Quasi-ideals in near-rings

Yakabe, Iwao
Department of Mathematics, College of General Education, Kyushu University

https://doi.org/10.15017/1449038

出版情報：九州大学教養部数学雑誌. 14 (1), pp.41-46, 1983-12. 九州大学教養部数学教室
バージョン：
権利関係：

Math. Rep.
XIV-1, 1983.

Quasi-ideals in near-rings

Iwao YAKABE

(Received September 3, 1983)

1. Introduction

In ring theory the notion of quasi-ideal, introduced by O. Steinfeld in [2], has proved very useful. It is only natural to ask whether this notion may be extended to near-rings. The purpose of this note is to show that this is indeed the case.

We shall introduce the notion of quasi-ideal in near-rings and consider its elementary properties. Applying these properties, we shall characterize those near-rings which are near-fields, in terms of quasi-ideal.

2. Preliminaries on near-rings

By a *near-ring* we mean a non-empty set N in which an addition $+$ and a multiplication $\cdot$ are defined such that

(1) $(N, +)$ is a group,

(2) $(N, \cdot)$ is a semigroup,

(3) $(n+n')n''=nn''+n'n''\ (n, n', n''\in N)$.

In dealing with general near-rings the neutral element of $(N, +)$ will be denoted by 0.

In this section, N will denote a near-ring. The set $N_0=\{n\in N\,|\,n0=0\}$ is called *the zero-symmetric part* of N; $N_c=\{n\in N\,|\,n0=n\}$ is called *the constant part* of N. N is called *zero-symmetric* if $N=N_0$; N is called *constant* if $N=N_c$.

Let A and B be two non-empty subsets of N. We shall define two types of products:

$$AB=\{\sum a_ib_i\,|\,a_i\in A,\ b_i\in \mathrm{B}\},$$

and

$$A*B=\{\sum(a_i(a_i'+b_i)-a_ia_i')\,|\,a_i,\ a_i'\in A,\ b_i\in B\},$$

where $\sum$ denotes all possible additions of finite terms. In case that $B=\{b\}$, we denote AB by Ab, and so on.

A subgroup S of $(N, +)$ is called an ***N-subgroup*** of N if $NS \subseteq S$. A subgroup M of $(N, +)$ with $MM \subseteq M$ is called a ***subnear-ring***. A subnear-ring M of N is called ***invariant*** if $MN \subseteq M$ and $NM \subseteq M$. N_0 and N_c are subnear-rings of N.

A normal subgroup I of $(N, +)$ is called an ***ideal*** of N if

(a) $IN \subseteq I$,

(b) $N*I \subseteq I$.

A normal subgroup I of $(N, +)$ with (a) is called a ***right ideal*** of N, while a normal subgroup I of $(N, +)$ with (b) is said to be a ***left ideal*** of N.

An element d of N is called ***distributive*** if $d(n+n')=dn+dn'$ for all elements n, n' of N. The set of all distributive elements of N will be denoted by N_d.

3. Quasi-ideals and elementary properties

Let N be a near-ring. A subgroup Q of $(N, +)$ is called a *quasi-ideal* of N if $QN \cap NQ \cap N*Q \subseteq Q$.

It is clear that right ideals, left ideals, N-subgroups and invariant subnear-rings of N are quasi-ideals of N. In particular, the zero-symmetric part N_0 and the constant part N_c of N are quasi-ideals of N.

We have the following elementary properties of quasi-ideals.

PROPOSITION 1. *The set of all quasi-ideals of a near-ring N forms a Moore-system on N.*

PROOF. Let $Q_\lambda (\lambda \in \Lambda)$ be any set of quasi-ideals of N. Then $\bigcap_{\lambda \in \Lambda} Q_\lambda$ is clearly a subgroup of $(N, +)$. Moreover, for every $Q_\mu (\mu \in \Lambda)$, we have

$$\begin{aligned} D &= (\bigcap_{\lambda \in \Lambda} Q_\lambda) N \cap N(\bigcap_{\lambda \in \Lambda} Q_\lambda) \cap N*(\bigcap_{\lambda \in \Lambda} Q_\lambda) \\ &\subseteq Q_\mu N \cap NQ_\mu \cap N*Q_\mu \subseteq Q_\mu. \end{aligned}$$

Hence $D \subseteq \bigcap_{\lambda \in \Lambda} Q_\lambda$, that is, $\bigcap_{\lambda \in \Lambda} Q_\lambda$ is a quasi-ideal of N.

PROPOSITION 2. *The intersection of a quasi-ideal Q and a subnear-ring M of a near-ring N is a quasi-ideal of M.*

PROOF. $Q \cap M$ is clearly a subgroup of $(M, +)$. Moreover, we have

$$(Q\cap M)M\cap M(Q\cap M)\cap M^*(Q\cap M)$$
$$\subseteq(Q\cap M)M\cap M(Q\cap M)\subseteq MM\subseteq M$$

and

$$(Q\cap M)M\cap M(Q\cap M)\cap M^*(Q\cap M)$$
$$\subseteq QN\cap NQ\cap N^*Q\subseteq Q.$$

These imply that $Q\cap M$ is a quasi-ideal of M.

PROPOSITION 3. *Let N be a zero-symmetric near-ring. Then a subgroup Q of $(N,+)$ is a quasi-ideal of N if and only if $QN\cap NQ\subseteq Q$.*

PROOF. We first remark that $NQ\subseteq N^*Q$. In fact, for any elements n of N and q of Q, we have

$$nq=n(0+q)-n0,$$

since N is zero-symmetric. Hence $NQ\subseteq N^*Q$.

From this property, we have

$$QN\cap NQ\cap N^*Q=QN\cap NQ,$$

by which this proposition is easily seen.

Proposition 3 implies that, in the case of rings, quasi-ideals in our sense coincide with those in [2].

4. Quasi-ideals which are subnear-rings

It is known that each quasi-ideal of a ring R is a subring of R (see [3, Proposition 2.1]). This proposition has no analogue for near-rings, as we will give an example later. In this section, we are going to characterize those quasi-ideals which are subnear-rings.

Let N be a near-ring and Q a quasi-ideal of N. By Proposition 2, $Q_0=Q\cap N_0$ is a quasi-ideal of N_0 and $Q_c=Q\cap N_c$ is a quasi-ideal of N_c. If $Q=Q_0+Q_c=\{q+q'\mid q\in Q_0, q'\in Q_c\}$, we say that Q is *of the first kind.*

We now characterize those quasi-ideals which are subnear-rings:

THEOREM 1. *Let N be a near-ring and Q a quasi-ideal of N. Then the following conditions are equivalent:*

(1) *Q is a subnear-ring of N;*

(2) $Q0 \subseteq Q$;

(3) *Q is of the first kind.*

PROOF. (1) $\Rightarrow$ (2): In fact, $Q0 \subseteq QQ \subseteq Q$.

(2) $\Rightarrow$ (3): For any element q of Q, we have

$$q = (q - q0) + q0,$$

where $q - q0 \in N_0$ and $q0 \in N_c$. By assumption $q0 \in Q$, so $q - q0 \in Q$. Hence $q - q0 \in Q_0$ and $q0 \in Q_c$, which imply that $Q \subseteq Q_0 + Q_c \subseteq Q$.

(3) $\Rightarrow$ (1): For any element q of Q, we can write $q = q_o + q_c$ with $q_o \in Q_o$ and $q_c \in Q_c$.

Then, for any elements q, q' of Q, we have

$$qq' = (q_0 + q_c)q' = q_0 q' + q_c q' = q_0 q' + q_c.$$

Moreover, we have

$$q_0 q' = q_0(0 + q') - q_0 0 \in N^*Q,$$

whence $q_0 q' \in QN \cap NQ \cap N^*Q \subseteq Q$.

These imply that $qq' \in Q$, so $QQ \subseteq Q$.

COROLLARY *If N is zero-symmetric or constant, then each quasi-ideal of N is a subnear-ring of N.*

PROOF. For each quasi-ideal Q of N, if N is zero-symmetric then $Q0 = \{0\} \subseteq Q$. If N is constant then $Q0 = Q$.

The following example due to [**1**, 2.19 Remarks] shows that there exists a quasi-ideal which is not a subnear-ring: Let Z be the ring of integers and $Z[x]$ the ring of polynomials in x over Z. Then $N = (Z[x], +, \circ)$ is a near-ring, where $+$ means usual addition and $\circ$ usual substitution. Consider

$$L = \{\sum a_i x^i \mid \sum a_i \in 2Z\}.$$

It is easy to see that L is a quasi-ideal of N, but it is not of the first kind.

5. Characterizations of near-rings which are near-fields

A near-ring N is called a ***near-field***, if the set of all non-zero elements of N forms a group under multiplication.

Let Z_2 be the integers modulo 2. We recall that $(Z_2, +)$ with $0\cdot 0=0\cdot 1=0$, $1\cdot 0=1\cdot 1=1$ is a near-field (see [1, 1.15 Examples]). As usual, in this section, we will exclude those near-fields which are isomorphic to this near-field.

There are some characterizations of those 'zero-symmetric' near-rings which are near-fields (see [1, 8.3 Theorem]). In this section we are going to characterize those 'general' near-rings which are near-fields.

Let N be a near-ring. Clearly $\{0\}$ and N are quasi-ideals of N. If N has no quasi-ideals except $\{0\}$ and N, we say that N is *Q-simple.*

PROPOSITION 4. *A near-ring N is Q-simple, then either N is zero-symmetric or N is constant.*

PROOF. Since the zero-symmetric part N_0 of N is a quasi-ideal of N, either $N_0=N$ or $N_0=\{0\}$, that is, either N is zero-symmetric or N is constant.

We now characterize those near-rings which are near-fields:

THEOREM 2. *Let N be a near-ring with more than one element. Then the following conditions are equivalent*:

(1) *N is a near-field*;

(2) *N is Q-simple and N has a left identity*;

(3) *N is Q-simple, $N_d\neq\{0\}$ and for each non-zero element n of N there exists an element n' of N such that $n'n\neq 0$.*

PROOF. (1)$\Rightarrow$(2): Clearly N has a left identity and N is zero-symmetric. Let Q be a quasi-ideal of N and q a non-zero element of Q, then $N=qN=Nq$. Hence we have by Proposition 3

$$N=qN\cap Nq\subseteq QN\cap NQ\subseteq Q,$$

whence $Q=N$.

(2)$\Rightarrow$(3): If N has a left identity e, then e is non-zero and distributive. Hence $N_d\neq\{0\}$ and $en=n\neq 0$ for every non-zero element n of N.

(3)$\Rightarrow$(1): $N_d \neq \{0\}$ implies that N is not constant. Hence N is zero-symmeric by Proposition 4. Moreover, let n be a non-zero element of N, then Nn is a quasi-ideal of N and $n'n \in Nn$, where n' is an element of N such that $n'n \neq 0$. Hence $Nn = N$.

Therefore, it follows from [1, 8.3 Theorem] that N is a near-field.

References

[1] G. PILZ: Near-rings, North-Holland, Amsterdam, Revised edition, 1983.

[2] O. STEINFELD: *On ideal-quotients and prime ideals*, Acta Math. Acad. Sci. Hung. 4(1953), 289-298.

[3] ——————: Quasi-ideals in rings and semigroups, Akadémiai Kiadó, Budapest, 1978.