(p-m)/(m-1)$.*

(i) *If $\beta>0$, i.e. $p>m$, then*

$$|(1+t)^{\gamma}u(x,t)-f_0^*(x)|\leqq Cf_0^*(x)\omega_{\beta}(t) \qquad \text{in } \bar{Q}. \tag{6.5}$$

(ii) *If* $\beta=0$, *i.e.* $p=m$, *then*

$$|(1+t)^{r}u(x,t)-f_{\lambda}^{*}(x)|\leqq Cf_{\lambda}^{*}(x)(1+t)^{-1} \qquad in\ \bar{Q}. \tag{6.6}$$

(iii) *If* $\beta<0$, *i.e.* $1<p<m$, *then at every point* $x_0\in\Omega$

$$|(1+t)^{1/(p-1)}u(x_0,t)-\bar{c}\chi_{P(\infty)}| \text{ tends to } 0 \text{ as}\to\infty. \tag{6.7}$$

Here $f_{\lambda}^{*}(x)$ *is the positive solution of Problem* $(\text{VII})_{\lambda}$,

$$\omega_{\beta}(t)=\begin{cases}(1+t)^{-1} & \beta>1\\ (1+t)^{-1}[1+\log(1+t)] & \beta=1\\ (1+t)^{-\beta} & 0<\beta<1,\end{cases}$$

$\bar{c}=[\lambda(p-1)]^{-1/(p-1)}$, $\chi_{P(\infty)}$ *is the characteristic function of the set* $P(\infty)=\bigcup_{t\geq 0}P(t)$, *and the constant* C *depends only on the data.*

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References

[1] N.D. ALIKAKOS: *L^p bounds of solutions of reaction-diffusion equations*, Comm, Partial Diff. Eq. 4(8) (1979), p. 827-868.
[2] D.G. ARONSON, M.G. CRANDALL and L.A. PELETIER: *Stabilization of solutions of a degenerate nonlinear diffusion problem*, Nonlinear Analysis T.M.A. 6 (1982), p. 1001-1022.
[3] D.G. ARONSON and L.A. PELETIER: *Large time behaviour of solutions of the porous medium equation in bounded domains*, Jour. Diff. Eq. 39 (3) (1981), p. 378-412.
[4] M. BERTSCH, T. NANBU and L.A. PELETIER: *Decay of solutions of a dgenerate nonlinear diffusion equation*, Nonlinear Analysis T.M.A. 6 (1982), p. 539-554.
[5] Y. EBIHARA and T. NANBU: *Global classical solutions to* $u_t-\Delta(u^{2m+1})+\lambda u=f$, Jour. Diff. Eq. 38 (1980), p. 260-277.
[6] A. FRIEDMAN: Variational Principles and Free-boundary Problems, Willey and Sons (1982)
[7] B.H. GILDING: *A nonlinear degenerate parabolic equation*, Annali della Scuola Norm. Pisa 4 (3) (1977), p. 393-432.
[8] A.S. KALASHNIKOV: *On a nonlinear equation appearing in the theory of non-stationary filtration*, Trudy Seminar I.G. Petrovskii (1978), p. 137-146.
[9] R. KERSHNER: *Localization conditions for thermal perturbations in a semibounded moving medium with absorpton*, Moscow Univ. Math. Bull. 31 (4) (1976), p. 52-58.
[10] O.A. LADYZHENSKAYA, V.A. SOLONOKOV and N.N. URAL'CEVA: Linear and quasilinear equations of parabolic type, Translation M.M. 23, Amer. Math. Soc. (1968).
[11] M. NAKAO: *Existence and decay of global solutions of some nonlinear degenerate equations*, to appear.

[12] M. NAKAO: *On solutions to the initial-boundary value problem for an equation in the theory of infiltration*, Nonlinear Analysis T.M.A. **8** (1984), p. 39-48.
[13] T. NANBU: *The Cauchy problem for quasi-linear degenerate parabolic equations*, Math. Rep. Coll. Gen. Ed. Kyushu Univ. **11** (1977), p. 83-96.
[14] L.A. PELETIER: *The porous media equation*, in Application of Nonlinear Analysis in the Physical Science (Eds. H. Amann, N. Bazley and K. Kirchgassner), Pitman (1981), p. 229-241.
[15] J. SERRIN: *The problem of Dirichlet for quasilinear elliptic differential equations with many independent variables*, Philos. Trans. Roy. Soc. London Ser. A **264** (1969), p. 413-496.