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Localization of global existence of holomorphic exact forms with complex parameters

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Dedicated to Professor Shigeo Nakano on his sixtieth birthday

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0. Introduction

Let D be a domain in the product space $Z \times S$ of an m -dimensional complex manifold Z of variable z and an n -dimensional complex space S of parameter s . Let \mathcal{O} be the sheaf over D of germs of holomorphic functions and, for $p=1, 2, \dots, m$, $\mathcal{O}_{z,s}^p$ be the sheaf of p -forms

$$g = \sum_{1 \leq i_1 < i_2 < \dots < i_p \leq m} g_{i_1 \dots i_p}(z, s) dz_{i_1} \wedge dz_{i_2} \wedge \dots \wedge dz_{i_p}$$

where $g_{i_1 i_2 \dots i_p}(z, s)$'s are germs of holomorphic functions of the variables of $z \in Z$ and $s \in S$. Then, each $\mathcal{O}_{z,s}^p$ is a locally free \mathcal{O} -sheaf and hence is an analytic coherent sheaf over D . We introduce differentials with respect to the variable z by putting

$$d_z^p g = \sum_{1 \leq i_1 < i_2 < \dots < i_p \leq m} d_z g_{i_1 i_2 \dots i_p}(z, s) dz_{i_1} \wedge dz_{i_2} \wedge \dots \wedge dz_{i_p}$$

where d_z is the complex differential with respect to the variable z defined by

$$d_z g_{i_1 i_2 \dots i_p}(z, s) = \sum_{j=1}^m \frac{\partial g_{i_1 i_2 \dots i_p}}{\partial z_j} dz_j$$

for $p=1, 2, \dots, m$, and z_1, z_2, \dots, z_m are local holomorphic coordinates in Z . Especially we put

$$T = d_z^p.$$

The short exact sequence of sheaves

$$0 \rightarrow \text{Ker } T \xrightarrow{i} \mathcal{O} \xrightarrow{T} \text{Im } T \rightarrow 0$$

gives the long exact sequence of cohomology groups

$$\dots \rightarrow H^0(D, \mathcal{O}) \xrightarrow{T} H^0(D, \text{Im } T) \rightarrow H^1(D, \text{Ker } T) \rightarrow H^1(D, \mathcal{O}) \rightarrow \dots$$

If D is a Stein domain, we have $H^1(D, \mathcal{O}) = 0$ by the theorem of Cartan-

Oka-Serre. Hence we have

$$H^1(D, \text{Ker } T) \cong H^0(D, \text{Im } T) / TH^0(D, \mathcal{O})$$

and $H^1(D, \text{Ker } T)$ accounts for obstacles in seeking a solution $f \in H^0(D, \mathcal{O})$ of overdetermined system

$$Tf = g$$

globally for any given $g \in H^0(D, \text{Im } T)$, that is, for any given

$$g = \sum_{j=1}^m g_j(z, s) dz_j \in H^0(D, \mathcal{O}_{z,s}^1) \text{ satisfying } d_z^1 g = 0, \text{ i. e.,}$$

$$\frac{\partial g_j}{\partial z_k} = \frac{\partial g_k}{\partial z_j} \quad (j, k = 1, 2, \dots, m)$$

for any local holomorphic coordinates z_1, z_2, \dots, z_m .

Moreover, we have the exact sequence

$$0 \rightarrow \text{Ker } T \rightarrow \mathcal{O}_{z,s}^0 \xrightarrow{r=d_z^0} \mathcal{O}_{z,s}^1 \xrightarrow{d_z^1} \dots \rightarrow \mathcal{O}_{z,s}^m \xrightarrow{d_z^m} 0$$

of sheaves over D . If D is a Stein domain, we have $H^q(D, \mathcal{O}_{z,s}^p) = 0$ for $q=1, 2, \dots$ by the theorem B of Cartan-Serre. So the above exact sequence is a resolution of the sheaf $\text{Ker } T$, locally constant with respect to the variable z , over the Stein space D . By a generalized Dolbeault's isomorphism, we have a Serre's isomorphism with parameter $s \in S$

$$H^p(D, \text{Ker } T) \cong \{f \in H^0(D, \mathcal{O}_{z,s}^p); d_z^p f = 0\} / d_z^{p-1} H^0(D, \mathcal{O}_{z,s}^{p-1}).$$

Let $\psi: D \rightarrow S$ be the canonical projection, $D(z, s)$ be the connected component of $\psi^{-1}(s)$ in the manifold $Z \times S$ containing $(z, s) \in D$, \tilde{D} be the set of all $D(z, s)$, (z, s) running over D , $\eta: D \rightarrow \tilde{D}$ be the canonical mapping defined by

$$\eta(z, s) = D(z, s) \in \tilde{D}$$

for $(z, s) \in D$. We introduce in D an equivalence relation \sim such that (z, s) and (z', s') are equivalent if $s=s'$ and $D(z, s) = D(z', s')$. Then \tilde{D} is regarded as the quotient set of D by this relation. As we stated in the previous paper [10], \tilde{D} equipped with the quotient topology is a T_1 -space. If two distinct connected components join in one connected component as s varies in S , then the T_1 -space \tilde{D} bifurcates and is not a T_2 -space. But the canonical mapping $\phi: \tilde{D} \rightarrow S$ is a local homeomorphism and the pair (\tilde{D}, ϕ) is called a T_1 -space over the complex space S . Moreover, we can speak of

holomorphic objects as in the previous paper [10]. We denote by $\tilde{\mathcal{O}}$ the sheaf over \tilde{D} of germs of holomorphic functions by using the type \sim in order to distinguish it from the sheaf \mathcal{O} of germs of holomorphic function of variables z and s . If \tilde{D} is a Hausdorff space, then \tilde{D} is equipped with a structure of a complex space and ϕ is a locally biholomorphic mapping. Then (\tilde{D}, ϕ) is called a *domain over the complex space S*.

In the present paper, we prove that the countability at most of the cohomology of $H^0(D, \text{Ker } T)$ -modules $H^p(D, \text{Ker } T)$ for $p \in \{1, 2, \dots, n\}$ implies the Steinness of the space \tilde{D} of cuts of D , in case that D is a Stein domain and that S is an n -dimensional Stein manifold. We also give the following characterization of the existence of global solutions to the above equations:

THEOREM. *Let D be a Stein domain in the product manifold of an m -dimensional complex manifold Z and an n -dimensional Stein manifold S . Assume that D has a regular boundary and finitely generated C-cohomology. Then the following properties (1) and (2) are equivalent:*

(1) *For any $p=1, 2, \dots, m$ and any $g \in H^0(D, \mathcal{O}_{z,s}^p)$ satisfying $d_z^p g = 0$, there exists a $f \in H^0(D, \mathcal{O}_{z,s}^{p-1})$ satisfying $d_z^{p-1} f = g$.*

(2) *For any $(z, s) \in D$, we have $H^p(D(z, s), \mathbb{C}) = 0$ for $p=1, 2, \dots, m$, \tilde{D} is a Hausdorff space and, for the natural complex structure, \tilde{D} is a Stein manifold.*

Since the property (2) is local in the parameter space S , we have a local characterization of the property (1) in the parameter space S . Roughly talking, we have a local characterization of the global existence of solutions of the overdetermined system $d_z^{p-1} f = g, d_z^p g = 0$. This would be an important principle in the theory of partial differential equations in complex domains.

1. Steinness of schlicht domains

Hereafter, if we use the notations $D, \tilde{D}, \mathcal{O}, \tilde{\mathcal{O}}$ and so on, then D is a domain in the product space $Z \times S$ of an m -dimensional complex manifold Z and an n -dimensional complex space S , and $\tilde{D}, \mathcal{O}, \tilde{\mathcal{O}}, \phi, \psi, \eta, T$ etc are the ones introduced in the introduction where D, Z and S are not necessarily Stein. The cohomology used is the Čech cohomology.

LEMMA 1. *If $N = \dim H^1(D, \text{Ker } T) + 1 < +\infty$, then we have $\dim H^1(\tilde{D}, \tilde{\mathcal{O}}) < N$.*

PROOF. Let $\mathcal{U} = \{U_i; i \in I\}$ be an open covering of \tilde{D} and $\{g_i^a; i, j \in I\}$ ($a=1, 2, \dots, N$) be N cocycles of $Z^1(\mathcal{U}, \tilde{\mathcal{O}})$. Each cocycle $\{g_i^a; i, j \in I\}$ is an element of $Z^1(\eta^{-1}(\mathcal{U}), \text{Ker } T)$ where $\eta^{-1}(\mathcal{U}) = \{\eta^{-1}(U_i); i \in I\}$. Since the mapping $H^1(\eta^{-1}(\mathcal{U}), \text{Ker } T) \rightarrow H^1(D, \text{Ker } T)$ is injective, we have $\dim H^1(\eta^{-1}(\mathcal{U}), \text{Ker } T) < N$. There exist a non zero vector $(c_a) \in \mathbb{C}^N$ and a cochain $\{f_i; i \in I\}$ of $C^0(\mathcal{U}, \text{Ker } T)$ whose coboundary is $\{\sum_a c_a g_i^a; i, j \in I\}$. Since each f_i is a holomorphic function of the variables $z \in Z$ and $s \in S$, locally independent of the first variable z , for any $i \in I$, there exists an element g_i of $H^0(U_i, \tilde{\mathcal{O}})$ such that $f_i = g_i \circ \eta$ in $\eta^{-1}(U_i)$ for any $i \in I$. Since $\{\sum_{a=1}^N c_a g_i^a; i, j \in I\}$ is the coboundary of the cochain $\{g_i; i \in I\}$, we have $\dim H^1(\mathcal{U}, \tilde{\mathcal{O}}) < N$. Taking into account of the direct limit, we have $\dim H^1(\tilde{D}, \tilde{\mathcal{O}}) < N$.

THEOREM 1. *If S is a two dimensional Stein space, if ϕ is injective and if the dimension of $H^1(D, \text{Ker } T)$ is finite or countably infinite, then \tilde{D} is a Stein space.*

PROOF. By Theorem 4 of Y. T. Siu [21], $\dim H^1(D, \text{Ker } T) < +\infty$. Let f be a holomorphic function in S which is not constant on any irreducible component of S . By R. Narasimhan [15], the analytic set $\text{loc } f = \{s \in S; f(s) = 0\}$ with the reduced structure is a Stein space. By Theorem 2 of E. Ballico [1], \tilde{D} is a Stein space.

COROLLARY. *If D is a Stein domain, if ϕ is injective and if for any $g \in H^0(D, \mathcal{O}^{\frac{1}{2}, s})$ with $d_1^{\frac{1}{2}} g = 0$, there exists $f \in H^0(D, \mathcal{O}^{\frac{0}{2}, s})$ with $d_1^{\frac{0}{2}} f = g$, then \tilde{D} is a Stein space.*

PROOF. Since D is a Stein domain, we have $H^1(D, \text{Ker } T) = H^0(D, \text{Im } T) / \text{TH}^0(D, \mathcal{O})$.

2. Descent of dimension

LEMMA 2. *Let f be a holomorphic function on S which is not constant on any irreducible component of S . If p is an integer with $p \geq 1$, and if $N_p = \dim H^p(D, \text{Ker } T) + 1 < +\infty$, $N_{p+1} = \dim H^{p+1}(D, \text{Ker } T) + 1 < +\infty$, then we have*

$$\dim H^p(D, \text{Ker } T / (f \circ \phi) \text{Ker } T) < N_p N_{p+1}.$$

PROOF. The multiplication by $f \circ \psi$ induces canonically the sheaf homomorphism $\mu: \text{Ker } T \rightarrow \text{Ker } T$. The short exact sequence of sheaves over D

$$0 \rightarrow \text{Ker } T \xrightarrow{\mu} \text{Ker } T \rightarrow \text{Ker } T / (f \circ \psi) \text{Ker } T \rightarrow 0$$

gives the long exact sequence of cohomology $H^p(D, \text{Ker } T)$ -modules

$$\cdots \rightarrow H^p(D, \text{Ker } T) \xrightarrow{\mu} H^p(D, \text{Ker } T) \rightarrow H^p(D, \text{Ker } T / (f \circ \psi) \text{Ker } T) \rightarrow H^{p+1}(D, \text{Ker } T) \rightarrow \cdots$$

Hence we have the lemma.

The situation is slite different in case that $p=0$ which will be discussed in Lemma 10. In case that $m=0$, we have the following Lemma.

LEMMA 3. *Let ϕ be a local homeomorphism of a T_1 -space Ω into a complex space S , f be a holomorphic function on S which is not constant on any irreducible component of S and $\tilde{\mathcal{O}}$ be the sheaf over Ω of germs of holomorphic functions. If $p \geq 1$ and if $N_p = \dim H^p(\Omega, \tilde{\mathcal{O}}) + 1 < +\infty$, and $N_{p+1} = \dim H^{p+1}(\Omega, \tilde{\mathcal{O}}) + 1 < +\infty$, then we have $\dim H^p(\Omega, \tilde{\mathcal{O}} / (f \circ \phi) \tilde{\mathcal{O}}) < N_p N_{p+1}$.*

3. Separation of topology

LEMMA 4. *If S is an n -dimensional Stein space and if the dimension of $H^p(D, \text{Ker } T)$ is finite or countably infinte for any $p=1, 2, \dots$, then \tilde{D} is a T_2 -space.*

PROOF. By Theorem 4 of Y. T. Siu [21], $N_p = \dim H^p(D, \text{Ker } T) + 1 < +\infty$ for any $p=1, 2, \dots$. Assume that \tilde{D} were not a T_2 -space. There would exist two distinct points x' and x'' of \tilde{D} such that the union of the filters $\mathcal{V}(x')$ and $\mathcal{V}(x'')$ of the systems of neighborhoods of, respectively, x' and x'' generates a filter \mathcal{V} on \tilde{D} . Since $\phi: \tilde{D} \rightarrow S$ is continuous at x' and x'' and the filter \mathcal{V} converges to x' and x'' , the filter basis $\phi(\mathcal{V})$ converges to $\phi(x')$ and $\phi(x'')$ in S . Since S is a Hausdorff space, we have $\phi(x') = \phi(x'') = s_0$. Since ϕ is a local homeomorphism, there exist open neighborhoods V_1, V_2 and V of, respectively, x', x'' and s_0 such that the restriction $\phi|_{V_1}$ and $\phi|_{V_2}$ are homeomorphisms of V_1 and V_2 onto a connected Stein subdomain V of S . By the assumption $V_1 \cap V_2$ is not empty. Let x''' be its point. We put $s_1 = \phi(x''') \in V$.

Since S is an n -dimensional Stein space, there exist $n-1$ holomorphic functions f_1, f_2, \dots, f_{n-1} such that, for any $k \in \{1, 2, \dots, n-1\}$, the analytic set

$$\text{loc}(f_1, f_2, \dots, f_k) = \{s \in S; f_i(s) = 0, 1 \leq i \leq k\}$$

in S is of pure dimension $n-k$ and that $A = \text{loc}(f_1, f_2, \dots, f_{n-1})$ contains s_0 and s_1 . By R. Narasimhan [15], the one dimensional complex space A without compact irreducible components is a Stein space with respect to the reduced structure. Since $\tilde{\mathcal{O}} / \sum_{i=1}^{n-1} f_i \tilde{\mathcal{O}}$ is an analytic coherent sheaf on the Stein space A , we have $H^1(S, \tilde{\mathcal{O}} / \sum_{i=1}^{n-1} f_i \tilde{\mathcal{O}}) = H^1(A, \tilde{\mathcal{O}} / \sum_{i=1}^{n-1} f_i \tilde{\mathcal{O}}) = 0$ by

the the theorem B of Cartan-Serre. Hence the complex space $(A, \tilde{\mathcal{O}} / \sum_{i=1}^{n-1} f_i \tilde{\mathcal{O}})$ is a Cousin-I space. There exists a meromorphic function t in the

above complex space A such that $t \in H^0(A - \{s_0\}, \tilde{\mathcal{O}} / \sum_{i=1}^{n-1} f_i \tilde{\mathcal{O}})$, and t^{-1} is holomorphic and has the value zero at s_0 . By induction and Lemmata 1 and 2, we have $N = \dim H^1(\phi^{-1}(A), \tilde{\mathcal{O}} / \sum_{i=1}^{n-1} (f_i \circ \phi) \tilde{\mathcal{O}}) + 1 \leq \dim H^1(\phi^{-1}(A),$

$\text{Ker } T / \sum_{i=1}^{n-1} (f_i \circ \phi) \text{Ker } T) + 1 < +\infty$. We put $U_1 = \phi^{-1}(A) \cap V_1$ and $U_2 = \phi^{-1}(A) - \{x'\}$ and consider the open covering $\mathfrak{U} = \{U_1, U_2\}$ of $\phi^{-1}(A)$. For any $a \in \{1, 2, \dots, N\}$, the holomorphic function t^a in $U_1 \cap U_2 = \phi^{-1}(A) \cap V_1 - \{x'\}$ defines an element of $Z^1(\mathfrak{U}, \tilde{\mathcal{O}} / \sum_{i=1}^{n-1} (f_i \circ \phi) \tilde{\mathcal{O}})$. Since $\dim H^1(\phi^{-1}(A), \tilde{\mathcal{O}} / \sum_{i=1}^{n-1} (f_i \circ \phi) \tilde{\mathcal{O}}) < N$, there exist functions g and h holomorphic, respectively, in U_1 and U_2 , and a non zero vector $(c_a) \in C^N$ such that

$$\sum_{a=1}^N c_a t^a = g - h$$

in $U_1 \cap U_2 = \phi^{-1}(A) \cap V_1 - \{x'\}$.

Since ϕ is a local homeomorphism, the set

$$E = \{s \in A \cap V; (\phi|V_1)^{-1}(s) = (\phi|V_2)^{-1}(s)\}$$

is an open subset of $A \cap V$ and s_0 is its boundary point. We consider two holomorphic functions $h_0(\phi|V_1)^{-1}$ and $h_0(\phi|V_2)^{-1}$ in $A \cap V - \{s_0\}$. Since they coincide with each other on the non empty open subset E of the connected open subset $V \cap \phi^{-1}(A) - \{s_0\}$, they coincide with each other on $V \cap \phi^{-1}(A) - \{s_0\}$ by the theorem of identity. Since $h \in H^0(V_2, \tilde{\mathcal{O}})$, $h_0(\phi|V_2)^{-1}$ is holomorphic at s_0 . Hence $h_0(\phi|V_1)^{-1}$ is also holomorphic at s_0 too.

Now we consider the relation

$$\sum_{a=1}^N c_a t^a = g \circ (\phi|V_1)^{-1} - h_0(\phi|V_1)^{-1}$$

in $V \cap \phi^{-1}(A) - \{s_0\}$. As we have seen above, the right-hand side is holomorphic at s_0 because g is holomorphic in V_1 . But the left hand side has a pole at s_0 . This is a contradiction. Thus we have proved that \tilde{D} is a Hausdorff space.

In case that $m=0$, we have the following lemma:

LEMMA 5. *Let ϕ be a local homeomorphism of a T_1 -space Ω in an n -dimensional Stein space S and $\tilde{\mathcal{O}}$ be the sheaf over Ω of germs of holomorphic functions. If the dimension of $H^p(\Omega, \tilde{\mathcal{O}})$ is finite or countably infinite for any $p=1, 2, \dots$, then Ω is a T_2 -space.*

4. Domains over Stein manifolds

LEMMA 6. *Let f_1, f_2, \dots, f_{n-1} be holomorphic functions on an n -dimensional complex space S such that, for any $k \in \{1, 2, \dots, n-1\}$, the analytic set $\text{loc}(f_1, f_2, \dots, f_k) = \{s \in S; f_i(s) = 0, 1 \leq i \leq k\}$ in S is of pure dimension $n - k$. If the dimension of $H^p(D, \text{Ker } T)$ is finite or countably infinite for any $p \in \{1, 2, \dots, n\}$, then we have*

$$H^1(\tilde{D}, \tilde{\mathcal{O}} / \sum_{i=1}^k (f_i \circ \phi) \tilde{\mathcal{O}}) = 0$$

for any $k \in \{0, 1, \dots, n-1\}$ where $\sum_{i=1}^0 (f_i \circ \phi) \mathcal{O} = 0$.

PROOF. By Theorem 4 of Y. T. Siu [21], we have $\dim H^p(D, \text{Ker } T) < +\infty$ for any $p \in \{1, 2, \dots, n\}$. By Lemma 4, \tilde{D} is a T_2 -space. So \tilde{D} is equipped with complex structure such that ϕ is a locally biholomorphic mapping of the complex space \tilde{D} into the n -dimensional complex space S . By Lemma 2 we have $\dim H^p(D, \text{Ker } T / (f_0 \phi) \text{Ker } T) < N_p N_{p+1}$. By induction and Lemma 1, we have

$$\begin{aligned} \dim H^1(\tilde{D}, \tilde{\mathcal{O}} / \sum_{i=1}^k (f_i \circ \phi) \tilde{\mathcal{O}}) &\leq \dim H^1(D, \text{Ker } T / \sum_{i=1}^k (f_i \circ \phi) \text{Ker } T) \\ &< +\infty \end{aligned}$$

for any $k \in \{1, 2, \dots, n-1\}$. We prove the lemma by backward induction with respect to k . In case that $k=n-1$, if $\phi^{-1}(\text{loc}(f_1, f_2, \dots, f_{n-1}))$ had a compact irreducible component K , the image $\phi(K)$ of the proper mapping $\phi|_K: K \rightarrow S$, restriction of ϕ to K , would be a one dimensional compact analytic set of the Stein space S by R. Remmert [19]. This is a contradic-

tion. The one dimensional complex space $A = \phi^{-1}(\text{loc}(f_1, f_2, \dots, f_{n-1}))$ with reduced structure and without compact irreducible component is a Stein space by R. Narasimhan [15]. Since $\tilde{\mathcal{O}} / \sum_{i=1}^{n-1} (f_i \circ \phi) \tilde{\mathcal{O}}$ is an analytic coherent sheaf, we have

$$\begin{aligned} H^1(\tilde{D}, \tilde{\mathcal{O}} / \sum_{i=1}^{n-1} (f_i \circ \phi) \tilde{\mathcal{O}}) &\cong H^1(\phi^{-1}(\text{loc}(f_1, f_2, \dots, f_{n-1})), \tilde{\mathcal{O}} / \sum_{i=1}^{n-1} (f_i \circ \phi) \\ &\tilde{\mathcal{O}}) = 0 \end{aligned}$$

by Theorem B of Cartan-Serre.

We suppose $H^1(\tilde{D}, \tilde{\mathcal{O}} / \sum_{i=1}^k (f_i \circ \phi) \tilde{\mathcal{O}}) \ni a \neq 0$ for $k \in \{2, 3, \dots, n-1\}$. Let \mathcal{A} be the set of entire functions on the complex plane C not identically zero on any irreducible component of A . For any $\alpha \in \mathcal{A}$, we consider multiplication $\mu(\alpha)$ by $\alpha \circ f_{k+1} \circ \phi$. The short exact sequence of sheaves over \tilde{D}

$$\begin{aligned} 0 \rightarrow \tilde{\mathcal{O}} / \sum_{i=1}^k (f_i \circ \phi) \tilde{\mathcal{O}} &\xrightarrow{\mu(\alpha)} \tilde{\mathcal{O}} / \sum_{i=1}^k (f_i \circ \phi) \tilde{\mathcal{O}} \rightarrow \\ \tilde{\mathcal{O}} / ((\alpha \circ f_{k+1} \circ \phi) \tilde{\mathcal{O}} &+ \sum_{i=1}^k (f_i \circ \phi) \tilde{\mathcal{O}}) \rightarrow 0 \end{aligned}$$

gives the long exact sequence of cohomology modules

$$\begin{aligned} \dots \rightarrow H^1(\tilde{D}, \tilde{\mathcal{O}} / \sum_{i=1}^k (f_i \circ \phi) \tilde{\mathcal{O}}) &\xrightarrow{\mu(\alpha)} H^1(\tilde{D}, \tilde{\mathcal{O}} / \sum_{i=1}^k (f_i \circ \phi) \tilde{\mathcal{O}}) \\ \rightarrow H^1(\tilde{D}, \tilde{\mathcal{O}} / ((\alpha \circ f_{k+1} \circ \phi) \tilde{\mathcal{O}} &+ \sum_{i=1}^k (f_i \circ \phi) \tilde{\mathcal{O}})) \rightarrow \dots \end{aligned}$$

By the assumption of our induction, we have

$$H^1(\tilde{D}, \tilde{\mathcal{O}} / ((\alpha \circ f_{k+1} \circ \phi) \tilde{\mathcal{O}} + \sum_{i=1}^k (f_i \circ \phi) \tilde{\mathcal{O}})) = 0.$$

By the above exact sequence, the C -linear mapping

$$\mu(\alpha); H^1(\tilde{D}, \tilde{\mathcal{O}} / \sum_{i=1}^k (f_i \circ \phi) \tilde{\mathcal{O}}) \rightarrow H^1(\tilde{D}, \tilde{\mathcal{O}} / \sum_{i=1}^k (f_i \circ \phi) \tilde{\mathcal{O}})$$

is surjective. Since $\mu(\alpha)$ is a C -linear mapping of the finite dimensional C -vector space $H^1(\tilde{D}, \tilde{\mathcal{O}} / \sum_{i=1}^k (f_i \circ \phi) \tilde{\mathcal{O}})$ onto its self, $\mu(\alpha)$ is bijective.

Since the mapping $\mathcal{A} \ni \alpha \rightarrow \mu(\alpha)(a) \in H^1(\tilde{D}, \tilde{\mathcal{O}} / \sum_{i=1}^k (f_i \circ \phi) \tilde{\mathcal{O}})$ is injective, we have

$$+\infty > \dim H^1(\tilde{D}, \tilde{\mathcal{O}} / \sum_{i=1}^k (f_i \circ \phi) \tilde{\mathcal{O}}) \geq \dim \mathcal{A} = +\infty.$$

This is a contradiction. Thus we have proved

$$H^1(\tilde{D}, \tilde{\mathcal{O}} / \sum_{i=1}^k (f_i \circ \phi) \tilde{\mathcal{O}}) = 0$$

for any $k \in \{0, 1, \dots, n-1\}$.

LEMMA 7. *Let f_1, f_2, \dots, f_n be holomorphic functions on an n -dimensional complex space S such that, for any $k \in \{1, 2, \dots, n\}$, the analytic set $\text{loc}(f_1, f_2, \dots, f_k)$ in S is of pure dimension $n-k$. If $H^p(\phi^{-1}(f_1, f_2, \dots, f_n), C) = 0$ for any $p \geq 1$ and if the dimension of $H^p(D, \text{Ker } T)$ is finite or countably infinite for any $p \geq 1$, we have*

$$H^p(D, \text{Ker } T / \sum_{i=1}^k (f_i \circ \phi) \text{Ker } T) = 0$$

for any $p \geq 1$ and $k \in \{0, 1, \dots, n\}$ where $\sum_{i=1}^0 (f_i \circ \phi) \text{Ker } T = 0$.

PROOF. By Theorem 4 of Y. T. Siu [21], we have $\dim H^p(D, \text{Ker } T) < +\infty$ for any $p \geq 1$. We prove the lemma by backward induction with respect to k . In case that $k=n$, we have the lemma by the assumption that $H^p(D, \text{Ker } T / \sum_{i=1}^n (f_i \circ \phi) \text{Ker } T) = H^p(\phi^{-1}(f_1, f_2, \dots, f_n), C) = 0$ for any $p \geq 1$. Since the short exact sequence

$$\begin{aligned} 0 \rightarrow \text{Ker } T / \sum_{i=1}^k (f_i \circ \phi) \text{Ker } T \xrightarrow{(\alpha)} \text{Ker } T / \sum_{i=1}^k (f_i \circ \phi) \text{Ker } T \rightarrow \\ \text{Ker } T / ((\alpha f_{k+1} \circ \phi) \text{Ker } T + \sum_{i=1}^k (f_i \circ \phi) \text{Ker } T) \rightarrow 0 \end{aligned}$$

and the assumption of our induction give the injective mapping of the set of entire functions \mathcal{A} on C into $H^p(D, \text{Ker } T / \sum_{i=1}^k (f_i \circ \phi) \text{Ker } T)$ for any $p \geq 1$, we have the lemma.

LEMMA 8. *Let D be a Stein domain such that $H^p(D(z, s), C) = 0$ for $p \in \{1, 2, \dots, m\}$. Then the following properties (1) and (2) are equivalent.*

- (1) *For any $p \geq 1$, we have $H^p(D, \text{Ker } T) = 0$.*
- (2) *For any $p \in \{1, 2, \dots, m\}$, the dimension of $H^p(D, \text{Ker } T)$ is finite or countably infinite.*

PROOF. (2) \rightarrow (1). Since D is a Stein domain, we have the Serre's isomorphism with complex parameter $s \in S$ which is obtained by the generalized Dolbeault's isomorphism

$$H^p(D, \text{Ker } T) \cong \{g \in H^0(D, \mathcal{O}_D^p) \mid d_{\bar{z}}^p g = 0\} / d_{\bar{z}}^{p-1} H^0(D, \mathcal{O}_D^{p-1})$$

for any $p \geq 1$. Hence we have $H^p(D, \text{Ker } T) = 0$ for $p \geq 1$. Y. T. Siu [22]

proved that $H^p(\Omega, \mathcal{F})=0$ for $p > m$ for an analytic coherent sheaf \mathcal{F} over an m -dimensional complex space Ω . So the situation is quite similar. Moreover, the lemma is similar to the results of Y. T. Siu [21]. Now, since S is a Stein space of dimension n , there exist holomorphic functions f_1, f_2, \dots, f_n such that $\text{loc}(f_1, f_2, \dots, f_k)$ is of pure dimension $n-k$ for any $k \in \{1, 2, \dots, n\}$. Then $\phi^{-1}(f_1, f_2, \dots, f_n)$ is a discrete set $\{s_\mu; \mu=1, 2, \dots\}$. For each s_μ , the connected components of $\phi^{-1}(s_\mu)$ are at most countable. There exists a countable subset $\{z_{\mu\nu}\}$ of Z such that $\phi^{-1}(s_\mu) = \bigcup_{\nu=1}^{\infty} D(z_{\mu\nu}, s_\mu)$ for any $\mu \geq 1$. By the assumption, we have

$$H^p(\phi^{-1}(f_1, f_2, \dots, f_n), \text{Ker } T) \cong \bigcup_{\mu, \nu} H^p(D(z_{\mu\nu}, s_\mu), \mathcal{C}) = 0.$$

for $p \leq m$. Since $\phi^{-1}(\text{loc}(f_1, f_2, \dots, f_n))$ is a Stein manifold, we have $H^p(\phi^{-1}(\text{loc}(f_1, f_2, \dots, f_n), \text{Ker } T) \cong H^p(\phi^{-1}(\text{loc}(f_1, f_2, \dots, f_n), \mathcal{C}) = 0$ by the Serre's isomorphism without parameter s . Applying Lemma 7 in case that $k=0$, we have $H^p(D, \text{Ker } T) = 0$ for any $p \geq 1$.

As the case that $m=0$ in Lemma 7, we have the following lemma:

LEMMA 9. *Let ϕ be a local homeomorphism of a T_1 -space Ω into an n -dimensional complex space S and f_1, f_2, \dots, f_{n-1} be holomorphic functions on S such that $\text{loc}(f_1, f_2, \dots, f_{n-1})$ is of pure dimension $n-k$ for any $k \in \{1, 2, \dots, n-1\}$. If the dimension of $H^p(\Omega, \tilde{\mathcal{O}})$ is finite or countably infinite, then we have $H^p(\Omega, \tilde{\mathcal{O}} / \sum_{i=1}^k (f_i \circ \phi) \tilde{\mathcal{O}}) = 0$ for any $p \geq 1$.*

LEMMA 10. *Let f_1, f_2, \dots, f_m be holomorphic functions on an n -dimensional complex space Ω with structure group $\tilde{\mathcal{O}}$ such that, for any $k \in \{1, 2, \dots, m\}$, $\text{loc}(f_1, f_2, \dots, f_k)$ is of pure dimension $n-k$. If $H^1(\Omega, \tilde{\mathcal{O}} / \sum_{i=1}^k (f_i \circ \phi) \tilde{\mathcal{O}}) = 0$ for any $k \in \{0, 1, \dots, m\}$ where $\tilde{\mathcal{O}} / \sum_{i=1}^0 (f_i \circ \phi) \tilde{\mathcal{O}} = 0$, then the canonical mapping*

$$H^0(\Omega, \tilde{\mathcal{O}}) \rightarrow H^0(\Omega, \tilde{\mathcal{O}} / \sum_{i=1}^m (f_i \circ \phi) \tilde{\mathcal{O}})$$

is surjective.

PROOF. For any $k \in \{0, 1, \dots, m\}$, the short exact sequence of sheaves over Ω

$$0 \rightarrow \tilde{\mathcal{O}} / \sum_{i=1}^{k-1} f_i \tilde{\mathcal{O}} \xrightarrow{\mu} \tilde{\mathcal{O}} / \sum_{i=1}^{k-1} f_i \tilde{\mathcal{O}} \rightarrow \tilde{\mathcal{O}} / \sum_{i=1}^k f_i \tilde{\mathcal{O}} \rightarrow 0$$

defined by the multiplication μ by f_k gives the long exact sequence of cohomology modules

$$\cdots \rightarrow H^0(\Omega, \tilde{\mathcal{O}} / \sum_{i=1}^{k-1} f_i \tilde{\mathcal{O}}) \rightarrow H^0(\Omega, \tilde{\mathcal{O}} / \sum_{i=1}^k f_i \tilde{\mathcal{O}}) \rightarrow H^1(\Omega, \tilde{\mathcal{O}} / \sum_{i=1}^{k-1} f_i \tilde{\mathcal{O}}) \rightarrow \cdots.$$

Since $H^1(\Omega, \tilde{\mathcal{O}} / \sum_{i=1}^{k-1} f_i \tilde{\mathcal{O}}) = 0$, the \mathcal{C} -linear mapping $H^0(\Omega, \tilde{\mathcal{O}} / \sum_{i=1}^{k-1} f_i \tilde{\mathcal{O}}) \rightarrow H^0(\Omega, \tilde{\mathcal{O}} / \sum_{i=1}^k f_i \tilde{\mathcal{O}})$ is surjective where $\tilde{\mathcal{O}} / \sum_{i=1}^0 f_i \tilde{\mathcal{O}} = \tilde{\mathcal{O}}$.

Hence the composite mapping $H^0(\Omega, \tilde{\mathcal{O}}) \rightarrow H^0(\Omega, \tilde{\mathcal{O}} / \sum_{i=1}^m f_i \tilde{\mathcal{O}})$ is surjective.

THEOREM 2. *If S is an n -dimensional Stein space and if the dimension of $H^p(D, \text{Ker } T)$ is finite or countably infinite for any $p \in \{1, 2, \dots, n\}$, then \tilde{D} is holomorphically separable and (\tilde{D}, ϕ) is a domain of holomorphy over S .*

PROOF. By Theorem 4 of Y. T. Siu [21], we have $\dim H^p(D, \text{Ker } T) < +\infty$ for any $p \geq 1$. By Lemma 4 \tilde{D} is a T_2 -space and equipped with complex structure such that ϕ is a locally biholomorphic mapping of the complex space \tilde{D} into the Stein manifold S . Let x' and x'' be two distinct points of D with $\phi(x') = \phi(x'') = s_0$. Let $\text{id}(x'')$ be the sheaf of ideals over \tilde{D} of germs of holomorphic functions vanishing at x'' . Since S is an n -dimensional Stein space, there exist $n-1$ holomorphic functions f_1, f_2, \dots, f_{n-1} with $s_0 \in \text{loc}(f_1, f_2, \dots, f_{n-1})$ satisfying the condition of Lemma 6. By R. Narasimhan [15], the one dimensional complex space $\phi^{-1}(\text{loc}(f_1, f_2, \dots, f_{n-1}))$ with reduced structure and without compact irreducible analytic component is a Stein space. Since the sheaf $\text{id}(x'') / \sum_{i=1}^{n-1} (f_i \circ \phi) \text{id}(x'')$ is an analytic coherent sheaf, the sections of the sheaf $\text{id}(x'') / \sum_{i=1}^{n-1} (f_i \circ \phi) \text{id}(x'')$ over the Stein space $\phi^{-1}(\text{loc}(f_1, f_2, \dots, f_{n-1}))$ generates its stalk at x' by Theorem A of Cartan-Serre. There exists a $g \in H^0(\tilde{D}, \text{id}(x'') / \sum_{i=1}^{n-1} (f_i \circ \phi) \text{id}(x''))$ such that $g(x') \neq 0$ and $g(x'') = 0$. The function g on $\phi^{-1}(\text{loc}(f_1, f_2, \dots, f_{n-1}))$ is regarded as an element of $H^0(\tilde{D}, \tilde{\mathcal{O}} / \sum_{i=1}^{n-1} (f_i \circ \phi) \tilde{\mathcal{O}})$. By Lemmata 6 and 10, g can be extended to a holomorphic function on \tilde{D} which separates the points x' and x'' . Thus we have proved that \tilde{D} is holomorphically separable.

Let x_0 be a boundary point of the domain (\tilde{D}, ϕ) over the Stein space S . There exist $n-1$ functions f_1, f_2, \dots, f_{n-1} such that x_0 is a boundary point of

the one dimensional domain $(\phi^{-1}(\text{loc}(f_1, f_2, \dots, f_{n-1})), \phi)$ over the one dimensional Stein space $\phi^{-1}(\text{loc}(f_1, f_2, \dots, f_{n-1}))$. Then $\phi^{-1}(\text{loc}(f_1, f_2, \dots, f_{n-1}))$ is a Stein space with reduced structure. We put $\mathcal{F} = \widetilde{\mathcal{O}} / \sum_{i=1}^{n-1} (f_i \circ \phi) \widetilde{\mathcal{O}}$. Let $(x_\nu)_{\nu=1}^\infty$ be a sequence of points in $\phi^{-1}(\text{loc}(f_1, f_2, \dots, f_{n-1}))$ which tends to the ideal boundary x_0 . Let \mathcal{S} be the subsheaf of \mathcal{F} whose stalks are given by

$$\mathcal{S}_x = \begin{cases} \text{id}(x_\nu) & x = x_\nu (\nu \geq 1) \\ \mathcal{F}_x & x \in \{x_1, x_2, \dots, x_\nu, \dots\}. \end{cases}$$

Then the sheaf $\mathcal{F} / \mathcal{S}$ is an analytic coherent sheaf over the one dimensional Stein space $\phi^{-1}(\text{loc}(f_1, f_2, \dots, f_{n-1}))$. The short exact sequence of sheaves over $\phi^{-1}(\text{loc}(f_1, f_2, \dots, f_{n-1}))$

$$0 \rightarrow \mathcal{S} \rightarrow \mathcal{F} \rightarrow \mathcal{F} / \mathcal{S} \rightarrow 0$$

gives the long exact sequence of cohomology modules of

$$\begin{aligned} \dots \rightarrow H^0(\phi^{-1}(\text{loc}(f_1, f_2, \dots, f_{n-1})), \mathcal{F}) \rightarrow H^0(\phi^{-1}(\text{loc}(f_1, f_2, \dots, f_{n-1})), \\ \mathcal{F} / \mathcal{S}) \rightarrow H^1(\phi^{-1}(\text{loc}(f_1, f_2, \dots, f_{n-1})), \mathcal{S}) \rightarrow \dots \end{aligned}$$

Since we have $H^1(\phi^{-1}(\text{loc}(f_1, f_2, \dots, f_{n-1})), \mathcal{S}) = 0$ by Theorem B of Cartan-Serre, by Lemma 10 and the above exact sequence, the canonical mapping $H^0(D, \mathcal{O}) \rightarrow H^0(\phi^{-1}(\text{loc}(f_1, f_2, \dots, f_{n-1})), \mathcal{F} / \mathcal{S}) = H^0(\{x_\nu\}, \mathcal{C})$ is surjective. Hence there exists a holomorphic function h on D with $h(x_\nu) = \nu (\nu \geq 1)$. Thus we have proved that the domain (\tilde{D}, ϕ) over the Stein space S is a domain of holomorphy.

Moreover, if D is a Stein domain, we have the isomorphism

$$H^0(D, \text{Ker } T) \cong \{g \in H^0(D, \mathcal{O}_{\tilde{z}}^p, \ast); d_z^p g = 0\} / d_z^{p-1} H^0(D, \mathcal{O}_{\tilde{z}}^{p-1}).$$

Hence we have the following corollary:

COROLLARY. *Let D be a Stein domain in the product space $Z \times S$ of an m -dimensional complex manifold Z and an n -dimensional Stein space S . If, for any $p \in \{1, 2, \dots, n\}$ and for any $g \in H^0(D, \mathcal{O}_{\tilde{z}}^{p-1})$ with $d_z^p g = 0$, there exists a $f \in H^0(D, \mathcal{O}_{\tilde{z}}^{p-1})$ with $d_z^{p-1} f = g$, then \tilde{D} is holomorphically separable and (\tilde{D}, ϕ) is a domain of holomorphy over S .*

In case that $m=0$ of Lemma 6, we have the following Lemma:

LEMMA 11. *Let ϕ be a local homeomorphism of a T_1 -space Ω into an n -dimensional Stein space S . If the dimension of $H^p(\Omega, \tilde{\mathcal{O}})$ is finite or countably infinite for the sheaf $\tilde{\mathcal{O}}$ over Ω of germs of holomorphic functions and for any $p \in \{1, 2, \dots, n\}$, then Ω is holomorphically separable and (Ω, ϕ) is a domain of holomorphy over S .*

THEOREM 3. *If S is an n -dimensional Stein manifold and if the dimension of $H^p(D, \text{Ker } T)$ is finite or countably infinite for any $p \in \{1, 2, \dots, n\}$, then \tilde{D} is a Stein manifold.*

PROOF. Let U be a Stein subdomain of S which is mapped biholomorphically onto a domain of holomorphy V of \mathbb{C}^n . Since (\tilde{D}, ϕ) is a domain of holomorphy over S , $(\phi^{-1}(U), \phi)$ is a domain of holomorphy over $U \cong V \subset \mathbb{C}^n$. By K. Oka [17], $\phi^{-1}(U)$ is a Stein manifold. The pseudoconvex domain (\tilde{D}, ϕ) over the Stein manifold S is a Stein manifold by Docquier-Grauert [3].

COROLLARY *Let D be a Stein domain in the product space $Z \times S$ of an m -dimensional complex manifold Z and an n -dimensional Stein manifold S . If, for any $p \in \{1, 2, \dots, \min(m, n)\}$ and for any $g \in H^0(D, \mathcal{O}_{z, s}^p)$ with $d_z^* g = 0$, there exists a $f \in H^0(D, \mathcal{O}_{z, s}^{p-1})$ with $d_z^{p-1} f = g$, then \tilde{D} is a Stein manifold.*

In case that $m=0$ in Theorem 3, we have the following lemma which is exactly the, so called, "if" part of Theorem B of Y. T. Siu [21] in case that the T_2 -ness of Ω is assured, to whom we owe Theorem 2.

LEMMA 12. *Let ϕ be a local homeomorphism of a T_1 -space Ω into an n -dimensional Stein manifold S . If the dimension of $H^p(\Omega, \tilde{\mathcal{O}})$ is finite or countably infinite for the sheaf $\tilde{\mathcal{O}}$ over Ω of germs of holomorphic functions and for any $p \in \{1, 2, \dots, n\}$, then Ω is a Stein manifold.*

In Theorem 2 and 3, or Lemmata 10 and 11, it suffices to assume the countability at most of $H^p(D, \text{Ker } T)$ or $H^p(\Omega, \tilde{\mathcal{O}})$ for p belonging to $\{1, 2, \dots, n-1\}$ in case that the T_2 -ness of \tilde{D} or Ω is assured. In order to obtain the isomorphism

$$H^p(D, \text{Ker } T) \cong H^p(\tilde{D}, \tilde{\mathcal{O}})$$

and to establish the converse of Theorem 3, we defined in the previous paper

[10] the notion of a Stein domain with regular boundary and finitely generated C -cohomology:

LEMMA 13. *Let p be an integer with $1 \leq p \leq m$ and D be a Stein domain with regular boundary and finitely generated C -cohomology such that $H^q(D(z, s), C) = 0$ for any q with $1 \leq q \leq p$ and for any $(z, s) \in D$. Then we have*

$$H^p(D, \text{Ker } T) \cong H^p(\tilde{D}, \tilde{\mathcal{O}}).$$

THEOREM 4. *Let D be a Stein domain with regular boundary and finitely generated C -cohomology in the product space of $Z \times S$ of an m -dimensional complex manifold Z and an n -dimensional Stein manifold S . Then the following properties (1), (2) and (3) are equivalent:*

(1) *For any $p=1, 2, \dots, m$ and for any $g \in H^0(D, \mathcal{O}_{z, s}^p)$ with $d_z^p g = 0$, there exists a $f \in H^0(D, \mathcal{O}_{z, s}^{p-1})$ with $d_z^{p-1} f = g$.*

(2) *\tilde{D} is a Stein manifold with $H^p(D(z, s), C) = 0$ for any $p=1, 2, \dots, m$.*

(3) *For any $p=1, 2, \dots, m$, we have $H^p(D, \text{Ker } T) = 0$.*

PROOF. Since D is a Stein domain, we have

$$H^p(D, \text{Ker } T) \cong \{g \in H^0(D, \mathcal{O}_{z, s}^p); d_z^p g = 0\} / d_z^{p-1} H^0(D, \mathcal{O}_{z, s}^{p-1})$$

for any $p \geq 1$. Hence (1) and (3) are equivalent.

(2) \rightarrow (3). By Lemma 13, we have

$$H^p(D, \text{Ker } T) \cong H^p(\tilde{D}, \tilde{\mathcal{O}})$$

for any $p \geq 1$. Since \tilde{D} is a Stein manifold, we have $H^p(\tilde{D}, \tilde{\mathcal{O}}) = 0$ for any $p \geq 1$ by the theorem of Cartan-Oka-Serre. Then we have (3) by the above isomorphism.

(3) \rightarrow (2). By the above isomorphism, we have $H^p(D, \text{Ker } T) = 0$ for any $p \geq 1$. By Lemma 2, we have $H^p(D(z, s), \text{Ker } T) = 0$ for any $p \geq 1$ and for any $(z, s) \in D$. By Theorem 3, \tilde{D} is a Stein manifold.

COROLLARY. *If, for any point s of S , there exists a Stein neighborhood $U(s)$ of s in S such that, for any $p=1, 2, \dots, m$ and for any $g \in H^0(\phi^{-1}(U(s)), \mathcal{O}_{z, s}^p)$ with $d_z^p g = 0$, there exists $f \in H^0(\phi^{-1}(U(s)), \mathcal{O}_{z, s}^{p-1})$ with $d_z^{p-1} f = g$, then (1) holds globally in D .*

PROOF. Since $\phi^{-1}(U(s))$ is a Stein manifold by the theorem for any

$s \in S$, (\tilde{D}, ϕ) is an unramified pseudoconvex domain over the Stein manifold S . By Docquier–Grauert [3], \tilde{D} is a Stein manifold. Since we have $H^p(U(s), \text{Ker } T) = 0$ for any $p \geq 1$ and for any $s \in S$, we have (2). By the theorem, we have (1).

The above corollary is a local characterization of the existence of global solutions f of the overdetermined system $d_z^{p-1} f = g$ satisfying the integrability condition. Although, the set $\{1, 2, \dots, m\}$ of degrees p in the theorem does not depend on the dimension n of the dimension of the manifold S but m , that of the dimension of Z , we have the Steinness of the T_1 -space \tilde{D} over S because of the above Serre's isomorphism with parameter $s \in S$.

5. Domains over Stein spaces

LEMMA 14. *Let Ω_1 and Ω_2 be T_1 -spaces such that Ω_1 is a relatively compact open subset of Ω_2 , and ϕ be a local homeomorphism of Ω_2 into an n -dimensional Stein space S . If the dimensions of $H^p(\Omega_1, \tilde{\mathcal{O}})$ and $H^p(\Omega_2, \tilde{\mathcal{O}})$ are finite or countably infinite for $p=1, 2, \dots, n$ and for the sheaf $\tilde{\mathcal{O}}$ over Ω_2 of germs of holomorphic functions. Then Ω_2 is holomorphically separable and Ω_1 is a Stein space.*

PROOF. By Theorem 4 of Y. T. Siu [21], we have $\dim H^p(\Omega_1, \tilde{\mathcal{O}}) < +\infty$ and $H^p(\Omega_2, \tilde{\mathcal{O}}) < +\infty$ for $p=1, 2, \dots, n$. By Lemma 11, Ω_1 and Ω_2 are holomorphically separable. Applying Lemma 10 to the case $k=0$. We have $H^p(\Omega_1, \tilde{\mathcal{O}}) = 0$ for $p=1, 2, \dots, n$. Since Ω_1 is a relatively compact open subset of the holomorphically separable space Ω_2 satisfying $H^p(\Omega_1, \tilde{\mathcal{O}}) = 0$ for $p=1, 2, \dots, n-1$, Ω_1 is a Stein space by Theorem 1.3 of S. Coen [2].

LEMMA 15. *Let $\{\Omega_\nu; \nu=0, 1, \dots\}$ be a sequence of T_1 -spaces such that, for any $\nu=1, 2, \dots$, Ω_ν is a relatively compact open subset of $\Omega_{\nu+1}$ and that $\Omega_0 = \bigcup_{\nu=1}^{\infty} \Omega_\nu$, and ϕ be a local homeomorphism of Ω_0 into an n -dimensional Stein space S . If the dimensions of $H^p(\Omega_\nu, \tilde{\mathcal{O}})$ are finite or countably infinite for $p=1, 2, \dots, n, \nu=0, 1, \dots$ and for the sheaf $\tilde{\mathcal{O}}$ over Ω_0 of germs of holomorphic functions, then Ω_ν is a Stein space for $\nu=0, 1, \dots$.*

PROOF. By Lemma 5, Ω_0 is a Hausdorff space. By Theorem 4 of Y. T. Siu [21], we have $\dim H^p(\Omega_\nu, \tilde{\mathcal{O}}) < +\infty$ for $p=1, 2, \dots, n$ and $\nu=0, 1,$

... By Lemma 11 Ω_0 is holomorphically separable and Ω_ν is a Stein space for $\nu=1, 2, \dots$. Applying Lemma 10 to the case that $k=0$, we can prove that $H^1(\Omega_0, \tilde{\mathcal{O}}) = 0$. Since Ω_0 is a complex space with $H^1(\Omega_0, \tilde{\mathcal{O}}) = 0$ which is the limit of the increasing sequence of Stein spaces Ω_ν , Ω_0 is a Stein space by A. Markoe [14].

THEOREM 5. *Let D_1 and D_2 be domains of the product space $Z \times S$ of an m -dimensional complex manifold Z and a two dimensional Stein space S such that D_1 is a relatively compact subdomain of D_2 . If the dimensions of $H^p(D_1, \text{Ker } T)$ and $H^p(D_2, \text{Ker } T)$ are finite or countably infinite for $p=1, 2$, then \tilde{D}_2 is holomorphically separable and \tilde{D}_1 is a Stein space.*

PROOF. By Theorem 4 of Y. T. Siu [21], we have $\dim H^p(D_\nu, \text{Ker } T) < +\infty$ for $p, \nu \in \{1, 2\}$. By Lemma 4 D_ν 's are T_2 -spaces. By Lemma 1, we have $\dim H^1(\tilde{D}_1, \tilde{\mathcal{O}}) < +\infty$ and $\dim H^1(\tilde{D}_2, \tilde{\mathcal{O}}) < +\infty$. By Lemma 11, \tilde{D}_2 is holomorphically separable and \tilde{D}_1 is a Stein space.

COROLLARY. *Let D_1 and D_2 be Stein domains of the product space $Z \times S$ of an m -dimensional complex manifold Z and a two dimensional Stein space S such that D_1 is relatively compact subdomain of D_2 . If, for any $p, \nu \in \{1, 2\}$ and for any $g \in H^0(D, \mathcal{O}_{\nu, \cdot}^p)$ with $d_\nu^p g = 0$, there exists $f \in H^0(D, \mathcal{O}_{\nu, \cdot}^{p-1})$ with $d_\nu^{p-1} f = g$, then \tilde{D}_2 is holomorphically separable and D_1 is a Stein space.*

PROOF. Since D_1 and D_2 are Stein, we have

$$H^p(D_\nu, \text{Ker } T) \cong \{g \in H^0(D_\nu, \mathcal{O}_{\nu, \cdot}^p)\} / d_\nu^{p-1} H^0(D_\nu, \mathcal{O}_{\nu, \cdot}^{p-1})$$

for $p, \nu \in \{1, 2\}$. Hence we have the corollary by the theorem.

THEOREM 6. *Let $\{D_\nu; \nu=0, 1, \dots\}$ be a sequence of domains of the product space $Z \times S$ of an m -dimensional complex manifold and a two dimensional Stein space S such that, for any $\nu=1, 2, \dots$, D_ν is a relatively compact subdomain of $D_{\nu+1}$ and that $D_0 = \bigcup_{\nu=1}^{\infty} D_\nu$. If, for $p=1, 2$ and for any $\nu=0, 1, 2, \dots$, the dimensions of $H^p(D_\nu, \text{Ker } T)$ are finite or countably infinite, \tilde{D}_ν is a Stein space for $\nu=0, 1, 2, \dots$.*

PROOF. By Theorem 4 of Y. T. Siu [21], we have $\dim H^p(D_\nu, \text{Ker } T) < +\infty$ for $p=1, 2$ and for $\nu=0, 1, \dots$. By Lemma 4, \tilde{D}_ν 's are T_2 -spaces.

By Lemma 1, we have $H^1(\tilde{D}_\nu, \tilde{\mathcal{O}}) = 0$ for $\nu = 0, 1, \dots$. By the proof of Lemma 15, \tilde{D}_ν 's are Stein spaces.

COROLLARY. *Let $\{D_\nu; \nu = 0, 1, \dots\}$ be a sequence of Stein domains of the product space $Z \times S$ of an m -dimensional complex manifold Z and a two dimensional Stein space S such that, for any $\nu = 1, 2, \dots$, D_ν is a relatively compact subdomain of $D_{\nu+1}$ and that $D_0 = \bigcup_{\nu=1}^{\infty} D_\nu$. If, for $p = 1, 2$ and for $\nu = 0, 1, \dots$ and for any $g \in H^0(D, \mathcal{O}_{\tilde{D}_\nu}^p)$ with $d_{\tilde{D}_\nu}^p g = 0$, there exists $f \in H^0(D, \mathcal{O}_{\tilde{D}_\nu}^{p-1})$ with $d_{\tilde{D}_\nu}^{p-1} f = g$, then D_ν is a Stein space for $\nu = 0, 1, \dots$.*

6. Ramified domains over C^n .

THEOREM 7. *Let (S, σ) be a ramified domain over C^n . If the dimension of $H^p(D, \text{Ker } T)$ is finite or countably infinite for $p = 1, 2, \dots, n$, then \tilde{D} is holomorphically separable and the ramified domain $(\tilde{D}, \sigma\phi)$ is a domain of holomorphy over C^n .*

PROOF. *Let x' and x'' be two distinct point in \tilde{D} with $\sigma\phi(x') = \sigma\phi(x'') = s^0 = (s_1^0, s_2^0, \dots, s_n^0)$ in C^n . By the proof of Lemmata 6 and 10, the canonical mapping $H^0(\tilde{D}, \tilde{\mathcal{O}}) \rightarrow H^0(\tilde{D}, \tilde{\mathcal{O}} / \sum_{i=2}^n (s_i - s_i^0) \sigma\phi \tilde{\mathcal{O}})$ is surjective. Since the complex space $(\sigma\phi)^{-1}(\text{loc}(s_2 - s_2^0, s_3 - s_3^0, \dots, s_n - s_n^0))$ is a Stein space, there exists a $g \in H^0(\tilde{D}, \tilde{\mathcal{O}} / \sum_{i=2}^n (s_i - s_i^0) \sigma\phi \tilde{\mathcal{O}})$ satisfying $g(x') \neq 0$ and $g(x'') = 0$, as we stated in the proof of Theorem 2. Then g can be extended to a holomorphic function $h \in H^0(\tilde{D}, \tilde{\mathcal{O}})$ satisfying $h(x') \neq 0$ and $h(x'') = 0$.*

Hence \tilde{D} is holomorphically separable.

Let x^0 be a boundary point of the ramified domain $(\tilde{D}, \sigma\phi)$ over C^n . We may assume that x^0 is a boundary point of the ramified domain $(\tilde{D}, \sigma\phi)$ over C^n . We may assume that x^0 is a boundary point of the ramified domain $((\sigma\phi)^{-1}(\text{loc}(s_2, s_3, \dots, s_n)), \sigma\phi)$ which is a domain of holomorphy over the complex plane and on which there exists a holomorphic function unbounded at x^0 . As stated above, it can be extended to a holomorphic function on \tilde{D} unbounded at x^0 , that is, unbounded in any open subset of \tilde{D} belonging to the filter x^0 . Thus we have proved that $(\tilde{D}, \sigma\phi)$ is a ramified domain of holomorphy.

Making use of Theorem 7, we can develop similar discussion to the preceding paragraph.

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