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Existence of strong and smooth periodic solutions of some nonlinear evolution equations

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Introduction

In this paper we are concerned with the existence of strong or smooth periodic solutions of some nonlinear evolution equations. 'Smooth' means that the solutions under consideration are smoother than the so-called strong solutions and become classical under some reasonable situations.

The first equation we consider is a semilinear parabolic equation of higher order;

$$(E_1) \qquad \frac{\partial}{\partial t}u + Au + Bu = f(x, t), \text{ on } \Omega\times R,$$

$$D^{\gamma}u|_{\partial\Omega}=0 \quad \text{for} \quad 0\leqq|\gamma|\leqq m-1,$$

where Ω is a bounded domain in R^N with smooth boundary $\partial\Omega$ and A, B are operators given by

$$(1) \qquad Au = \sum_{\substack{0\leqq|\alpha|\leqq m \\ 0\leqq|\beta|\leqq m}} (-1)^{|\alpha|} D^{\alpha}(a_{\alpha,\beta}(x)D^{\beta}u)$$

and

$$(2) \qquad Bu = \sum_{i=1}^{N} \frac{\partial}{\partial x^i} h_i(u) + g(u).$$

Throughout this paper we set $|\alpha| = \sum_{i=1}^{N} \alpha_i$ and $D^{\alpha} = \prod_{i=1}^{N} (\frac{\partial}{\partial x^i})^{\alpha_i}$ for $\alpha = (\alpha_1, \alpha_2, \cdots \alpha_N)$. We assume that the coefficients $a_{\alpha,\beta}(x)$ are smooth functions on $\bar{\Omega}$ and A is uniformly elliptic, $h_i(u), g(u)$ are smooth functions with appropriate growth order and $f(x, t)$ is a smooth function with period ω in t.

When $m=1$, the existence problem of ω-periodic solution for (E_1) have been investigated by many authors (Bange [3], Biroli [5], Fife [8], Kusano [16], Nakao [21], Nakao-Nanbu [25], Prodi [26], Smulev [30], etc.), while the case $m>1$ seems not to be considerd sufficiently. If $h_i(u)\equiv 0$

the existence of strong solution is proved similarly as in [5] (see also [21]) even for $m>1$, and moreover the existence of classical solution is shown rather easily if we assume $f(t)$ is small in a certain sense (cf. [25]).

Our purpose is to show the existence of strong and smooth periodic solution for (E_1) without any smallness condition on $f(t)$.

In fact, we could treat more general nonlinear operator than (2), say,

$$Bu=\sum_{i=1}^{N}\sum_{|\alpha|\leq m-1}\frac{\partial}{\partial x^i}D^\alpha h_{\alpha,i}(D^\alpha u)+\sum_{|\alpha|\leq m-1}D^\alpha g(D^\alpha u).$$

However the calculation would become too complicated for such general form, and we restrict ourselves to the simple one (2) in order to clearify the essential feature of argument.

The second equation is the modified Navier-Stokes equations

$$(E_2)\qquad \frac{\partial}{\partial t}u+(-\Delta)^{1+2\varepsilon}u+(u\cdot\nabla)u=f-\nabla p \qquad \text{on } \Omega\times R,\ (\varepsilon>0),$$

$$\text{div } u=0 \quad \text{and} \quad u|_{\partial\Omega}=0,$$

where Ω is a bounded domain in R^3 with smooth boundary $\partial\Omega$, $u=(u^1,u^2,u^3)$ is the velocity field, p is the pressure and $f=(f^1,f^2,f^3)$ is the external force. The precise definition of the operator $(-\Delta)^{1+2\varepsilon}$ will be given later.

We are interested again in the existence of smooth ω-periodic solution for (E_2) when f is smooth and ω-periodic in t. If $\varepsilon=0$, (E_2) is the usual system of Navier-Stokes equations and has been considered by a number of authors (see Ladyzhenskaia [17], and the references cited there). But the existence problem of global strong solution is not still solved. While, Lions [18] introduced the modified problem (E_2) with $\varepsilon>0$ to prove that if $\varepsilon\geq\frac{1}{8}$, the problem (E_2) with initial value $u(0)=u_0$ has a unique global solution in a certain function space. Bardos et al [4] also considered related problems.

Our object concerning (E_2) is to prove the existence of strong and smooth ω-periodic solutions when $\varepsilon>\frac{1}{8}$. If we assume $f(t)$ is small it is not difficult to show the existence of smooth ω-periodic solution even for $\varepsilon=0$ (cf. Shinbrot [29]), while here we do not require any smallness condition on f.

It is true that the problem (E_2) is very artificial, but this equation may be regarded as a model which describe the motion of fluid with strong viscosity. We adapt this problem mainly for the demonstration of our technique to get smooth periodic solution.

Finally we treat the following semilinear wave equation

$$(E_3)\qquad \begin{cases} \dfrac{\partial^2}{\partial t^2}u+Au+\nu\dfrac{\partial}{\partial t}u+g(u)=f(x,t), \\ D^{\gamma}u|_{\partial\Omega}=0 \qquad \text{for } |\gamma|\leqq m-1, \end{cases}$$

where ν is a positive constant and A is the operator defined in (1).

If f is small or if $g(u)$ is replaced by $\varepsilon g(u)$ (ε; small) the existence of smooth ω-periodic solution for (E_3) is shown rather easily under appropriate conditions on g. (cf. Ficken & Fleishman [7], K. Masuda [19], Wahl [32], Nakao [23], Matsumura [20]). Here we want to prove such solution without smallness condition on f. (For the case $N=1$, $m=1$ see [24])

Unfortunately, however, our result excludes the important case; $m=1$, $g(u)=u^3$ and $N=3$, and this case remains still open. For generalized solutions of the related problems see Amerio-Prouse [1], Kakita [13], Clements [6], Nakao [22], etc.

We apply a common method to the problem (E_1), (E_2), and (E_3). That is, starting from energy inequalities which are given directly from the assumptions we derive the a priori estimates for higher order derivatives of solutions. The existence of strong or smooth solutions follows from these estimates. Thus the main purpose of this paper is to show how to get such estimates. During the preparation of this work we learned the paper [15] by Kielhöfer, where the same idea is applied to the initial-boundary value problem for nonlinear parabolic equations. In [15] the theory of linear semi-groups is used extensively, while here we employ the Galerkin's method. We note also that the treatment of periodicity problems is, in general, more delicate than the initial-boundary value problems.

Notation

Let X be a Banach space. Then we denote by $L^p(\omega;X)$ $(1\leqq p\leqq\infty)$ the set of ω-periodic X-valued measurable functions f on R such that

$$\|f\|_{L^p(\omega;X)}=\left(\int_0^{\omega}\|f(t)\|_X^p\,dt\right)^{\frac{1}{p}}<\infty$$

with trivial modification for $p=\infty$. For nonnegative integer k, and $p(1\leqq$

$p \leqq \infty$), $W^{k,p}(\omega;X)$ denotes the set of functions f which together with the derivatives up to the order k belongs to $L^p(\omega;X)$. $f \in C^{k,\alpha}(\omega;X)$ (k; non-negative integer, $0 \leqq \alpha \leqq 1$) if and only if f is an X-valued ω-periodic function with the continuous derivatives up to the order k and $\frac{d^k}{dt^k}f$ is Hölder continuous with exponent α. Corresponding norms are defined in usual way for above function spaces. We denote L^p-norm on Ω by $\|\cdot\|_p$, $\|\cdot\|$ being L^2-norm on Ω. Finally we note that the functions considered are all real valued.

Chapter 1. Semi-linear parabolic equations

1.1 Assumptions and results

Let

$$Au = \sum_{\substack{0 \leqq |\alpha| \leqq m \\ 0 \leqq |\beta| \leqq m}} (-1)^{|\alpha|} D^\alpha (a_{\alpha,\beta}(x)) D^\beta u.$$

Then we make the following assumptions.

H_1 : $a_{\alpha,\beta}$'s are sufficiently smooth, A is formally selfadjoint and coercive on $\mathring{H}_m$,

i.e.

$$C_0^2 \|u\|_{\mathring{H}_m}^2 \leqq \langle Au, u \rangle \leqq C_1^2 \|u\|_{\mathring{H}_m}^2 \quad \text{for } u \in \mathring{H}_m(\Omega),$$

where $C_i (i=0,1)$ are positive constants and we set

$$\langle Au, u \rangle = \sum_{\alpha,\beta} \int_\Omega a_{\alpha,\beta}(x) D^\alpha u(x) D^\beta u(x) dx.$$

H_2 : $g(u)$ satisfies

$$|g(u)| \leqq C(1+|u|^{r_0}),$$
$$-(b_1 + b_0|u|^{\nu_0}) \leqq g(u)u \leqq C(1+|u|^{r_0+1})$$

for some positive constants b_0, b_1, C and ν_0, r_0 with $0 < \nu_0 < 2$ and

$$0 < r_0 \begin{cases} < \dfrac{N+2m}{N-2m} & \text{if } N > 2m \\ < \infty & \text{if } N \leqq 2m. \end{cases}$$

Further assumptions will be made for $g(u)$. Our results of this chapter are as follows.

THEROEM 1.1 (*existence of strong solution*)
In addition to the hypotheses H_1-H_2 *suppose that* $f\in W^{1,2}(\omega;L^2)$, $g(u)\in C^1(R)$, $h_i(u)\in C^1(R)$ $(i=1,2,\cdots,N)$ *and*

$$-g'(u)\leqq C(1+|u|^{r_1})$$
$$|h_i^{(k)}(u)|\leqq C(1+|u|^{\gamma_k}) \qquad (k=0,1)$$

with $C>0$ *and* r_k, γ_k *such that*

$$0\leqq\gamma_0\leqq\frac{2m+N}{N},\qquad 0\leqq r_0\begin{cases}<\dfrac{N+2m}{N-2m} & \text{if } N>2m\\ <\infty & \text{if } N\leqq 2m,\end{cases}$$

$$0\leqq\gamma_1\begin{cases}<\dfrac{4m-2}{N-2m+2} & \text{if } N>2m\\ <2m-1 & \text{if } N\leqq 2m\end{cases}$$

and

$$0\leqq r_1\begin{cases}<\dfrac{4m}{N-2m+2} & \text{if } N>2m\\ <2m & \text{if } N\leqq 2m.\end{cases}$$

Then the problem (E_1) *has an* ω*-periodic solution* $u(t)$ *such that*

$$u\in W^{1,\infty}(\omega;L^2)\cap W^{1,2}(\omega;\mathring{H}_m)\cap L^\infty(\omega;H_{2m}\cap\mathring{H}_m)$$

and

$$\operatorname*{ess.sup}_t(\|u_t(t)\|+\|u(t)\|_{H_{2m}})+\int_0^\omega\|u_t(s)\|^2_{\mathring{H}_m}ds$$
$$\leqq C(M_1)<\infty,$$

where we set $M_i=\|f\|_{W^{i,2}(\omega;L^2)}$ $(i=1,2)$ *and generally* $C(M)$ *denotes positive constants depending on* M.

THEOREM 1.2 (*existence of classical solution*)
Let $N<4m$, *and in addition to the assumtions in theorem* 1.1, *suppose* $f\in W^{2,2}(\omega;L^2)\cap W^{1,\infty}(\omega;\mathring{H}_m\cap H_{2m})$ *and* $g, h_i\in C^2(R)$ $(i=1,2,\cdots,N)$.
Then the solution $u(t)$ *in theorem* 1.1 *belongs to*

$$W^{3,2}(\omega;L^2)\cap W^{2,2}(\omega;H_{2m}\cap\mathring{H}_m)\cap W^{1,\infty}(\omega;H_{4m})$$

and, in particular,

$$u\in C^{1,\frac{1}{2}}(\omega;C(\bar{\Omega}))\cap C^{0,1}(\omega;C^{2m}(\bar{\Omega})).$$

The corresponding norms for u are dominated by $C(M_2)+C\sup_t\|f(t)\|_{H_{2m}}$.

Remark. The assumtion $N<4m$ in theorem 1.2 is made mainly for simplicity. In fact we could consider the case $N\geqq 4m$ (cf. Chap. 3).

1.2 Approximate solutions and the proof of theorem 1.1

We denote by A also the Friedrichs extension in L^2 of the operator A with the Dirichlet condition. It is known that $D(A^{\frac{1}{2}})=\mathring{H}_m$ and $D(A)=\mathring{H}_m\cap H_{2m}$. Let w_j(j=1, 2, ⋯) be the basis of L^2 consisting of the eigen vectors of A and consider the system of ordinary differential equations:

$$\text{(1.1)}\qquad \begin{cases}(u_{nt}(t)+Au_n(t)+Bu_n(t),w_j)=(f(t),wj), \quad j=1,2,\cdots n,\\ u_n(t)=u_n(t+\omega)\end{cases}$$

where, $u_n(t)=\sum_{i=1}^{n}\alpha_{i,n}(t)w_i$.

Using the standard argument due to Leray-Schauder's theory we see that the problem (1.1) has a solution $\alpha_{i,n}(t)\in C^1(\omega;R)$ with $\alpha_{i,n}\in L^2(0,\omega)$ $(i=1,2,\cdots,n)$ if we can show

$$\text{(1.2)}\qquad \|u_n(t)\|\leqq C$$

for all possible solutions of (1.1) with Bu_n replaced by $\lambda\, Bu_n$, $0\leqq\lambda\leqq 1$, where C is a constant independent of λ. But the estimate (1.2) follows immediately from lemma 1.1 below.

LEMMA 1.1 *Let $u_n(t)$ be a solution of* (1.1). *Then,*

$$\text{(1.3)}\qquad \int_0^\omega\|u_n(s)\|^2_{\mathring{H}_m}ds\leqq C\bar{M}_0^2$$

and

$$\text{(1.4)}\qquad \sup_t\|u_n(t)\|\leqq C\bar{M}_0$$

where, $\bar{M}_0^2=(M_0^2+b_1+b_0^{\frac{2}{2-\nu_0}})$.

Proof. By (1.1) we have easily

$$\text{(1.5)}\qquad \frac{1}{2}\frac{d}{dt}\|u_n\|^2+\|u_n\|^2_{\sqrt{A}}+\int_\Omega g(u_n)u_n dx$$

$$+\sum_{i=1}^{N}\int_{\Omega}h_i(u_n)\frac{\partial}{\partial x^i}u_n dx=\int_{\Omega}fu_n dx,$$

where we set $\|u\|_{\sqrt{A}}=\langle Au, u\rangle=(Au, u)$. Note that the last term of the lefthand side of (1.5) equals to zero, and integrating (1.5) over $[0, \omega]$ we have from the assumption on g

$$\int_0^{\omega}\|u_n(s)\|_{\sqrt{A}}^2 ds\leqq\int_0^{\omega}\int_{\Omega}(|f|\,|u_n|+b_1+b_0|u_n|^{\nu_0})dxds$$

$$\leqq C\int_0^{\omega}(\|f(s)\|^2+b_1+b_0^{\frac{2}{2-\nu_0}})ds,$$

$$+\frac{1}{2}\int_0^{\omega}\|u_n(s)\|_{\sqrt{A}}^2 ds,$$

which implies (1.3).

From (1.3) we see that there exists $t^*\in[0, \omega]$ such that

$$\|u_n(t^*)\|^2\leqq C\|u_n(t^*)\|_{\sqrt{A}}^2\leqq C\bar{M}_0^2$$

and integrating (1.5) from t^* to $t+\omega$ $(t\in[0, \omega])$

$$\|u_n(t)\|^2\leqq\|u_n(t^*)\|^2+\int_{t^*}^{t+\omega}\int_{\Omega}(|f|\,|u_n|+b_1+b_0|u_n|^{\nu_0})dxds$$

$$+\int_{t^*}^{t+\omega}\|u_n(s)\|_{\sqrt{A}}^2 ds\leqq C\bar{M}_0^2 \qquad \text{for } \forall t\in[0, \omega].$$

q. e. d.

We shall derive the estimates of the higher derivatives of $u_n(t)$.

LEMMA 1.2 *Assume that* $0\leqq\eta_0\leqq\dfrac{2m+N}{N}$.

Then,

$$\text{(1.6)} \qquad \|u_n(t)\|_{H_m}^2+\int_0^{\omega}\|u_{nt}(t)\|^2 dt$$

$$\leqq C(M_0)\,(1+\int_0^{\omega}\|u_{nt}\|_{H_m}^2 dt)^{\frac{1}{m+1}}.$$

PROOF. Multiplying (1.1) by $\dot{\alpha}_{i,n}(t)$, summing up over i and integrating we have

$$\text{(1.7)} \qquad \int_0^{\omega}\|u_{nt}\|^2 dt+\frac{1}{2}\int_0^{\omega}\frac{d}{dt}(Au_n(t), u_n(t))dt$$

$$+\int_0^{\omega}\int_{\Omega}g(u_n)u_{nt}dxdt+\int_0^{\omega}\int_{\Omega}\sum_{i=1}^{N}\Big(\frac{\partial}{\partial x^i}h_i(u_n)\Big)u_{nt}dxdt$$

$$=\int_0^\omega\int_\Omega f u_{nt}dxdt$$

and

$$\int_0^\omega \|u_{nt}\|^2dt \leqq \int_0^\omega\int_\Omega \Big(\sum_{i=1}^N |h_i(u_n)|\,|\nabla u_{nt}| + |f|\,|u_{nt}|\Big)dxdt$$

$$\leqq C\int_0^\omega\int_\Omega (1+|u_n|^{\gamma_0})\,|\nabla u_{nt}|\,dxdt + \frac{1}{2}M_0^2 + \frac{1}{2}\int_0^\omega \|u_{nt}\|^2dt$$

and hence, using Nirenberg-Gagliardo inequality (cf. Henry[34])

$$\int_0^\omega \|u_{nt}\|^2dt \leqq C\Big(\int_0^\omega (1+\|u_n\|_{2\gamma_0}^{\gamma_0})\|u_{nt}\|_{\dot{H}_m}^{\frac{1}{m}}\|u_{nt}\|^{\frac{m-1}{m}}dt + M_0^2\Big)$$

$$\leqq C\Big\{\Big[\int_0^\omega (1+\|u_n\|_{2\gamma_0}^{\gamma_0})^{\frac{2m}{m+1}}\|u_{nt}\|_{\dot{H}_m}^{\frac{2}{m+1}}\Big]^{\frac{m+1}{2m}}\Big[\int_0^\omega \|u_{nt}\|^2dt\Big]^{\frac{m-1}{2m}} + M_0^2\Big\}$$

which implies

(1.8) $$\int_0^\omega \|u_{nt}\|^2dt$$

$$\leqq C\Big(\int_0^\omega (1+\|u_n\|_{2\gamma_0}^{\gamma_0})^{\frac{2m}{m+1}}\|u_{nt}\|_{\dot{H}_m}^{\frac{2}{m+1}}dt + M_0^2\Big).$$

By the way

$$\|u_n\|_{2\gamma_0} \leqq C\|u_n\| \leqq C\bar{M}_0 \qquad \text{if } \gamma_0 \leqq 1$$

and

(1.9) $$\|u_n\|_{2\gamma_0} \leqq C\|u_n\|^{(1-\theta)}\|u_n\|_{\dot{H}_m}^{\theta} \leqq C(M_0)\|u_n\|_{\dot{H}_m}^{\theta} \qquad \text{if } \gamma_0 > 1$$

with $\theta = \frac{N}{m}\Big(\frac{1}{2} - \frac{1}{2\gamma_0}\Big)$.

From (1.8) and (1.9) we have easily

(1.10) $$\int_0^\omega \|u_{nt}\|^2dt \leqq C(M_0)\Big\{\Big[\int_0^\omega (1+\|u_n\|_{\dot{H}_m})^{2\theta\gamma_0}dt\Big]^{\frac{m}{m+1}}\Big[\int_0^\omega \|u_{nt}\|_{\dot{H}_m}^2dt\Big]^{\frac{1}{m+1}} + 1\Big\}$$

$$\leqq C(M_0)\Big(1+\int_0^\omega \|u_{nt}\|_{\dot{H}_m}^2dt\Big)^{\frac{1}{m+1}},$$

where we have used the fact $2\theta\gamma_0 \leqq 2$ and (1.3). To estimate $\|u_n(t)\|_{\dot{H}_m}$ we note that by (1.3)

(1.11) $$\|u_n(t^*)\|_{\dot{H}_m}^2 \leqq C\bar{M}_0^2 \qquad \text{for some } t^* \in [0, \omega]$$

and by an equality similar to (1.7)

$$\|u_n(t)\|^2_{\sqrt{A}}+\int_\Omega\int_0^{u_n(t)} g(\eta)\,d\eta dx$$
$$=\|u_n(t^*)\|^2_{\sqrt{A}}+\int_\Omega\int_0^{u_n(t^*)} g(\eta)\,d\eta dx-\int_{t^*}^{t+\omega}\|u_{nt}\|^2dt$$
$$+\int_{t^*}^{t+\omega}\int_\Omega\Big(fu_{nt}+\sum_{i=1}^N\Big(\frac{\partial}{\partial x^i}h_i(u_n)\Big)u_{nt}dxdt$$

for $\forall t\in[0,\omega]$. The same argument obtaining (1.10), the inequality (1.11) and the assumption H_2 yield

$$(1.12)\qquad \|u_n(t)\|^2_{\sqrt{A}}\leqq C(M_0)\Big(1+\int_0^\omega\|u_{nt}\|^2_{\mathring{H}_m}dt\Big)^{\frac{1}{m+1}}. \qquad \text{q. e. d.}$$

LEMMA 1.3 *Under the hypotheses of theorem* 1.1 *we have*

$$\int_0^\omega\|u_{nt}\|^2_{\mathring{H}_m}dt+\|u_n(t)\|^2_{\mathring{H}_m}+\|u_{nt}(t)\|^2\leqq C(M_1).$$

PROOF. Differentiating the equation of (1.1),

$$(1.13)\qquad \Big(D_t^2u_n+Au_{nt}+g'(u_n)u_{nt}+\sum_{i=1}^N\frac{\partial}{\partial x^i}(h_i'(u_n)u_{nt}),\quad w_j\Big)$$
$$=(f_t,w_j),\qquad j=1,2,\cdots,n,$$

where we set $D_t^k=\dfrac{\partial^k}{\partial t^k}$ $(k=1,2,\cdots)$. From this it follows immediately that

$$(1.14)\qquad \int_0^\omega\|u_{nt}\|^2_{\sqrt{A}}dt=\int_0^\omega\int_\Omega\Big\{-g'(u_n)\,|u_{nt}|^2+\sum_{i=1}^N\Big(h_i'(u_n)u_{nt}\frac{\partial}{\partial x^i}u_{nt}\Big)$$
$$+f_tu_{nt}\Big\}dxdt.$$

The first two terms of the right hand side of (1.14) are treated as follows.

$$\int_0^\omega\int_\Omega-g'(u_n)\,|u_{nt}|^2dxdt$$
$$\leqq C\int_0^\omega\int_\Omega(1+|u_n|^{r_1})\,|u_{nt}|^2dxdt$$
$$\leqq C\Big\{\int_0^\omega\|u_{nt}\|^2dt+\sup_t\|u_n(t)\|^{r_1}_{\mathring{H}_m}\int_0^\omega\|u_{nt}\|^{2(1-\theta_1)}\|u_{nt}\|^{2\theta_1}_{\mathring{H}_m}dt\Big\}$$

$$\text{with }\ \theta_1=\begin{cases}\dfrac{(N-2m)r_1}{4m} & \text{if } N>2m\\ \text{close to } 0 & \text{if } N=2m\\ 0 & \text{if } N<2m\end{cases}$$

and by (1.6)

$$(1.15)\qquad \leq C(M_0)\ \{1+\int_0^\infty \|u_{nt}\|^2_{\mathring{H}_m}\|dt\}^{\nu_1}$$

with $\nu_1=\frac{r_1}{2(m+1)}+\frac{1-\theta_1}{m+1}+\theta_1=\frac{r_1+2(1+m\theta_1)}{2(m+1)}$

$$(1.16)\qquad \int_0^\infty\int_\Omega \sum_{i=1}^N |h_i'(u_n)u_{nt}\frac{\partial}{\partial x^i}u_{nt}|\,dxdt$$

$$\leq C\int_0^\infty\int_\Omega (1+|u_n|^{\eta_1})\,|u_{nt}|\,|\nabla u_{nt}|\,dxdt$$

$$\leq C\left\{\int_0^\infty \|u_{nt}\|\|\nabla u_{nt}\|dt+\sup_t \|u_n(t)\|^{\eta_1}_{\mathring{H}_m}\int_0^\infty \|u_{nt}|\nabla u_{nt}|\,\|_q dt\right\}$$

with $q=\begin{cases}\dfrac{2N}{2N-(N-2m)\eta_1} & \text{if } N>2m\\ \text{closo to } 1 & \text{if } N=2m\\ 1 & \text{if } N<2m.\end{cases}$

If q<2,

$$\int_0^\infty \|u_{nt}|\nabla u_{nt}|\,\|_q dt\leq C\int_0^\infty \|u_{nt}\|\,\|\nabla u_{nt}\|_{\frac{2q}{2-q}}dt$$

$$\leq C\int_0^\infty \|u_{nt}\|\,\|u_{nt}\|^{1-2\theta_2}\|u_{nt}\|^{2\theta_2}_{\mathring{H}_m}\ dt$$

$$(1.17)\qquad \leq C\left\{\int_0^\infty \|u_{nt}\|^2 dt\right\}^{1-\theta_2}\left\{\int_0^\infty \|u_{nt}\|^2_{\mathring{H}_m}dt\right\}^{\theta_2}$$

with $\theta_2=\begin{cases}\dfrac{1}{4m}(2+(N-2m)\eta_1) & \text{if } N>2m\\ \text{close to } \dfrac{1}{2m} & \text{if } N=2m\\ \dfrac{1}{2m} & \text{if } N<2m\end{cases}$

If q≧2, we have $N>2m$ and

$$(1.18)\qquad \int_0^\infty \|u_{nt}|\nabla u_{nt}|\,\|_q dt$$

$$\leq C\int_0^\infty \|u_{nt}\|_{\frac{2N}{N-2m}}\|\nabla u_{nt}\|_{\frac{2N}{N+2m-(N-2m)\eta_1}}dt$$

$$\leq C\left(\int_0^\infty \|u_{nt}\|^2_{\mathring{H}_m}dt\right)^{\frac{1}{2}}\left(\int_0^\infty \|u_{nt}\|^2 dt\right)^{\frac{1-\bar\theta_2}{2}}\left(\int_0^\infty \|u_{nt}\|^2_{\mathring{H}_m}dt\right)^{\frac{\bar\theta_2}{2}},$$

with $\bar\theta_2=\frac{1}{2m}(2-2m+(N-2m)\eta_1)$.

From (1.16)–(1.18) we have

$$(1.19)\qquad \int_0^{\omega}\int_{\Omega}\sum_{i=1}^{N}|h_i'(u_n)u_{nt}\frac{\partial}{\partial x^i}u_{nt}|\,dxdt$$
$$\leqq C(M_1)(1+\int_0^{\omega}\|u_{nt}\|^2_{\dot{H}_m}dt)^{\nu_2}$$

with $\nu_2=\dfrac{\eta_1+m\bar{\theta}_2+m+2}{2(m+1)}$.

From (1.14), (1.15), and (1.19) we have

$$(1.20)\qquad \int_0^{\omega}\|u_{nt}\|^2_{\dot{H}_m}dt\leqq C(M_1)(1+\int_0^{\omega}\|u_{nt}\|^2_{\dot{H}_m}dt)^{\mathrm{Max}(\nu_1,\nu_2)}.$$

Now it is easy to check that our assumptions on r_1 and η_1 implies $\max(\nu_1, \nu_2)<1$ and we conclude

$$(1.21)\qquad \int_0^{\omega}\|u_{nt}\|^2_{\dot{H}_m}dt\leqq C(M_1).$$

By lemma 1.2 the same estimate holds for $\|u_n(t)\|^2_{\dot{H}_m}$ and $\|u_{nt}(t)\|^2$. q.e.d.

We need one more lemma for the proof of theorem 1.1.

LEMMA 1.4 *Under the hypotheses of theorem* 1.1,

$$(1.22)\qquad \|u_n(t)\|_{H_{2m}}\leqq C(M_1)\qquad \textit{for any } t.$$

Proof. By (1.1) and lemma 1.3

$$(1.23)\qquad \|Au_n(t)\|\leqq\|u_{nt}(t)\|+\|g(u_n(t))\|+\|\sum_{i=1}^{N}h_i'(u_n(t))\frac{\partial}{\partial x^i}u_n(t)\|$$
$$+\|f(t)\|$$
$$\leqq C(M_1)+C(\|u_n(t)\|^{r_0}_{2r_0}+\||u_n|^{\eta_1}|\nabla u_n|\|).$$

But, since $r_0<\dfrac{N+2m}{N-2m}$, if $N>2m$,

$$\|u_n\|^{r_0}_{2r_0}\leqq\|u_n\|^{r_0(1-\theta_0)}_{\dot{H}_m}\|u_n\|^{r_0\theta_0}_{H_{2m}}\leqq C(M_1)\|u_n\|^{r_0\theta_0}_{H_{2m}}$$

with $\theta_0=\dfrac{1}{2m}(N-2m-\dfrac{N}{r_0})$ and $\theta_0r_0<1$. Also, taking $1\leqq p, q\leqq\infty$, such that $\dfrac{1}{p}+\dfrac{1}{q}=1$ and

$$q=\begin{cases}\text{arbitrarily close to } \dfrac{N}{N+2-4m} & \text{if } N>4m-2\\ \text{arbitrarily large} & \text{if } N\leqq 4m-2\end{cases}$$

we have

$$\||u_n|^{\eta_1}|\nabla u_n|\|\leqq\|u_n\|_{2p\eta_1}^{\eta_1}\|\nabla u_n\|_{2q}$$

$$\leqq C\|u_n\|_{\mathring{H}_m}^{\eta_1}\|u_n\|_{\mathring{H}_m}^{1-\theta_1}\|u_n\|_{H_{2m}}^{\theta_1}$$

$$\leqq C(M_1)\|u_n\|_{H_{2m}}^{\theta_1} \quad , \tag{1.24}$$

with $0<\theta_1=\frac{1}{m}(1-\frac{N}{2q}+\frac{N-2m}{2})<1.$

From (1.23), (1.24) and the fact $\|u_n(t)\|_{H_{2m}}\leqq C\|Au_n(t)\|$

we obain (1.22) $\|u_n(t)\|_{H_{2m}}\leqq C(M_1)$. q.e.d.

Finally we must show a subsequence of $\{u_n(t)\}$ converges to a desired solution in theorem 1.1. After the estimates in lemmas 1.3 and 1.4 have been established the proof of theorem 1.1 is straightforward by a standard compactness argument (cf. Lions [18], Biroli [5], etc.) and we omit the details.

Remark. The assumption on r_0 is used only for the derivation of (1.22). Therefore we can assert:

THEOREM 1.1′ *Under the hypotheses of theorem* 1.1 *except for the one concerning* r_0, *the problem* (E_1) *has an* ω*-periodic solution* $u(t)$ *belonging to* $W^{1,\infty}(\omega;L^2)\cap W^{1,2}(\omega;\mathring{H}_m)$.

1.3 Proof of theorem 1.2

We shall carry out further estimation of $u_n(t)$ and give the proof of theorem 1.2.

LEMMA 1.5 *Under the assumptions of theorem* 1.2 *we have*

$$\int_0^\infty(\|D_t^2u_n\|_{\mathring{H}_m}^2+\|u_{nt}\|_{H_{2m}}^2)dt\leqq C(M_2).$$

PROOF. By the equations (1.13) we get easily

$$\int_0^\omega \|D_t^2 u_n\|^2 dt \leqq C\int_0^\omega\int_\Omega (1+|u_n|^{\gamma_1})|u_{nt}|u_{ntt}|dxdt \tag{1.25}$$
$$+C\int_0^\omega\int_\Omega (1+|u_n|^{\gamma_1})|u_{nt}||\nabla u_{ntt}|dxdt$$
$$+M_1^2.$$

Since $N<4m$, $\|u\|_\infty \leqq C\|u\|_{H_{2m}}$ for $u\in \mathrm{H}_{2m}$,

and

$$\int_0^\omega\int_\Omega |u_n|^{\gamma_1}|u_{nt}||u_{ntt}|\,dxdt$$

$$\leqq C\sup_t \|u_n(t)\|_{H_{2m}}^{\gamma_1}\Big(\int_0^\omega \|u_{nt}\|^2 dt\Big)^{\frac{1}{2}}\Big(\int_0^\omega \|D_t^2 u_n\|^2 dt\Big)^{\frac{1}{2}}$$

$$\leqq C(M_1)+\frac{1}{4}\int_0^\omega \|D_t^2 u_n\|^2 dt. \qquad \text{(by lemma 1.3, 1.4).} \tag{1.26}$$

Also,

$$\int_0^\omega\int_\Omega |u_n|^{\gamma_1}|u_{nt}||\nabla u_{ntt}|dxdt$$

$$\leqq C\sup_t \|u_n(t)\|_{H_{2m}}^{\gamma_1}\Big(\int_0^\omega \|u_{nt}\|^2 dt\Big)^{\frac{1}{2}}\Big(\int_0^\omega \|\nabla u_{ntt}\|^2\, dt\Big)^{\frac{1}{2}}$$

$$\leqq C(M_1)\Big(\int_0^\omega \|u_{ntt}\|^{\frac{2(m-1)}{m}}\|u_{ntt}\|_{\overset{\circ}{H}_m}^{\frac{2}{m}} dt\Big)^{\frac{1}{2}}$$

$$\leqq C(M_1)\Big(\int_0^\omega \|D_t^2 u_n\|^2 dt\Big)^{\frac{m-1}{2m}}\Big(\int_0^\omega \|D_t^2 u_n\|_{\overset{\circ}{H}_m}^2 dt\Big)^{\frac{1}{2m}}$$

$$\leqq C(M_1)\Big(\int_0^\omega \|D_t^2 u_n\|_{\overset{\circ}{H}_m}^2 dt\Big)^{\frac{1}{m+1}}+\frac{1}{4}\int_0^\omega \|D_t^2 u_n\|^2 dt. \tag{1.27}$$

From (1.25)-(1.27) we have

$$\int_0^\omega \|D_t^2 u_n\|^2 dt \leqq C(M_1)\Big(1+\int_0^\omega \|D_t^2 u_n\|_{\overset{\circ}{H}_m}^2 dt\Big)^{\frac{1}{m+1}}. \tag{1.28}$$

This inequality corresponds to (1.10). We have also by (1.13)

$$\|Au_{nt}\| \leqq \|D_t^2 u_n\|+C(M_1)(\|u_{nt}\|+\|\nabla u_n\|\|u_{nt}\|+\|\nabla u_{nt}\|+1)$$

and, using lemma 1.3,

$$\int_0^\omega \|Au_{nt}\|^2 dt \leqq C(M_1)\Big(1+\int_0^\omega \|D_t^2 u_n\|^2\, dt\Big)$$

$$(1.29) \qquad \leqq C(M_1)(1+\int_0^\infty \|D_t^2 u_n\|_{\mathring{H}_m}^2 dt)^{\frac{1}{m+1}}.$$

To estimate $\int_0^\infty \|D_t^2 u_n\|_{\mathring{H}_m}^2 dt$ we utilize the two time differentiated equations:

$$(D_t^3 u_n + AD_t^2 u_n + g''(u_n) u_{nt}{}^2 + g'(u_n) D_t^2 u_n$$
$$+ \sum_{i=1}^N \frac{\partial}{\partial x^i}(h_i''(u_n) u_{nt}{}^2 + h_i'(u_n) D_t^2 u_n), \qquad w_j)$$
$$(1.30) \qquad = (f_{tt}, w_j), \qquad j=1, 2, \cdots n.$$

From this, we get easily

$$(1.31) \qquad \int_0^\infty \|D_t^2 u_n\|_{\mathring{H}_m}^2 dt$$
$$\leqq C(M_1)\int_0^\infty\int_\Omega (|u_{nt}|^2 |D_t^2 u_n| + |D_t^2 u_n|^2 + |u_{nt}|^2 |\nabla D_t^2 u_n|$$
$$+ |D_t^2 u_n| |\nabla D_t^2 u_n|) dx dt + CM_2^2.$$

Each term of the right hand side of (1.31) is treated as follows.

$$\int_0^\infty\int_\Omega |u_{nt}|^2 |D_t^2 u_n| dx dt$$
$$\leqq \int_0^\infty \|u_{nt}\|^{2-\frac{N}{4m}} \|u_{nt}\|_{H_{2m}}^{\frac{N}{4m}} \|D_t^2 u_n\| dt$$
$$\leqq C(M_1)(\int_0^\infty \|u_{nt}\|_{H_{2m}}^2 dt)^{\frac{N}{8m}} (\int_0^\infty \|D_t^2 u_n\|^2 dt)^{\frac{1}{2}}$$
$$\leqq C(M_1)(1+\int_0^\infty \|D_t^2 u_n\|_{\mathring{H}_m}^2 dt)^{\frac{4m+N}{8m(m+1)}},$$

$$\int_0^\infty\int_\Omega |u_{nt}|^2 |\nabla D_t^2 u_n| dx dt$$
$$\leqq C(M_1)\int_0^\infty \|u_{nt}\|_{H_{2m}}^{\frac{N}{4m}} \|D_t^2 u_n\|^{1-\frac{1}{m}} \|D_t^2 u_n\|_{\mathring{H}_m}^{\frac{1}{m}} dt$$
$$\leqq C(M_1)(1+\int_0^\infty \|D_t^2 u_n\|_{\mathring{H}_m}^2 dt)^{\frac{8m+N}{8m(m+1)}},$$

$$\int_0^\infty\int_\Omega |D_t^2 u_n| |\nabla D_t^2 u_n| dx dt$$
$$\leqq \int_0^\infty \|D_t^2 u_n\|^{2-\frac{1}{m}} \|D_t^2 u_n\|_{\mathring{H}_m}^{\frac{1}{m}} dt$$

$$\leqq C(M_1)(1+\int_0^\infty \|D_t^2 u_n\|_{\dot{H}_m}^2 dt)^{\frac{3}{2(m+1)}}.$$

Thus we obtain from (1.31)

$$\int_0^\infty \|D_t^2 u_n\|_{\dot{H}_m}^2 dt \leqq C(M_1)(1+\int_0^\infty \|D_t^2 u_n\|_{\dot{H}_m}^2 dt)^{\mu} + CM_2^2,$$

with $\mu = \max(\frac{8m+N}{8m(m+1)}, \frac{3}{2(m+1)}) = \frac{3}{2(m+1)} < 1$,

which implies

$$\int_0^\infty \|D_t^2 u_n\|_{\dot{H}_m}^2 dt \leqq C(M_2),$$

and also by (1.29)

$$\int_0^\infty \|u_{nt}\|_{H_{2m}}^2 dt \leqq C(M_2). \qquad \text{q. e. d.}$$

LEMMA 1.6 *Under the assumptions of theorem* 1.2,

$$(1.32) \qquad \int_0^\infty (\|D_t^3 u_n\|^2 + \|D_t^2 u_n\|_{H_{2m}}^2) dt$$
$$+ \|D_t^2 u_n(t)\|_{\dot{H}_m} + \|u_{nt}(t)\|_{H_{2m}}$$
$$+ \|u_n(t)\|_{H_{4m}} + \|u_{nt}(t)\|_{H_{4m}} \leqq C(M_2).$$

PROOF. Using lemma 1.5, the estimates of the terms of (1.32) except for $\|u_n(t)\|_{H_{4m}}$ and $\|u_{nt}(t)\|_{H_{4m}}$ are given similarly as in the one of lemma 1.4 (in fact, much simpler) and the details are omitted. As for $\|u_n(t)\|_{H_{4m}}$ and $\|u_{nt}(t)\|_{H_{4m}}$ we have only to apply the regularity result of elliptic equations and the so called boutstrapping method to the equations (1, 1) and (1.2). q. e. d.

After the lemmas 1.4 and 1.6 have been established the proof of theorem 1.2 is quite standard and we omit it.

Chapter 2 Modified Navier-Stokes equations.

2.1 Formulation of the problem and statement of the results.

Here we treat vector-valued functions $u=(u^1, u^2, u^3)$, $f=(f^1, f^2, f^3)$ etc. and $H=L^2(\Omega)$, $H_m(\Omega)$ etc. denote the spaces of such vector-valued functions equipped with usual norms. We shall begin with the introduction of the function spaces and the operators to consider the problem (E_2).

Let Ω be a bounded domain in R^3 with sufficiently smooth boundary $\partial\Omega$. Setting

$$C_{0,\sigma}^{\infty}(\Omega)=\{\varphi \mid \varphi\in C_0^{\infty}(\Omega) \text{ and } \operatorname{div}\varphi=0\},$$

we define H_σ, $H_{0,\sigma}^m$, P and A as follows:

H_σ=the closure of $C_{0,\sigma}^{\infty}(\Omega)$ in $L^2(\Omega)$,

$H_{0,\sigma}^m$=the closure of $C_{0,\sigma}^{\infty}(\Omega)$ in $H_m(\Omega)$,

P=the orthogonal projection from H onto H_σ,

and

A=the Friedrichs extension of the symmetric operator

$-P\Delta$ in H_σ defined for $u\in C_{0,\sigma}^{\infty}(\Omega)$.

It is known that $u\in D(A)$, $w\in H_\sigma$ and $Au=w$ if and only if $u\in H_{0,\sigma}^1$, $w\in H_\sigma$ and $(\nabla u,\nabla v)=(w,v)$ for any $v\in H_{0,\sigma}^1$, and it holds that

$$D(A^{\frac{1}{2}})=H_{0,\sigma}^1 \text{ and } \|A^{\frac{1}{2}}u\|=\|\nabla u\| \qquad \text{for } u\in H_{0,\sigma}^1.$$

$A^k(k=1,2,\cdots)$ are defined in natural way, while $A^\theta(\theta\neq$ integer) are defined as interpolation operator between $A^{[\theta]}$ and $A^{[\theta]+1}$. Note that $D(A^\theta)$ is a Hilbert space with norm $(\|u\|^2+\|A^\theta u\|^2)^{\frac{1}{2}}$. Then our problem (E_2) is formulated as an ordinary differential equation in the Hilbert space H_σ:

$$(E_2') \qquad \frac{du}{dt}+A^{1+2\varepsilon}u+P(u\cdot\nabla)u=Pf(t).$$

Let us state a lemma which will be essential for the estimation of solutions.

Lemma 2.1

(2.1) $\quad (u\cdot\nabla)u\in H \quad$ *if* $u\in D(A)$.

(2.2) $\quad \|P(u\cdot\nabla)v\|\leqq C\|A^\theta u\|\|A^\rho v\|$

if $\theta+\rho\geqq\frac{5}{4}$ *and* $\rho>\frac{1}{2}$.

For a proof of above lemma see Fujita [9], Giga [10] and Henry [34] etc.

Now our results read as follows.

THEOREM 2.1 (*strong solution*) *Let* $\varepsilon > \frac{1}{8}$ *and* $f \in W^{1,2}(\omega; L^2)$. *Then* (E_2') *has an* ω-*periodic solution* $u(t)$ *belonging to*

$$W^{2,2}(\omega; H_\sigma) \cap W^{1,\infty}(\omega; D(A^{\frac{1}{2}+\varepsilon})) \cap L^\infty(\omega; D(A^{1+2\varepsilon})).$$

THEOREM 2.2 (*smooth solution*) *Let* $\varepsilon > \frac{1}{8}$ *and* $f \in W^{2,2}(\omega; L^2)$. *Then the solution* u *of* theorem 2.1 *belongs to*

$$W^{3,2}(\omega; L^2) \cap W^{2,\infty}(\omega; D(A^{\frac{1}{2}+\varepsilon})) \cap W^{1,\infty}(\omega; D(A^{1+2\varepsilon})).$$

COROLLARY 2.1 *Let* $\varepsilon > \frac{1}{8}$ *and* $f \in W^{2,2}(\omega; L^2) \cap C^{0,1}(\omega; H_1)$. *Then the solution* $u(t)$ *satisfies*

$$A^{2\varepsilon} u \in C^{0,1}(\omega; H_3).$$

2.2 Approximate solutions

We employ again the Galerkin's method. Let $\{w_j\}$ $(j=1, 2, \cdots)$ be a basis in $D(A^{\frac{1}{2}+\varepsilon})$. We may assume that they are included in $D(A^n)$ for any n and orthonormal in H_σ. Let us consider the system of ordinary differential equations;

$$(2.3) \qquad \begin{aligned} &(u_{nt}, w_j) + (A^{\frac{1}{2}+\varepsilon} u_n, A^{\frac{1}{2}+\varepsilon} w_j) + (P(u_n \cdot \nabla) u_n, w_j) \\ &= (Pf, w_j), \qquad\qquad j=1, 2, \cdots n, \end{aligned}$$

with $u_n(t) = u_n(t+\omega)$, where $u_n(t) = \sum_{i=1}^{n} \alpha_{i,n}(t) w_i$.

The existence of solution of (2.3) is an immediate consequence of the following a priori estimate.

LEMMA 2.1 *Let* $u_n(t)$ *be an* ω-*periodic solution of* (2.3). *Then*

$$(2.4) \qquad \int_0^\omega \|A^{\frac{1}{2}+\varepsilon} u_n\|^2 dt + \|u_n(t)\|^2 \leqq C M_0^2 \quad \textit{for any } t \in R.$$

PROOF. Noting $(P(u_n \cdot \nabla) u_n, u_n) = 0$, the estimate (2.4) is derived quite similarly as in lemma 1.1. q. e. d.

We proceed the further estimations of $u_n(t)$. The arguments are pararell to Chap. 1 and the details are sometimes omitted. We use similar notation as in Chap. 1.

LEMMA 2.2 *Let* $\varepsilon>\frac{1}{8}$, $f\in W^{1,2}(\omega;L^2)$ *and* $u_n(t)$ *be an* ω*-periodic solution of* (2.3) *Then*

$$\|A^{\frac{1}{2}+\varepsilon}u_n(t)\|+\|A^{\frac{1}{2}+\varepsilon}u_{nt}(t)\|+\int_0^\omega\|u_{ntt}\|^2dt \tag{2.5}$$
$$\leqq C(\varepsilon, M_1) \qquad \text{for } \forall t\in R.$$

Proof. From (2.3)

$$\int_0^\omega\|u_{nt}\|^2dt\leqq M_0^2+2\int_0^\omega\int_\Omega\Big|\sum_{i,k=1}^3 u_n^i\frac{\partial u_n^k}{\partial x^i}u_{nt}^k\Big|dxdt \tag{2.6}$$
$$\leqq M_0^2+C\Big(\int_0^\omega\|\nabla u_n\|^2dt\Big)^{\frac{3}{4}}\Big(\int_0^\omega\|\nabla u_{nt}\|^4dt\Big)^{\frac{1}{4}}\sup_t\|u_n(t)\|^{\frac{1}{2}}$$

where we have used the inequality

$$\|u\|_4^4\leqq C\|\nabla u\|^3\|u\| \qquad \text{for } u\in\mathring{H}_1.$$

With the aid of lemma 2.1 we have from above

$$\int_0^\omega\|u_{nt}\|^2dt\leqq C(M_0)(1+\sup_t\|\nabla u_{nt}\|) \tag{2.7}$$

and also by this and (2.4)

$$\|A^{\frac{1}{2}+\varepsilon}u_n(t)\|^2\leqq C(M_0)(1+\sup_t\|\nabla u_{nt}\|). \tag{2.8}$$

Next differentiating (2.3)

$$(u_{ntt}, w_j)+(A^{\frac{1}{2}+\varepsilon}u_{nt}, A^{\frac{1}{2}+\varepsilon}w_j) \tag{2.9}$$
$$=(f_t-(u_{nt}\cdot\nabla)u_n-(u_n\cdot\nabla)u_{nt}, w_j), \qquad j=1, 2, \cdots n.$$

From this

$$\int_0^\omega\|A^{\frac{1}{2}+\varepsilon}u_{nt}\|^2dt\leqq M_1^2+C\int_0^\omega\|\nabla u_{nt}\|^{\frac{3}{2}}\|u_{nt}\|^{\frac{1}{2}}\|\nabla u_n\|dt \tag{2.10}$$
$$\leqq C(M_1)(1+\sup_t\|\nabla u_{nt}(t)\|^{\frac{7}{4}}),$$

where we have used (2.4) and (2.7).
To estimate $\|A^{\frac{1}{2}+\varepsilon}u_{nt}\|$ we shall use the equality

$$\|u_{ntt}(t)\|^2+\frac{1}{2}\frac{d}{dt}\|A^{\frac{1}{2}+\varepsilon}u_{nt}(t)\|^2 \tag{2.11}$$
$$=(f_t-(u_{nt}\cdot\nabla)u_n-(u_n\cdot\nabla)u_{nt}, u_{ntt}).$$

The right hand side of (2.11) is bounded by

$$\begin{aligned}&(\|f_t(t)\|+\|P(u_{nt}\cdot\nabla)u_n\|+\|P(u_n\cdot\nabla)u_{nt}\|)\|u_{ntt}\|\\ \text{(2.12)}\quad&\leqq(\|f_t(t)\|+C\|A^\theta u_{nt}\|\|A^\rho u_n\|+\|A^\theta u_n\|\|A^\rho u_{nt}\|)\|u_{ntt}\|\end{aligned}$$

with $\rho=\frac{1}{2}+\varepsilon-\delta$ and $\theta=\frac{3}{4}-\varepsilon+2\delta$ $(0<2\delta<\varepsilon)$.

Therefore

$$\text{(2.13)}\qquad \int_0^\omega\|u_{ntt}\|^2dt\leqq M_1^2+C\int_0^\omega(\|A^\theta u_{nt}\|^2\|A^\rho u_n\|^2+\|A^\theta u_n\|^2\|A^\rho u_{nt}\|^2)\,dt.$$

Since $\varepsilon>\frac{1}{8}$ we can choose $\delta(>0)$ such that $2\varepsilon>\frac{1}{4}+2\delta$.

Then, $\theta<\frac{1}{2}+\varepsilon$, and for $\theta_\varepsilon=(\frac{1}{2}+\varepsilon)^{-1}\theta$ and $\rho_\varepsilon=(\frac{1}{2}+\varepsilon)^{-1}\rho$,

$$\begin{aligned}\int_0^\omega\|u_{ntt}\|^2dt\leqq M_1^2+C\sup_t\{&\|A^{\frac{1}{2}+\varepsilon}u_{nt}(t)\|^{2\theta_\varepsilon}\|u_{nt}(t)\|^{2-2\theta_\varepsilon}\\ &+\|A^{\frac{1}{2}+\varepsilon}u_{nt}(t)\|^{2\rho_\varepsilon}\|u_{nt}(t)\|^{2-2\rho_\varepsilon}\}\int_0^\omega\|A^{\frac{1}{2}+\varepsilon}u_n\|^2dt,\end{aligned}$$

and by (2.4) and (2.10),

$$\text{(2.14)}\qquad \leqq C(M_1)(1+\sup_t\|A^{\frac{1}{2}+\varepsilon}u_{nt}(t)\|^\lambda)$$

with $\lambda=\frac{1}{4}(7+\max(\rho_\varepsilon,\theta_\varepsilon))<2$.

From (2.10), (2.12) and (2.14) we can obtain

$$\sup_t\|A^{\frac{1}{2}+\varepsilon}u_{nt}(t)\|^2\leqq C(M_1)(1+\sup_t\|A^{\frac{1}{2}+\varepsilon}u_{nt}(t)\|^\lambda),$$

which implies

$$\text{(2.15)}\qquad \|A^{\frac{1}{2}+\varepsilon}u_{nt}(t)\|^2\leqq C(\varepsilon,M_1).$$

By (2.8) and (2.14) we have also

$$\int_0^\omega\|u_{ntt}\|^2dt+\|A^{\frac{1}{2}+\varepsilon}u_n(t)\|^2\leqq C(\varepsilon,M_1)\ \text{ for }\forall t.$$

q. e. d.

LEMMA 2.3 *Let* $\varepsilon>\frac{1}{8}$ *and* $f\in W^{2,2}(\omega;L^2)$. *Then,*

$$\|A^{\frac{1}{2}+\varepsilon}u_{ntt}(t)\|+\int_0^\omega\|D_t^3u_n\|^2dt\leqq C(M_2).$$

PROOF. Differentiating (2.9) once more again, we have

$$(2.16)\qquad (D_t^3 u_n, w_j)+(A^{\frac{1}{2}+\varepsilon}u_{ntt}, A^{\frac{1}{2}+\varepsilon}w_j)$$
$$=(f_{tt}-(u_{ntt}\cdot\nabla)u_n-2(u_{nt}\cdot\nabla)u_{nt}-(u_n\cdot\nabla)u_{ntt}, w_j)$$
$$(j=1, 2, \cdots n).$$

First we obtain from the above

$$(2.17)\qquad \frac{1}{2}\frac{d}{dt}\|u_{ntt}\|^2+\|A^{\frac{1}{2}+\varepsilon}u_{ntt}\|^2$$
$$\leqq M_2\|u_{ntt}\|+C\|\nabla u_{ntt}\|^{\frac{3}{2}}\|u_{nt}\|^{\frac{1}{2}}\|\nabla u_n\|$$
$$+C\|\nabla u_{nt}\|^{\frac{3}{4}}\|u_{nt}\|^{\frac{1}{4}}\|\nabla u_{ntt}\|^{\frac{3}{4}}\|u_{ntt}\|^{\frac{1}{4}}\|\nabla u_{nt}\|$$

and integrating

$$\int_0^\omega\|A^{\frac{1}{2}+\varepsilon}u_{ntt}\|^2dt\leqq CM_2^2+C(M_1)\left(\int_0^\omega\|\nabla u_{ntt}\|^2dt\right)^{\frac{3}{4}}$$

which implies

$$(2.18)\qquad \int_0^\omega\|A^{\frac{1}{2}+\varepsilon}u_{ntt}\|^2dt\leqq C(M_2).$$

Next, we have again from (2.16) (see (2.12))

$$(2.19)\qquad \int_0^\omega\|D_t^3u_n\|^2dt$$
$$\leqq M_2^2+C\int_0^\omega(\|A^\theta u_{ntt}\|^2\|A^\rho u_n\|^2+\|A^\theta u_{nt}\|^2\|A^\rho u_{nt}\|^2$$
$$+\|A^\theta u_n\|^2\|A^\rho u_{ntt}\|^2)dt$$
$$\leqq C(M_2),$$

where we have used lemm 2.2 and (2.18). From (2.18) and (2.19) we obtain as is usual

$$\|A^{\frac{1}{2}+\varepsilon}u_{ntt}(t)\|\leqq C(M_2)\qquad\text{for } \forall t\in R.$$ q. e. d.

2.3 Convergence of approximate solutions

We shall prove that a subsequence of $\{u_n(t)\}$ is convergent to the desired solution of the problem (E_2'). First we consider the case $f\in W^{1,2}(\omega;L^2)$.

Since the estimates in lemmas 2.1 and 2.2 are valid, standard compactness argument implies that $\{u_n(t)\}$ coverges (along a subsequence) to a function $u(t)$ in such a way that

$$(2.20)\qquad u_n(t)\to u(t)\qquad \text{weakly* in } L^\infty(\omega;D(A^{\frac{1}{2}+\varepsilon})) \text{ and strongly in } L^\infty(\omega;H_\sigma),$$

and

$$(2.21)\qquad u_{nt}(t)\to u_t(t)\qquad \text{weakly* in } L^\infty(\omega;H_\sigma)\cap L^\infty(\omega;D(A^{\frac{1}{2}+\varepsilon})).$$

Moreover we see that $\mathrm{P}(u\cdot\nabla)u$ is well-defined and for $\forall w\in H^1_{0,\sigma}$

$$\begin{aligned}(2.22)\qquad &|(P(u_n\cdot\nabla)u_n-P(u\cdot\nabla)u, w)|\\ &\leqq|((u_n-u)\cdot\nabla u_n, w)|+|(u\cdot\nabla(u_n-u), w)|\\ &\leqq C\|u_n-u\|_4\|\nabla u_n\|\|w\|_4+C\|u\|_4\|\nabla w\|\|u_n-u\|_4\\ &\leqq C(M_1)\|u_n-u\|^{\frac{1}{4}}\to 0\qquad \text{as } n\to\infty \text{ (uniformly in } t).\end{aligned}$$

Thus we have that

$$\begin{aligned}&(u_t, w)+(A^{\frac{1}{2}+\varepsilon}u, A^{\frac{1}{2}+\varepsilon}w)+(P(u\cdot\nabla)u, w)\\ &=(Pf, w) \text{ for a.e. } t\in R \text{ and for } \forall w\in D(A^{\frac{1}{2}+\varepsilon}),\end{aligned}$$

and consequently $u(t)\in D(A^{1+2\varepsilon})$ and

$$(2.23)\qquad u_t(t)+A^{1+2\varepsilon}u(t)+P(u\cdot\nabla)u(t)=Pf(t) \text{ in } H_\sigma \text{ for a.e. t.}$$

We shall show that (2.23) is valid for any t. Since $A^{\frac{1}{2}+\varepsilon}u_t$, and $A^{\frac{1}{2}+\varepsilon}u$ belong to $L^\infty(\omega;H_\sigma)$,

$$\begin{aligned}&\|P(u(t)\cdot\nabla)u(t)-P(u(t+h)\cdot\nabla)u(t+h)\|\\ &\leqq C\|A^\theta(u(t)-u(t+h))\|\|A^\theta u(t)\|\\ &\quad+C\|A^\theta u(t+h)\|\|A^\theta(u(t+h)-u(t))\|\\ &\leqq C(M_1)\|A^{\frac{1}{2}+\varepsilon}(u(t)-u(t+h))\|\leqq C(M_1)|h|,\end{aligned}$$

that is $P(u\cdot\nabla)u\in C^{0,1}(\omega;H_\sigma)$. Also we have $u_t\in C^{\frac{1}{2}}(\omega;H_\sigma)$ because of $u_{tt}\in L^2(\omega;H_\sigma)$. Therefore by (2.33) we may assume that $u\in C^{\frac{1}{2}}(\omega;D(A^{1+2\varepsilon}))$ and (2.23) is valid for $\forall t\in R$. Moreover differentiating (2.23) we see $A^{1+2\varepsilon}u_t\in L^2(\omega;H_\sigma)$. The proof of theorem 2.1 is now completed.

The proof of theorem 2.2 follows immediately from lemma 2.3.

Finally we shall prove corollary 2.1. The solution $u(t)$ of theorem 2.2 satisfies

$u_t+A^{1+2\varepsilon}u+(u\cdot\nabla)u=f-\nabla p$, in L^2, for $\forall t$, with some function $p(t)$. Setting $A^{2\varepsilon}u(t)=w(t)$ we see $w(t)\in D(A)$ and

(2.24) $\qquad -\Delta w(t)+\nabla p(t)=g(t)$, and div $w(t)=0$

with $g(t)=-u_t-(u\cdot\nabla)u+f\in C^{0,1}(\omega;H_1)$ (by theorem 2.2). The system (2.24) with respect to $\{w,p\}$ is elliptic in the sense of Agmon-Douglis-Nirenberg [1] (see Kaniel-Shinbrot [14] Ladyzhenskaia [17]). Thus, applying the regularity result of elliptic boundary value problem we have $w=A^{2\varepsilon}u\in C^{0,1}(\omega;H_3)$, which proves corollary 2.1.

Remark (i) Repeating the argument in the proof of lemma 2.3, we could prove any higher order differentiabilities of u with respect to the time t, if $f(t)$ is sufficiently smooth in t. As for the regularity in x there occurs a delicate problem; Is it true that if $A^\alpha u=f\in C^m, \alpha>1$ then $u\in H^{m+\alpha'}$ for some $\alpha'\geqq 0$? If this is valid, we can use the boutstrapping method to get the smoothess of u in x if f is smooth in x. Thus we would have $u\in C^\infty(R\times\bar{\Omega})$ if $f\in C^\infty(R\times\bar{\Omega})$. But, we do not know whether such assertion on $A^\alpha u=f$ is valid or not.

(ii) For 2-dimensional case we could prove a smooth periodic solution for any $\varepsilon>0$. However, it is already proved by A. Takeshita [31] that the problem (E_2) with $\varepsilon=0$ admits a periodic (more generally, reproducing) strong solution. Taking the regularizing effect of the Navier-Stokes equations into account his strong solution becomes a smooth solution if $f(t)$ is so. We note that his method depends on the semi-group theory and is completely different from ours.

(iii) If M_1 is sufficiently small, the solution of theorem 2.1 is unique. The proof is easy and omitted.

Chapter 3. Semilinear wave equation

3.1. Assumptions and result

In this chapter we consider the periodicity problem for the semilinear wave equation

$$(E_3) \qquad \begin{cases} u_{tt}+Au+\nu u_t+g(u)=f(x,t) & \text{in } \Omega\times R \\ D^\gamma u|_{\partial\Omega}=0 & \text{for } |\gamma|\leq m-1, \end{cases}$$

where ν is a positive constant and A is the same elliptic operator as in Chapt. 1. Without loss of generality we may assume $\nu=1$. We make similar assumptions on $g(u)$ as in Chapt. 1, that is, H_1, $g(u)\in C^k(R)$ and the following inequalities are satisfied;

$$-(b_1+b_0|u|^{\nu_0})\leq g(u)u\leq c(1+|u|^{1+r_0})$$

and

$$|g^{(i)}(u)|\leq c(1+|u|^{r_i})$$

for some constants $r_i\geq 0$ $(i=0, 1, \cdots, k)$.

Our results read as follows.

THEOREM 3.1. *(Existence of strong solution)* *Let* $f\in W^{1,2}(\omega; L^2)$ *and let* H_1 *be fulfilled with* $k=1$ *and with* r_0, r_1 *such that*

$$r_0<\frac{N}{N-2m} \text{ if } N>2m, \quad r_0=\text{arbitrary if } N\leq 2m$$

and

$$r_1<\frac{2m}{N-2m} \text{ if } N>2m, \quad r_1=\text{arbitrary if } N\leq 2m.$$

Then, (E_3) *admits an* ω*-periodic solution* $u(t)$ *belonging to*

$$W^{2,\infty}(\omega; L^2)\cap W^{1,\infty}(\omega; \mathring{H}_m)\cap L^\infty(\omega; H_{2m})$$

with the estimate

$$\|u_{tt}(t)\|+\|u_t(t)\|_{A^{1/2}}+\|u(t)\|_A\leq c(M_1).$$

THEOREM 3.2. *Let* $N<8m$. *In addition to the hypotheses of* Theorem 3.1 *suppose that* $f\in W^{2,2}(\omega; L^2)$ *and* H_1 *is valid with* $k=2$ *and with* r_2 *such that*

$$r_2<\frac{8m-N}{N-4m} \text{ if } N>4m, \quad r_2=\text{arbitrary if } N\leq 4m.$$

Then the solution $u(t)$ *in Th.* 3.1 *belonging to*

$$W^{3,\infty}(\omega; L^2)\cap W^{2,\infty}(\omega; \mathring{H}_m)\cap W^{1,\infty}(\omega; H_{2m})$$

with the *further estimate*

$$\|D_t^3 u(t)\|+\|u_{tt}(t)\|_{A^{1/2}}+\|u_t(t)\|_A\leq c(M_2).$$

Corollary 3.1. *In addition, if we assume* $f\in L^\infty(\omega;\mathring{H}_m)$ *and*

$$\|g(u)\|_{\mathring{H}_m}\leq c(\|u\|_{H_{2m}})\|u\|_{H_{3m}}^\theta \qquad \text{for } u\in H_{3m}$$

with some $0\leq\theta<1$. *Then the solution* $u(t)$ *in* Th. 3.2 *belonging to* $L^\infty(\omega;H_{3m})$.

THEOREM 3.3. *Let* $N<6m$, $f\in W^{3,2}(\omega;L^2)$ *and the assumption* H_1 *be satisfied with* $k=3$ *and* r_2,r_3 *such that*

$$r_2\leq\frac{6m-N}{N-4m} \text{ and } r_3\leq\frac{2(6m-N)}{N-4m} \text{ if } N>4m$$

(no restriction is made on r_2,r_3 *if* $N\leq 4m$*). Then the solution* $u(t)$ *of* Theorem 3.2 *belongs to the space*

$$W^{4,\infty}(\omega;L^2)\cap W^{3,\infty}(\omega;\mathring{H}_m)\cap W^{2,\infty}(\omega;H_{2m})$$

with the corresponding estimate.

COROLLARY 3.2. *In addition to the assumptions of* Ths. 3.2, 3.3 *and* Corollary 3.1 *suppose that*

$$f\in W^{1,\infty}(\omega;\mathring{H}_m)\cap L^\infty(\omega;H_{2m}),$$
$$\|g'(u)v\|_{H_m}\leq c(\|u\|_{H_{3m}}\|v\|_{H_{2m}}) \text{ for } u\in H_{3m} \text{ and } v\in H_{2m}\cap\mathring{H}_m$$

and

$$\|g(u)\|_{H_{2m}}\leq c(\|u\|_{H_{3m}})\|u\|_{\mathring{H}_{4m}}^\theta \text{ for } u\in H_{4m}$$

with some $0\leq\theta<1$. *Then we have*

$$u(t)\in W^{1,\infty}(\omega;H_{3m})\cap L^\infty(\omega;H_{4m}).$$

THEOREM 3.4 *Let* $N\leq\frac{16}{3}m$ *and* $f\in W^{4,2}(\omega;L^2)$. *Suppose that the hyposesis* H_1 *is valid with* $k=4$ *and* $r_4\leq(16m-3N)/(N-4m)$ *(if* $N>4m$*) together with the conditions on* r_2, r_3 *of* Theorem 3.3. *Then, we have for the solution* $u(t)$,

$$u\in W^{5,\infty}(\omega;L^2)\cap W^{4,\infty}(\omega;\mathring{H}_m)\cap W^{3,\infty}(\omega;H_{2m})$$

with the corresponding estimate.

We could assert a corrolary of Theorem 3.4 which corresponds to Corollary 3.2. We coluld also give explicite conditions on g and g' which should imply the fulfillment of the assumptions of Corollaries 3.1 and 3.2.

Since they are rather standard we omit the details (cf. the final remarks). For convenience, however, we mention the following

COROLLARY 3.3. *Let $N<4m$ and $f\in W^{4,2}(\omega;L^2)\cap C(\omega;C^{\beta}(\bar{\Omega}))$ with $\beta>0$. Then the solution u of* Theorem 3.4 *is a classical periodic solution of (E_3).*

As was seen in Chapters 1 and 11 it suffices for the proofs of Theorems to derive corresponding a priori estimates for (assumed) smooth ω-periodic solutions.

3.2. A priori estimate (1)

Here we shall derive the a priori estimates needed for the proof of Theorem 3.1.

Multiplying the equation by u_t,

$$(3.1)\qquad \frac{d}{dt}\{\frac{1}{2}\|u_t(t)\|^2+\frac{1}{2}\|u(t)\|^2_{A^{1/2}}+\int_\Omega\int_0^{u(t)}g(\eta)d\eta dx$$

$$+\|u_t\|^2=(f,u_t)$$

and integrating the above

$$(3.2)\qquad \int_0^\omega\|u_t(s)\|^2ds\le M_0^2.$$

Next, multiplying the equation by u and using the assumption on $g(u)$ and (3.2) we have easily

$$(3.3)\qquad \int_0^\omega\|u(s)\|^2_{A^{1/2}}ds\le c(M_0).$$

From (3.2) and (3.3) we see that

$$\|u_t(t^*)\|^2+\|u(t^*)\|^2_{A^{1/2}}\le c(M_0)\quad\text{for some } t^*\in[0,\omega]$$

and for any $t\in[0,\omega]$, by (3.1),

$$\|u_t(t)\|^2+\|u(t)\|^2_{A^{1/2}}+2\int_\Omega\int_0^{u(t)}g(\eta)d\eta dx$$

$$\le\|u_t(t^*)\|^2+\|u(t^*)\|^2_{A^{1/2}}+\int_\Omega\int^{u(t^*)}g(\eta)d\eta dx+c(M_0)$$

which implies

$$(3.4)\qquad \|u_t(t)\|^2+\|u(t)\|^2_{A^{1/2}}\le c(M_0)\qquad\text{for } t\in[0,\omega],$$

where we have used the assumption $r_0 \leq \frac{2N}{N-2m}$ if $N>2m$.

Next, we utilize the differentiated equation

$$D_t^3 u + Au_t + u_{tt} + g'(u)u_t = f_t. \tag{3.5}$$

From this we obtain as is usual

$$\int_0^\infty \|D_t^2 u(s)\|^2 ds \leq M_1^2 + c\int_0^\infty \int_\Omega (1+|u|^{2r_1})|u_t|^2 dxds. \tag{3.6}$$

Here we see

$$\begin{aligned}\int_0^\infty \int_\Omega |u|^{2r_1}|u_t|^2 dxds &\leq \int_0^\infty \|u\|_q^{2r_1}\|u_t\|_{2q/(q-2r_1)}^2 ds \\ &\leq c(M_0)\int_0^\infty \|u_t\|_{2q/(q-2r_1)}^2 ds \\ &\leq c(M_0)\int_0^\infty \|u_t\|^{2(1-\theta)}\|u_t\|_{\dot{H}_m}^{2\theta} ds \\ &\leq c(M_0)\left(\int_0^\infty \|u_t\|_{\dot{H}_m}^2 ds\right)^\theta \end{aligned} \tag{3.7}$$

where we set

$$q=\begin{cases} 2N/(N-2m) & \text{if } N>2m \\ \text{arbitrarily large} & \text{if } N=2m \\ \infty & \text{if } N<2m \end{cases}$$

and

$$\theta=\begin{cases} r_1(N-2m)/2m & \text{if } N>2m \\ \text{arbitrarily close to 0} & \text{if } N=2m \\ 0 & \text{if } N<2m. \end{cases}$$

Thus we obtain

$$\int_0^\infty \|D_t^2 u(s)\|^2 ds \leq c(M_1)\left(1+\int_0^\infty \|u_t(s)\|_{\dot{H}_m}^2 ds\right)^\theta. \tag{3.8}$$

Multiplying (3.5) by u_t and integrating,

$$\begin{aligned}\int_0^\infty \|u_t(s)\|_{A^{1/2}}^2 ds &= \int_0^\infty \|D_t^2 u(s)\|^2 ds - \int_0^\infty (g'(u)u_t^2 + f_t u_t)dxds \\ &\leq c(M_1)\left(1+\int_0^\infty \|u_t(s)\|_{\dot{H}_m}^2 ds\right)^\theta \qquad \text{(by (3.8))}. \end{aligned} \tag{3.9}$$

Since $\theta<1$ by the assumption, we have from (3.9) and (3.8)

$$\int_0^\omega (\|u_t(s)\|^2_{A^{1/2}}+\|D_t^2 u(s)\|^2)ds\leq c(M_1). \tag{3.10}$$

Applying the usual technique getting (3.4) to the inequality (3.10) we can obtain

$$\|u_{tt}(t)\|+\|u_t(t)\|_{A^{1/2}}\leq c(M_1), \tag{3.11}$$

and hence by the equation (E_3)

$$\|u(t)\|_A\leq c(M_1). \tag{3.12}$$

The estimates (3.11) and (3.12) give the proof of Theorem 3.1.

3.3. A priori estimate (11)

We proceed to further estimation of solutions. Let us start here from the equation

$$D_t^4 u+AD_t^2 u+D_t^3 u+g''(u)u_t^2+g'(u)D_t^2 u=f_{tt}. \tag{3.13}$$

Multiplying the above by $D_t^3 u$,

$$\int_0^\omega \|D_t^3 u\|^2 ds\leq M_2{}^2+c\int_0^\omega\int_\Omega \{(1+|u|^{r_2})^2|u_t|^4 + (1+|u|^{r_1})^2|u_{tt}|^2\}dxds. \tag{3.14}$$

We shall consider the case $N>4m$, the other case being treated in a similar and simpler way, Setting $q=2N/(N\text{-}4m)$, we have

$$\begin{aligned}\int_0^\omega\int_\Omega |u|^{2r_2}|u_t|^4 dxds &\leq\int_0^\omega \|u(t)\|^{2r_2}_{H_{2m}}\|u_t\|^4_{4q/(q-2r_2)}ds\\ &\leq c(M_1)\int_0^\omega \|u_t\|^{4(1-\theta_0)}_{\mathring{H}_m}\|u_t\|^{4\theta_0}_{H_{2m}}ds\\ &\leq c(M_1)\int_0^\omega \|u_t\|^{4\theta_0}_{H_{2m}}ds \end{aligned} \tag{3.15}$$

with $\theta_0=\frac{N}{2m}\{\frac{1}{4}-\frac{m}{N}+\frac{(N-4m)r_2}{4N}\}$, where we have used the estimates in the previous section and the assumption $N<8m$. Note that $4\theta_0<2$ by the assumption on r_2.

Similarly we can prove

$$(3.16)\qquad \int_0^\infty\int_\Omega |u|^{2r_1}|u_{tt}|^2dxds\leq c(M_1)\int_0^\infty \|u_{tt}\|^{2(1-\theta_1)}\|u_{tt}\|_{\mathring{H}_m}^{2\theta_1}ds$$

$$\leq c(M_1)\int_0^\infty \|u_{tt}\|_{\mathring{H}_m}^{2\theta_1}ds$$

with $\theta_1=\frac{(N-4m)r_1}{2m}(<1)$.

From (3.14)-(3.16) it follows that

$$(3.17)\qquad \int_0^\infty \|D_t^3u\|^2ds\leq c(M_2)\{1+(\int_0^\infty \|u_t\|_{H_{2m}}^2ds)^{2\theta_0}+(\int_0^\infty \|u_{tt}\|_{\mathring{H}_m}^2ds)^{\theta_1}\}.$$

Here we see from the equation (3.5)

$$(3.18)\qquad \int_0^\infty \|u_t\|_{H_{2m}}^2ds\leq\int_0^\infty \|D_t^3u\|^2ds+c(M_1),$$

and hence, since $2\theta_0<1$, we obtain from (3.17)

$$(3.19)\qquad \int_0^\infty \|D_t^3u\|^2\,ds\leq C(M_2)\{1+\int_0^\infty \|u_{tt}\|_{\mathring{H}_m}^2\}^{\theta_1}.$$

On the other hand, multiplying (3.13) by u_{tt} and using (3.15) and (3.16)

$$\int_0^\infty \|u_{tt}\|_{\mathring{H}_m}^2ds\leq\int_0^\infty \|D_t^3u\|^2ds+c\int_0^\infty\int_\Omega \{1+|u|^{r_2})|u_t|^2|u_{tt}|ds$$

$$+(1+|u|^{r_1})|u_{tt}|^2\}dxds$$

$$(3.20)\qquad \leq\int_0^\infty \|D_t^3u\|^2ds+c(M_2)\{1+\int_0^\infty \|u_t\|_{H_{2m}}^2ds$$

$$+\int_0^\infty \|u_{tt}\|_{\mathring{H}_m}^2ds\}^{\max(2\theta_0,\ \theta_1)}.$$

From (3, 19) and (3.20) we can conclude

$$\int_0^\infty(\|D_t^3u(t)\|+\|D_t^2u(t)\|_{\mathring{H}_m}+\|u_t(t)\|_{H_{2m}})dt\leq c(M_2)$$

and hence, as is usual,

$$(3.21)\qquad \|D_t^3u(t)\|+\|D_t^2u(t)\|_{\mathring{H}_m}+\|u_t(t)\|_{H_{2m}}\leq c(M_2)\ \text{ for } \forall t,$$

which is the desired estimate for the proof of Theorem 3.2.

Under the assumption of Corollary 3.1 we see

$$\|A^{3/2}u(t)\|+\|u(t)\|\leq c\|-(u_{tt}+u_t+g(u))+f\|_{\mathring{H}_m}+\|u(t)\|$$

$$\leq c(M_2)+c(M_2)\|u(t)\|_{H_{3m}}^{\theta}+\|f(t)\|_{\mathring{H}_m},\ 0<\theta<1,$$

and since $\|u(t)\|_{H_{3m}} \leq c(\|A^{3/2}u(t)\|+\|u(t)\|)$ we obtain

$$\|u(t)\|_{H_{3m}} \leq c(M_2)+\sup_s \|f(s)\|_{\mathring{H}_m}.$$

3.4. A priori estimates (111)

Finally we shall derive estimates needed for the proofs of Theorem 3.3 ans 3.4.

Using the equality

$$(3.22) \qquad D_t^5 u + AD_t^3 u + D_t^4 u + g^{(3)}(u)u_t^3 + 3g''(u)u_t u_{tt} + g'(u)D_t^3 u = D_t^3 f.$$

Corresponding to (3.14) we have

$$\int_0^\omega \|D_t^4 u\|^2 ds \leq M_3^2 + c\int_0^\omega \int_\Omega \{(1+|u|^{2r_3})|u_t|^6 + (1+|u|^{2r_2})|u_t|^2|u_{tt}|^2 + (1+|u|^{2r_1})|D_t^3 u|^2\} dxds.$$

We consider again the case $N>4m$. The following estimates are given in a pararell way to (3.15) and (3.16).

$$\int_0^\omega \int_\Omega |u|^{2r_3}|u_t|^6 dxds \leq \int_0^\omega \|u(t)\|_{H_{2m}}^{2r_3} \|u_t\|_{\mathring{H}_{2m}}^6 ds \leq c(M_2),$$

where we used the assumptions that $N\leq 6m$ and $r_3 \leq 2(6m-N)/(N-4m)$.

$$\int_0^\omega \int_\Omega |u|^{2r_2}|u_t|^2|u_{tt}|^2 dxds \leq c\int_0^\omega \|u\|_{\mathring{H}_{2m}}^{2r_2} \|u_t\|_{\mathring{H}_{2m}}^2 \|u_{tt}\|_{\mathring{H}_m}^2 ds \leq c(M_2),$$

where we used $r_2 \leq (6m-N)/(N-4m)$.
And

$$\int_0^\omega \int_\Omega |u|^{2r_1}|D_t^3 u|^2 dxds \leq \int_0^\omega \|u\|_{H_{2m}}^{2r_1} \|D_t^3 u\|^{2(1-\theta)} \|D_t^3 u\|_{H_m}^2 ds$$

with a certain $\theta(<1)$.

Thus we obtain

$$(3.24) \qquad \int_0^\omega \|D_t^4 u\|^2 ds \leq c(M_3)\{1+\int_0^\omega \|D_t^3 u\|_{\mathring{H}_m}^2 ds\}^\theta$$

and hence

$$(3.25) \qquad \int_0^\omega \|D_t^3 u\|_{\mathring{H}_m}^2 ds + \int_0^\omega \|D_t^4 u\|^2 ds \leq c(M_3).$$

From (3.25) and the regularity result of elliptic equation we conclude

(3.26) $\|D_t^4 u(t)\| + \|D_t^3 u(t)\|_{\mathring{H}_m} + \|D_t^2 u(t)\|_{H_{2m}} \leq c(M_3)$,

which implies Theorem 3.3.

For the proof of Theorem 3.4 we utilize

(3.27) $D_t^6 u + A D_t^4 u + D_t^5 u + g^{(4)}(u) u_t^4 + g'(u) D_t^4 u + G(x, t) = D_t^4 f$

where

$$G(x, t) = 6g^{(3)}(u) u_t^2 u_{tt} + 3g''(u) u_{tt}^2 + 4g''(u) u_t D_t^3 u.$$

It is easy to see $\|G(t)\| \leq c(M_3)$ and also, by the assumption on r_4

$$\int_0^\omega \int_\Omega |u|^{2r_4} |u_t|^8 dx ds \leq c(M_1) \int_0^\omega \|u_t\|_{H_{2m}}^8 ds \leq c(M_1, M_3)$$

and

$$\int_0^\omega \int_\Omega |u|^{2r_1} |D_t^4 u|^2 dx ds \leq c(M_2) \left(\int_0^\omega \|D_t^4 u\|_{\mathring{H}_m}^2 ds\right)^\theta$$

for some $\theta < 1$. From these we can obtain, as is usual,

(3.28) $\|D_t^5 u(t)\| + \|D_t^4 u(t)\|_{\mathring{H}_m} + \|D_t^3 u(t)\|_{H_{2m}} \leq c(M_4)$

which proves Theorem 3.4.

The proof of Cor. 3.2 is similar to that of Cor. 3.1 and omitted. If $N < 4m$, $H_{2m} \subset C^\nu(\bar{\Omega})$ with $\nu = (4m - N)/2 > 0$, and the solution u in Th. 3.4 belongs to $C^2(\omega; C^\nu(\bar{\Omega}))$ and under the assumption of Cor. 3.3 we see $\mathrm{Au}(t) \in C(\omega; C^\gamma(\bar{\Omega}))$ for $\gamma = \min(\beta, \nu) > 0$. Therefore by Shauder's type estimate (Agmon-Douglis-Nirenberg [1]) we see $u \in C(\omega; C^{2m+\gamma}(\bar{\Omega}))$. That is, u is a classical solution.

Finally we note that the condition $N < 4m$ in Cor. 3.3 is made mainly for simplicity, and we could show the existence of classical periodic solution even for the case $N \geq 4m$ under some additional condition on $g(u)$. For illustration we shall consider the case $N = 4$ or 5 and $m = 1$. Let $g(0) = 0$ and $f \in \mathrm{W}^{2,2}(\omega; \mathring{H}_1 \cap H_2)$. In addition to the assumpions of Th. 3.4 suppose that $g \in C^5(\mathrm{R})$. Then, by Th. 3.4,

$$Au(t) = -u_{tt} - u_t - g(u) + f \in W^{1,2}(\omega; H_2 \cap \mathring{H}_1) = W^{1,2}(\omega; D(A))$$

and $u(t) \in W^{1,2}(\omega; D(A^2)) \subset C(\omega; C^\gamma(\bar{\Omega}))$, $\gamma > 0$.

Similarly we see $Au_t \in W^{1,2}(\omega; D(A))$ and $u_t \in C(\omega; C^\gamma(\bar{\Omega}))$.

Setting $U=D_t^5 u$ we see

$$U_{tt}+AU+U_t\in L^2(\omega;L^2)$$

and hence $U\in W^{1,\infty}(\omega;L^2)\cap L^\infty(\omega;\mathring{H}_1)$. In particular, $D_t^4 u\in C(\omega;\mathring{H}_1)$ and we have

$$Au_{tt}\in C(\omega;\mathring{H}_1)=C(\omega;D(A^{1/2}))$$

which implies $u_{tt}\in C(\omega;H_3)\subset C(\bar{\Omega}\times R)$. Thus we conclude that u is a classical solution.

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