Existence, nonexistence and some asymptotic behaviour of global solutions of a nonlinear degenerate parabolic equation

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Existence, nonexistence and some asymptotic behaviour of global solutions of a nonlinear degenerate parabolic equation

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§ 0. Introduction

In this paper we are concerned with the existence, nonexistence and some asymptotic behaviour of global solutions to the initial-boundary value problem for the rather typical nonlinear degenerate parabolic equation:

(0.1) $$\frac{\partial}{\partial t}u-\Delta u^{p+1}-u^{\alpha+1}=0 \quad \text{on } \Omega\times[0,T],$$

$$u(x,0)=u_0,\ u(x,t)|_{\partial\Omega}=0 \quad \text{for } t\in[0,T] \text{ and } u\geqq 0,$$

where Ω is a bounded domain R^n with smooth boundary $\partial\Omega$, Δ is the Laplacian, α and p are nonnegative number and T is an arbitrarily fixed positive time.

When $p=0$ and $\alpha>0$, as is well known, H. Fujita [10, 11] gave criteria on global existence and nonexistence (blowing up) of solutions to the problem (0.1). The result of Fujita was extended by M. Tsutsumi [28] to the following typical equation:

(0.2) $$\frac{\partial}{\partial t}u-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}(|\frac{\partial}{\partial x_i}u|^p\frac{\partial}{\partial x_i}u)-u^{\alpha+1}=0 \quad \text{on } \Omega\times[0,T]$$

$$u(x,0)=u_0,\ u(x,t)|_{\partial\Omega}=0 \quad \text{for } t\in[0,T] \text{ and } u\geqq 0,$$

where the setting is the same as in (0.1).

Roughly speaking Tsutsumi's result is stated as follows: If $p>\alpha$ the problem (0.2) has a global solution for each $u_0\in W_0^{1,p}(\Omega)$, $u_0\geqq 0$, and if $p<\alpha$ with $1/(\alpha+2)\geqq 1/(p+2)-1/n$ the global existence or nonexistence depends on the initial data, that is, if $u_0\in W$, the potential well associated with (0.2),

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(0.2) admits a global solution and if $u_0 \in W$ a solution blows up in a finite time under some extra condition. This result was generalized by H. Ishii [13]. For related topics see N. Alikakos [1], J. Ball [6], S. Kaplan [14], H. Levine [16], M. Nakao & T. Narazaki [21], M. Ôtani [22] and the references cited there.

The first objective of this paper is to derive a similar result as Tsutsumi's for the problem (0.1), and the second one is to show the decay property of solutions when $p<\alpha$ and the global existence is assured. The framework of our argument on existence and nonexistence is the same as in Tsutsumi [28], though our treatment of the nonlinearity Δu^{1+p} is more delicate. The method for deriving the decay estimate is similar to the author's earlier paper [20] where the decay estimate of $\|\nabla u^{p+1}(t)\|_{L^2}$ for the solutions of the problem (0.1) without the blowing up term $-u^{\alpha+1}$ is included.

The equation (0.1) without the blowing-up term is well known as porous medium equation which appears in various physical, chemical and biological situations, and has been studied by many people in various views of point (N. Alikakos [1], N, Alikakos & R. Rostamian [2, 3], D. Aronson [4], D. Aronson & L. Peletier [5], H. Brezis [7], M. Crandall [8], Y. Ebihara & T. Nanbu [9], B. Gilding & L. Peletier [12], Y. Konishi [15], J.L. Lions [17], O.A. Oleinik, A.S. Kalashnikov & Chzhou Yui-Lin [23] and a number of other interesting papers). In particular, concerning the decay of solutions Alikakos [1] and Aronson & Peletier [5] have proved the detailed estimates in L^∞-norm. It seems to be interesting to derive such estimates for the solutions to the problem (0.1) when their global existence is guaranteed. The result and the outline of proofs of this paper were announud in [32].

§1. Global existence for the case $p>\alpha$.

In this section we shall prove the existence of a global solution to the probelm (0.1) under the assumption $p>\alpha$. The function spaces we use are almost familiar and we omit the definitions except for Lip(θ, q), the Lipshitz space in $L^q(\Omega)$ with exponent θ, $0 \leqq \theta \leqq 1$.

Our result here reads as follows:

THEOREM 1.1. ($p>\alpha$) *Let* $T>0$ *be arbitrarily fixed. Suppose that* $|u_0|^p u_0 \in \mathring{H}_1(\Omega)$ *and* $u_0(x) \geqq 0$. *Then the problem* (0.1) *admits a* (*weak*) *solution* $u(x,t)$ *such that*

$$\frac{\partial}{\partial t}u^{p/2+1}(t)\in L^2([0,T];L^2(\Omega)),\ u^{p+1}(t)\in L^\infty([0,T];\mathring{H}_1(\Omega))$$

and the equation is satisfied in the sense that

$$(1.1)\quad \begin{aligned}&\int_0^T\int_\Omega\{-u(x,t)\phi_t(x,t)+\nabla(u^{p+1})\nabla\phi(x,t)-u^{\alpha+1}\phi(x,t)\}\,dxdt\\&-\int_\Omega u_0(x)\phi(x,0)\,dx=0\end{aligned}$$

for $\forall\phi\in C^1([0,T];\mathring{H}_1)$ *with* $\phi(T)=0$.

Moreover the following estimates hold:

$$\int_t^{t+1}\left\|\frac{\partial}{\partial t}(u^{p/2}u)(s)\right\|_{L^2}^2ds+\|u^{p+1}(t)\|_{\mathring{H}_1}^2\leqq C(\|u_0^{p+1}\|_{\mathring{H}_1})$$

for $0\leqq t\leqq T$, *where the constant* $C(\|u_0^{p+1}\|_{\mathring{H}_1})$ *is independent of* T.

REMARK 1.1. By a standard argument (see Lions [17]) we see (1.1) is equivalent to

$$(1.1)'\quad \begin{aligned}&\frac{d}{dt}u(t)+Au(t)-Bu(t)=0 \text{ in } H^{-1} \text{ a.e. } t\in[0,T]\\&u(0)=u_0\end{aligned}$$

where A, B are defined as follows

$$\begin{aligned}<Au,\ v>_{H^{-1}\times\mathring{H}_1}&\equiv\int_\Omega\Delta(u^{p+1})\nabla v\,dx \text{ for } v\in\mathring{H}_1(\Omega)\\&\equiv(\nabla(u^{p+1}),\nabla v)\end{aligned}$$

and

$$<Bu,\ v>_{H^{-1}\times\mathring{H}_1}\equiv\int_\Omega u^{+1}v\,dx \text{ for } v\in\mathring{H}_1(\Omega).$$

(For a weaker definition of A see Brezis [71]).

As in other works it is convenient to rewrite the equation by setting $u^{p+1}=U$. Then we have

$$(1.2)\quad \begin{aligned}&\frac{\partial}{\partial t}(|U|^{-p/(1+p)}U)-\Delta U-\Psi(U)-0\\&U(x,0)=U_0\equiv u_0^{1+p},\ U(x,t)\geqq0 \text{ and } U|_{\partial\Omega}=0,\end{aligned}$$

where we set

$$\Psi(U)=\begin{cases}0 & \text{if } U\leqq0\\ U^{(\alpha+1)/(p+1)} & \text{if } U\geqq0\end{cases}$$

For the proofs of Theorem 1.1 and other Theorems below we employ the Galerkin method. For this we must modify the equation (1.2) because the leading term $\frac{\partial}{\partial t}(|U|^{-p/(p+1)}U)=\frac{1}{p+1}|U|^{-p/(p+1)}\frac{\partial}{\partial t}U$ has singularities at $U=0$ and $U=\infty$, which is in fact only one difficulty to solve the problem when $p>\alpha$. Let us introduce the function

$$\beta'_\varepsilon(U)=(p+1)^{-1}(|U|+\varepsilon)^{-p/(p+1)},\ \varepsilon>0,$$

and set

$$\beta_\varepsilon(U)=\int_0^U \beta'_\varepsilon(\eta)d\eta.$$

We first consider the modified problem

$$\text{(1.3)}\quad \begin{aligned} &\frac{\partial}{\partial t}(\beta_\varepsilon(U)+\varepsilon U)-\Delta U-\Psi(U)=0,\quad \varepsilon>0\\ &U(x,0)=U_0\in \mathring{H}_1,\ \text{and}\ U|_{\partial\Omega}=0. \end{aligned}$$

Now let us begin to prove Theorem 1.1. We take $\{w_j\}_{j=1}^\infty$, a basis of $\mathring{H}_1(\Omega)$ (we may assume w_j, $j=1,2,\ldots$ are as smooth as wanted because of the smoothness assumption on $\partial\Omega$), and consider the approximate solutions $U_{m,\varepsilon}$, $m=1,2,3,\ldots$, constructed through the system of ordinary differential equations:

$$\text{(1.4)}\quad \begin{aligned} &((\beta'_\varepsilon(U_{m,\varepsilon}(t))+\varepsilon\frac{\partial}{\partial t}U_{m,\varepsilon}(t),w_j)+(\nabla U_{m,\varepsilon}(t),\nabla w_j)\\ &\qquad\qquad -(\Psi(U_{m,\varepsilon}(t)),w_j)=0\\ &\qquad\qquad (j=1,2,..,m)\\ &U_{m,\varepsilon}(0)=U_{0,m}\in[w_1,\ldots,w_m] \end{aligned}$$

where $U_{m,\varepsilon}(t)=\sum_{j=1}^m \lambda_{j,m}(t)w_j$ and the initial data $U_{0,m}$ should be chosen in such a way that

$$\text{(1.5)}\quad U_{0,m}\to U_0 \text{ in } \mathring{H}_1(\Omega) \quad \text{as } m\to\infty.$$

Of course (,) denotes L^2-inner product. The above system of equations with respect to $\lambda_{j,m}(t)$ has a solution on some interval, say, $[0,T_{m,\varepsilon}]$, because no singularity appears in (1.4), and $U_{m,\varepsilon}(t)$ is defined for each m and $\varepsilon(>0)$ on this interval. We shall derive estimates for $U_{m,\varepsilon}(t)$.

Multiplying (1.4) by $\lambda_{j,m}(t)$, summing up over j and integrating (we often say this procedure simply 'multiplying $U_{m,\varepsilon}(t)$ and integrating' etc.) we have

$$(1.6)\qquad \int_0^t\int_\Omega (\beta'_\varepsilon(U_{m,\varepsilon})+\varepsilon)\left|\frac{\partial}{\partial t}U_{m,\varepsilon}\right|^2 dxds+J(U_{m,\varepsilon}(0))=<c<\infty$$

for $t\in[0, T_{m,\varepsilon}]$, where we set

$$(1.7)\qquad J(U)=\frac{1}{2}\|\nabla U\|^2_{L^2}-\frac{p+1}{p+\alpha+2}\int_\Omega \Psi(U)Udx \qquad \text{for } U\in\mathring{H}_1.$$

Since $p>\alpha$ $J(U)$ is well defined and

$$\left|\int_\Omega \Psi(U)Udx\right|\leqq\int_\Omega |U|^{\frac{p+2+\alpha}{p+1}}dx\leqq c\|U\|_{\mathring{H}_1}^{(p+\alpha+2)/(p+1)}$$

$$\leqq c+\frac{1}{4}\|\nabla U\|^2_{L^2} \qquad \left(\text{note that } \frac{p+2+\alpha}{p+1}<2\right)$$

and hence we have

$$(1.8)\qquad J(U)\geqq\frac{1}{4}\|\nabla U\|^2_{L^2}-c.$$

Hereafter c denotes generic positive constants independent of m and ε. Thus we obtain from (1.6) and (1.8)

$$\int_0^t\int_\Omega (\beta'_\varepsilon(U_{m,\varepsilon})+\varepsilon)\left|\frac{\partial}{\partial t}U_{m,\varepsilon}\right|^2 dxds+\frac{1}{4}\|\nabla U_{m,\varepsilon}(t)\|^2_{L^2}\leqq c<\infty$$

for $t\in[0, T_{m,\varepsilon}]$, which implies that we can take $T_{m,\varepsilon}=T$ and

$$(1.9)\qquad \int_0^T\left\|\frac{\partial}{\partial t}\int_0^{U_{m,\varepsilon}(t)}\sqrt{\beta'_\varepsilon(\eta)}\,d\eta\right\|^2_{L^2}dt\leqq c,$$

$$(1.10)\qquad \varepsilon\int_0^T\left\|\frac{\partial}{\partial t}U_{m,\varepsilon}(t)\right\|^2_{L^2}dt\leqq c$$

and

$$(1.11)\qquad \|U_{m,\varepsilon}(t)\|_{\mathring{H}}\leqq c \qquad \text{for} \qquad t\in[0, T].$$

Setting $V_{m,\varepsilon}=\int_0^{U_{m,\varepsilon}}\sqrt{\beta'_\varepsilon(\eta)}\,d\eta$ we see easily $V_{m,\varepsilon}\in\mathring{H}_1$ for each t and

$$(1.12)\qquad \begin{aligned}\|V_{m,\varepsilon}(t)\|^2_{\mathring{H}_1}&\equiv\sum_{i=1}^n\left\|\sqrt{\beta'_\varepsilon(U_{m,\varepsilon}(t))}\frac{\partial}{\partial x_i}U_{m,\varepsilon}(t)\right\|^2_{L^2}\\&\leqq c\varepsilon^{-p/(p+1)}\|\nabla U_{m,\varepsilon}(t)\|^2_{L^2}\leqq c(\varepsilon)<\infty.\end{aligned}$$

From (1.9)−(1.12) we can conclude by a standard compactness argument that as $m\to\infty$ along a subsequence

(1.13) $V_{m,\varepsilon}(x,t)\to V_\varepsilon(x,t)$ strongly in $L^2([0,T];L^2(\Omega))$ and $a.e.$ on $\Omega\times[0,T]$

and

(1.14) $U_{m,\varepsilon}(x,t)\to U_\varepsilon(x,t)$ strongly in $L^q([0,T];\ L^r(\Omega))$ (for $q\geqq 1$ and $r\geqq 1$ with $\frac{1}{r}\gneqq\frac{1}{2}-\frac{1}{n}$) and $a.e.$ on $\Omega\times[0,T]$

where V_ε and U_ε are connected by

$$V_\varepsilon(x,t)=\int_0^{U_\varepsilon(x,t)}\sqrt{\beta'_\varepsilon(\eta)}\,d\eta.$$

In particular, from (1.14), we see easily

(1.15) $\beta_\varepsilon(U_{m,\varepsilon}(x,t))\to\beta_\varepsilon(U_\varepsilon(x,t))$ strongly in $L^1([0,T];L^1(\Omega))$.

Moreover we may assume

$$\frac{\partial}{\partial t}V_{m,\varepsilon}\to\frac{\partial}{\partial t}V_\varepsilon \text{ weakly in } L^2([0,T];\ L^2(\Omega)).$$

$$\frac{\partial}{\partial t}U_{m,\varepsilon}\to\frac{\partial}{\partial t}U_\varepsilon \text{ weakly in } L^2([0,T];\ L^2(\Omega)),$$

$$U_{m,\varepsilon}\to U_\varepsilon \quad \text{weakly* in } L^\infty([0,T];\ \mathring{H}_1(\Omega))$$

and the following estimates hold

(1.16)
$$\int_0^T\left\|\frac{\partial}{\partial t}V_\varepsilon(t)\right\|_{L^2}^2dt\leqq c,$$

$$\int_0^T\left\|\varepsilon\frac{\partial}{\partial t}U_\varepsilon(t)\right\|_{L^2}^2dt\leqq c\varepsilon,$$

and

$$\|\nabla U_\varepsilon(t)\|_{L^2}\leqq c.$$

Applying a familiar argument to (1.4) (see Lions [17]) we get

(1.17)
$$\int_0^T\int_\Omega\{-\beta_\varepsilon(U_\varepsilon(x,t))\phi_t(x,t)+\nabla U_\varepsilon\nabla\phi-\Psi(U_\varepsilon\}\,dxdt$$
$$+\int_\Omega\beta_\varepsilon(U_0(x))\phi(0)\,dx=0$$

for $\forall\phi\in C^1([0,T];\ \mathring{H}_1(\Omega))$ with $\phi(T)=0$. That is, $U_\varepsilon(x,t)$ is a solution of the modified problem (1.3).

Next, on the basis of the estimates (1.16), we shall study the convergence property of $U_\varepsilon(x,t)$ as $\varepsilon\to 0$. This time the estimate (1.12) has no effect since the right hand side is unbounded as $\varepsilon\to 0$. So, to derive the similar convergence as in (1.13) we must find another estimate of $V_\varepsilon(x,t)$ in some function space which is compactly imbedded in $L^2(\Omega)$. We employ $\mathrm{Lip}(\theta,q)$, $0\leqq\theta\leqq 1$, $1\leqq q<\infty$, as such a space. Let us recall (see A. Ono [22])

$$u\in \mathrm{Lip}(\eta,q)$$

if and only if

$$u\in L^q(\Omega) \text{ and } |u|_{\mathrm{Lip}(\theta,q)}\equiv\overline{\lim_{|h|\to 0}}\frac{\|u(\cdot+h)-u(\cdot)\|_{L^q}}{|h|^\theta}<\infty,$$

where $h=(h_1,\dots,h_n)$ and $u(x+h)$ must be defined appropriately if $x+h\notin\Omega$. We set for $u\in\mathrm{Lip}(\theta,q)$

$$\|u\|_{\mathrm{Lip}(\theta,q)}=\|u\|_{L^q}+|u|_{\mathrm{Lip}(\theta,q)}.$$

$\mathrm{Lip}(\theta,q)$ is a Banach space and we know $\mathrm{Lip}(\theta,2)\subset H_\theta(\Omega)$ and $\mathrm{Lip}(1,2)=H_1(\Omega)$. Moreover the imbedding relation (Nikolskii's Lemma)

$$\text{(1.18)}\qquad \mathrm{Lip}(\theta,q_1)\subset\mathrm{Lip}\left(\theta+\frac{n}{q_2}-\frac{n}{q_1},q_2\right)$$

holds if $\theta+\dfrac{n}{q_2}-\dfrac{n}{q_1}>0$ and $q_2\geqq q_1$, which will be used later in § 4.

Now, recalling the definition of $V_\varepsilon(x,t)$, we have

$$\begin{aligned}|V_\varepsilon(x+h,t)-V_\varepsilon(x,t)|&=\sqrt{(p+1)^{-1}}\left|\int_{U_\varepsilon(x,t)}^{U_\varepsilon(x+h,t)}(|\eta|+\varepsilon)^{-p/2(p+1)}d\eta\right|\\&=\frac{2\sqrt{p+1}}{p+2}|(|U_\varepsilon(x+h,t)|+\varepsilon)^{(p+2)/2(p+1)}\\&\quad-(|U_\varepsilon(x,t)|+\varepsilon)^{(p+2)/2(p+1)}|.\\&\leqq c|U_\varepsilon(x+h,t)-U_\varepsilon(x,t)|^{(p+2)/2(p+1)},\end{aligned}\qquad\text{(1.19)}$$

where $U_\varepsilon(x+h,t)$ and $V_\varepsilon(x+h,t)$ should be defined as 0 if $x+h\notin\Omega$. Thus we have

$$\begin{aligned}\int_\Omega|V_\varepsilon(x+h,t)-V_\varepsilon(x,t)|^{\frac{4(p+1)}{p+2}}dx&\leqq c\int_\Omega|U_\varepsilon(x+h,t)U_\varepsilon(x,t)|^2dx\\&\leqq c\|U_\varepsilon(t)\|_{H_1}^2|h|^2\end{aligned}$$

or

$$\|V_\varepsilon(\cdot+h,t)-V_\varepsilon(\cdot,t)\|_{L^{4(p+1)/(p+2)}}\leqq c\|U_\varepsilon(t)\|_{\mathring{H}_1}{}^{(p+2)/2(p+1)}|h|^{(p+2)/2(p+1)}$$

which implies $V_\varepsilon(x,t)\in\mathrm{Lip}(\theta,q)$ with $q=4(p+1)/(p+2)$ and $\theta=(p+2)/(p+1)$, and by (1.16)

$$(1.20)\qquad \|V_\varepsilon(t)\|_{\mathrm{Lip}(\theta,q)}\leqq c\|_{L^q}+\|U_\varepsilon(t)\|_{\mathring{H}_1}{}^{(p+2)/2(p+1)})<c<\infty.$$

Since $\mathrm{Lip}(\theta,2)$, $\theta>0$, is compactly imbedded in $L^2(\Omega)$ we have from (1.16) and (1.20) that

$$(1.21)\qquad V_\varepsilon(x,t)\to V(x,t) \text{ strongly in } L^r([0,T];\ L^2(\Omega)) \text{ for } r\geqq 1 \text{ and a.e. } \Omega\times[0,T]$$

as $\varepsilon\to 0$ (more precisely along a sequence $\{\varepsilon^j\}_{j=1}^\infty$ tending to 0). Further implications of (1.16) are

$$\frac{\partial}{\partial t}V_\varepsilon(t)\to\frac{\partial}{\partial t}V \text{ weakly in } L^2([0,T];\ L^2(\Omega)),$$

$$(1.22)\qquad \varepsilon\frac{\partial}{\partial t}U_\varepsilon(t)\to 0 \qquad \text{strongly in } L^2([0,T];\ L^2(\Omega)).$$

and

$$U_\varepsilon(x,t)\to U(x,t) \qquad \text{weakly* in } L^\infty([0,T];\ \mathring{H}_1(\Omega)) \text{ and a.e. in } \Omega\times[0,T]$$

where

$$V(x,t)=\frac{1}{\sqrt{p+1}}\int_0^{U(x,t)}|\eta|^{-p/2(p+1)}d\eta.$$

Moreover we have from the last assertion of (1.22) that

$$(1.23)\qquad \beta_\varepsilon(U_\varepsilon)\to U^{1/(p+1)} \quad \text{weakly in } L^2([0,T];\ L^2(\Omega))$$

and

$$\Psi(U_\varepsilon)\to\Psi(U) \qquad \text{weakly* in } L^\infty([0,T];\ L^{(p+\alpha+2)/(\alpha+1)}(\Omega)).$$

Using the first assertion of (1.22) we can say more about the convergence properties of $\beta_\varepsilon(U_\varepsilon)$ and $\Psi(U_\varepsilon)$, which is not needed at this stage (see § 4).

Any way, taking $\varepsilon\to 0$ along a sequence in (1.17) we can conclude that $U(x,t)$ is the desired solution of (1.2) apart from the nonnegativity. To show the nonnegativity of $u(x,t)=|U|^{-p/(p+1)}U(x,t)$ we note by (1.1)′ with u^{p+1} and $u^{\alpha+1}$ replaced by $|u|^p u$ and $|u|^\alpha u$, respectively, that

$$\frac{d}{dt}u(t)\in L^\infty([0,T];H^{-1}).$$

Then, setting $u^- = \min(u, 0)$ and using the method of mollifier we have

$$\frac{1}{p+2}\frac{d}{dt}\int_\Omega u^-(t)^{p+2}dx = <\frac{d}{dt}u, u^->_{H^{-1}\times\mathring{H}_1}$$
$$= -(\nabla|u|^p u, \nabla|u^-|^p u^-)$$
$$= -|\nabla(|u^-|^p u^-|^2 \leqq 0$$

and

$$\int_\Omega |u^-(t)|^{p+2}dx \leqq \int_\Omega |u^-(0)|^{p+2}dx = \int_\Omega u_0^-|^{p+2}dx = 0$$

for $t\in[0, T]$ which means $u(x, t\geqq 0$ a. e. $x\in\Omega$. We have now finished the proof of Theorem 1. 1.

§ 2. Global existence for the case $p<\alpha$

To discuss the existence of global solutions of (0. 1) for the case $p<\alpha$ we define the 'potential well (which was introduced by Sattinger [27].) In applying it to our equation we follow Tsutsumi [28].

Let us define

$$J(U) \equiv \frac{1}{2}\|\nabla U\|_{L^2}^2 - \frac{p+1}{p+\alpha+2}\int_\Omega \Psi(U)Udx \text{ for } U\in\mathring{H}_1$$

where

$$\Psi(U) = \begin{cases} 0 & \text{if } U<0 \\ U^{\frac{\alpha+1}{p+1}} & \text{if } U\geqq 0. \end{cases}$$

For $J(U)$ to be well defined we must make an assumption $(p+1)/(p+\alpha+2) \geqq 1/2-1/n$, i. e.,

$$(2.1)\quad \begin{aligned} &0\leqq p\leqq\frac{(n+2)+4}{n-2} && \text{if } n\geqq 3 \\ &0\leqq\alpha<\infty && \text{if } n=1, 2. \end{aligned}$$

We set

$$(2.2)\quad d \equiv \inf_{\substack{U\in\mathring{H}_1 \\ U\neq 0}} \sup_{\lambda\geqq 0} J(\lambda U).$$

Then it is easy to see that

$$(2.3)\quad \infty > d = \inf_{\substack{U\in\mathring{H}_1 \\ U\neq 0}} J(\lambda U)\,|_{\lambda=\left(\frac{\|\nabla U\|^2_{L^2}}{\int_\Omega \Psi(U)Udx}\right)^{(p+1)/(\alpha-p)}}$$

$$=\frac{\alpha-p}{2(p+\alpha+1)}\inf_{\substack{U\in\mathring{H}_1\\ U\neq 0}}\left(\frac{\|\nabla U\|^2{}_{L^2}}{\int_\Omega \Psi(U)Udx}\right)^{2(p+1)/(\alpha-p)}\|\Delta U\|_{L^2}^2$$

$$\geqq c_0/(\alpha-p)/2(p+\alpha+1)>0,$$

where c_0 is the Sobolev's constant such that

$$\int_\Omega |U|^{(p+\alpha+2)/(p+1)}dx\leqq c_0\|\nabla U\|_{L^2}{}^{(p+\alpha+2)/(p+1)}.$$

The potential well associated with (0.1) or (1.2) is defined as

$$W=\{U\in\mathring{H}^1\,|\,0\leqq J(\lambda U)<d \quad\text{for}\quad \lambda\in[0.1]\},$$

d being called 'the depth of the well $\mathfrak{w}$'. The following lemma is proved easily.

LEMMA 2.1 (Tsutsumi [28]). *It holds that*

$$W=W_*\cup\{0\},$$

where we set

$$W_*=\{U\in\mathring{H}_1\,|\,\|\nabla U\|_{L^2}^2-\int_\Omega \Psi(U)Udx>0 \quad and \quad J(U)<d\}.$$

The purpose of this sectioin is to prove that (0.1) admits a global solution if $u_0^{p+1}\in W$. Let us consider again the approximate solutions $\{U_{m,\varepsilon}(t)\}$ by (1.4). As is already seen in §1 it suffices for our purpose to derive the apriori estimates (1.9)–(1.11).

LEMMA 2.2. *Let $U_0\in W$. Then for large m the approximate solutions $\{U_{m,\varepsilon}(t)\}$ exist on $[0,T]$ and $U_{m,\varepsilon}(t)\in W$ for $t\in[0,T]$, and consequently we have* (1.9)–(1.11), *i.e.*,

$$\|U_{m,\varepsilon}(t)\|_{\mathring{H}_1}^2+\int_0^T\{\|\sqrt{\beta'_\varepsilon(U_{m,\varepsilon})}\frac{\partial}{\partial t}U_{m,\varepsilon}(s)\|_{L^2}^2+\varepsilon\|\frac{\partial}{\partial t}U_{m,\varepsilon}(s)\|_{L^2}^2\}ds \tag{2.4}$$

$$\leqq c,$$

where c is a constant independent of m, ε and U_0.

PROOF. If $U_{m,\varepsilon}(t)\in\mathfrak{w}$ it follows from Lemma 2.1 that

$$d>J(U_{m,\varepsilon}(t))\geqq\frac{\alpha-p}{2(p+\alpha+2)}\|\nabla U_{m,\varepsilon}(t)\|_{L^2}^2, \tag{2.5}$$

which together with the energy equality (1.6) yields (2.4) immediately.

The former part of the Lemma is essentially included in Tsutsumi [28]. For completeness, however, we reproduce the proof briefly.

First, note that $U_{m,\varepsilon}(0)\equiv U_{0,m}\in W$ for large m, because $U_0\in W$ and $J(U)$ is continuous in U with respect to the norm $\|\cdot\|_{\mathring{H}^1}^2$. Assuming that our assertion were false, we can choose $t^*>0$ such that

$$U_{m,\varepsilon}(t^*\in\partial W \text{ and } U_{m,\varepsilon}(t)\in W \text{ for } 0\leqq{}^\forall t\leqq t^*.$$

By Lemma 2.1 we have either

(i) $J(U_{m,\varepsilon}(t^*))=d$ or (ii) $\|\nabla U_{m,\varepsilon}(t^*)\|_{L^2}^2=\int_\Omega \Psi(U_{m,\varepsilon}(t^*))U_{m,\varepsilon}(t^*)dx.$

But the energy equality (1.6) with $t=t^*$ shows the case (i) is impossible. If the case (ii) was valid we have

$$d>J(U_{m,\varepsilon}(t^*))=\sup_{\lambda\geq 0} J(\lambda U_{m,\varepsilon}(t^*))\geqq d$$

which is also a contradiction. q. e. d.

By the above Lemma combined with the argument as in § 1 we obtain.

THEOREM 2.1. *If* $u_0\geqq 0$ *and* $u_0^{p+1}\in W$, *the problem* (0.1) ((1.1)) *has a global solution* $u(t)$ *such that* $u(t)^{p+1}\in W$ *for* ${}^\forall t\in[0, T]$ *and*

$$\|\nabla u^{p+1}(t)\|_{L^2}+\int_0^T\|\frac{\partial}{\partial t}u^{\frac{p}{2}+1}\|_{L^2}^2ds\geqq c<\infty$$

for ${}^\forall t\in[0, T]$.

REMARK 2.1. Using the result of the next section we have in fact $u(t)^{p+1}\in W$ for $t\in[0, T]$.

§ 3. Decay of solutions

In this section we derive a certain decay estimate of the energy $\|\nabla u^{p+1}(t)\|_{L^2}$ for the solution $u(t)$ in Theorem 2.1. As was mentioned in the introduction the equation without the perturbation $-u^{\alpha+1}$ was already treated by the author in [20] by use of elliptic regularization and energy inequalities. Here we shall prove the same result for the solutions when $u_0^{p+1}\in W$.

THEOREM 3.1. *Let* $\alpha>p$. *If* $u_0\geqq 0$ *and* $u_0^{p+1}\in W$, *the solution in Theorem* 2.1 *satisfies the decay estimate*

$$\int_t^{t+1}\|\frac{\partial}{\partial t}(u^{\frac{p}{2}+1})\|_{L^2}^2ds+\|\nabla u^{p+1}(t)\|_{L^2}^2\leqq c(1+t)^{-2(p+1)/p}$$

for $t\in[0, T]$, *where* c *is a constant independent of* T *which depends on* $d-J(U_0)$.

For the proof we prepare a lemma.

LEMMA 3.1. *Let* $U_0\in W$ *and* $0<\varepsilon_0<d-J(U_0)$. *Then there exists* $m_0>0$ *such that if* $m>m_0$ *we have*

$$\|\nabla U_{m,\varepsilon}(t)\|^2_{L^2}\geqq\left(\frac{d}{d-\varepsilon_0}\right)^{(\alpha-p)/2(p+1)}\int_\Omega\Psi(U_{m,\varepsilon}(t))U_{m,\varepsilon}(t)dx \tag{3.1}$$

where $\{U_{m,\varepsilon}\}$ *are approximate solutions in* (1.4).

PROOF. By (1.6) and the continuity of $J(U)$ we have

$$J(U_{m,\varepsilon}(t))\leqq J(U_{m,\varepsilon}(0))<d-\varepsilon_0 \tag{3.2}$$

for sufficiently large m. By the way (2.3) and (2.5) imply

$$\frac{\alpha-p}{2(p+2+\alpha)}\|\nabla U_{m,\varepsilon}(t)\|^2_{L^2}\leqq J(U_{m,\varepsilon}(t))$$

$$\leqq\frac{\alpha-p}{2(p+2+\alpha)}\|\nabla U_{m,\varepsilon}(t)\|^2_{L^2}\left(\frac{\|\nabla U_{m,\varepsilon}(t)\|^2_{L^2}}{\int_\Omega\Psi(U_{m,\varepsilon})U_{m,\varepsilon}}(t)\,dx\right)^{2(p+1)/(\alpha-p)}-\varepsilon_0 \tag{3.3}$$

and hence

$$\frac{(\alpha-p)}{2(p+2+\alpha)}\|\nabla U_{m,\varepsilon}(t)\|^2\left\{\left(\frac{\|\nabla U_{m,\varepsilon}(t)\|^2_{L^2}}{\int_\Omega\Psi(U_{m,\varepsilon}(t))U_{m,\varepsilon}(t)}dx\right)^{2(p+1)/(\alpha-p)}-1\right\}\geqq\varepsilon_0.$$

It follows again from (3.2) and (3.3) that

$$\left(\frac{\|\nabla U_{m,\varepsilon}(t)\|^2_{L^2}}{\int_\Omega\Omega(U_{m,\varepsilon}(t))U_{m,\varepsilon}}(t)\,dx\right)^{2(p+1)/(\alpha-p)}-1\geqq\frac{\varepsilon_0}{d-\varepsilon_0}$$

which shows (3.1). q. e. d.

PROOF of Theorem 3.1. It suffices to derive the estimate for approximate solutions $\{u_{m,\varepsilon}(t)\}$, $u_{m,\varepsilon}(t)\equiv|U_{m,\varepsilon}(t)|^{-p/(p+1)}U_{m,\varepsilon}(t)$, for large m. As was already seen $U_{m,\varepsilon}(t)\in W$ for $t\in R^+$ (we take $T=\infty$ for simplicity), and we have (2.5) and (3.1). As in (1.6) we have

$$\int_t^{t+1}\int_\Omega(\beta'_\varepsilon(U_{m,\varepsilon})+\varepsilon)\left|\frac{\partial}{\partial t}U_{m,\varepsilon}\right|^2dxds \tag{3.4}$$
$$=J(U_{m,\varepsilon}(t))-J(U_{m,\varepsilon}(t+1))\equiv D(t)^2.$$

On the other hand, mulitplying (1.4) by $U_{m,\varepsilon}(t)$ and integrating, we have

$$\int_t^{t+1}\|\nabla U_{m,\varepsilon}(s)\|^2{}_{L^2}\,ds=-\int_t^{t+1}\int_\Omega(\beta'\varepsilon(U_{m,\varepsilon})+\varepsilon)\frac{\partial}{\partial t}U_{m,\varepsilon}U_{m,\varepsilon}dxds$$

$$+\int_t^{t+1}\int_\Omega\Psi(U_{m,\varepsilon})U_{m,\varepsilon}dxds$$

$$\leqq\ \{\int_t^{t+1}\int_\Omega(\beta'\varepsilon(U_{m,\varepsilon})+\varepsilon)\,|\frac{\partial}{\partial t}U_{m,\varepsilon}|^2dxds\}^{1/2}$$

$$\times\int_t^{t+1}\{\int_\Omega\beta'\varepsilon(U_{m,\varepsilon})+\varepsilon)\,|U_{m,\varepsilon}|^2dx\}^{1/2}ds$$

$$+(\frac{d-\varepsilon_0}{d})^{(\alpha-p)/(p+1)}\int_t^{t+1}\|\nabla U_{m,\varepsilon}(s)\|^2_{L^2}ds,$$

where we have used Lemma 3.1. Thus we obtain by (3.4)

$$\int_t^{t+1}\|\nabla U_{m,\varepsilon}(s)\|^2_{L^2}ds\leqq c_{\varepsilon_0}D(t)\int_t^{t+1}\{\int_\Omega(\beta'\varepsilon(U_{m,\varepsilon})+\varepsilon)\,|U_{m,\varepsilon}|^2dx\}^{1/2}ds, \tag{3.5}$$

where c_{ε_0} denotes constants tending to ∞ as $\varepsilon_0\to 0$. The integrating term of (3.5) is estimated as

$$\int_t^{t+1}\{\int_\Omega(\beta'\varepsilon(U_{m,\varepsilon})+\varepsilon)\,|U_{m,\varepsilon}|^2dx\}^{1/2}ds$$

$$=\int_t^{t+1}\{\int_\Omega(p+1)(|U_{m,\varepsilon}|+\varepsilon)^{-p/(p+1)}|U_{m,\varepsilon}|^2dx+\varepsilon\int_\Omega|U_{m,\varepsilon}|^2dx\}^{1/2}ds$$

$$\leqq c\{\int_t^{t+1}\|\nabla U_{m,\varepsilon}(s)\|_{L_2}^{(p+2)/2(p+1)}+\sqrt{\varepsilon}\int_t^{t+1}\|U_{m,\varepsilon}(s)\|_{L^2}ds\}. \tag{3.6}$$

From (3.3), (3.5) and (3.6) we obtain

$$\int_t^{t+1}J(U_{m,\varepsilon}(s))ds\leqq c_{\varepsilon_0}D(t)\int_t^{t+1}\{J(U_{m,\varepsilon}(s))^{(p+2)/4(p+1)}+\sqrt{\varepsilon J(U_{m,\varepsilon}(s))}\}ds$$

and, applying Young's inequality,

$$\int_t^{t+1}J(U_{m,\varepsilon}(s))ds\leqq c_{\varepsilon_0}(D(t)^{4(p+1)/(2+3p)}+\varepsilon D(t)^2). \tag{3.7}$$

Since $J(U_{m,\varepsilon}(t)$ is nonincreasing in t we have from (3.4) and (3.7)

$$\sup_{t\leqq s\leqq t+1}J(U_{m,\varepsilon}(s))\leqq J(U_{m,\varepsilon}t))$$

$$=J(U_{m,\varepsilon}(t+1))+\int_t^{t+1}\int_\Omega(\beta'\varepsilon(U_{m,\varepsilon})+\varepsilon)\,|\frac{\partial}{\partial t}U_{m,\varepsilon}|^2dxds$$

$$\leqq c_{\varepsilon_0}\{D(t)^{4(p+1)/(2+3p)}+\varepsilon D(t)^2+D(t)^2\}$$

$$\leqq c_{\varepsilon_0}D(t)^{4(p+1)/(2+3p)} \tag{3.8}$$

where in the last step we have used the facts that $0 \leqq D(t)^2 < d$ and $0 < \varepsilon << 1$. From (3.8) we obtain immediately

(3.9) $$\sup_{t \leq s \leq t+1} J(U_{m,\varepsilon}(s))^{1+\frac{p}{2(p+1)}} \leqq c_{\varepsilon_0}\{J(U_{m,\varepsilon}(t)) - J(U_m(t+1))\}.$$

Thus, applying the Lemma below with $g(t) \equiv 0$ we finish the proof of Theorem 3.1.

LEMMA 3.2. *Let $\phi(t)$ be a nonnegative function on R^+ satisfying*

$$\sup_{t \leq t \leq s+1} \phi(s)^{1+r} \leqq c_0(\phi(t) - \phi(t+1)) + g(t)$$

with $r \geqq 0$, $c_0 \geqq 0$ and $g(t) \geqq 0$. Then,

i) *if $r > 0$ and $\lim_{t\to\infty} (1+t)^{1+\frac{1}{r}} g(t) = 0$, we have*

$$\phi(t) \leqq c(\phi(0))(1+t)^{-\frac{1}{r}}$$

and

(ii) *if $r = 0$ and $g(t) \leqq c_1 \exp\{-\lambda t\}$ for some $c_1, \lambda > 0$, we have*

$$\phi(t) \leqq c(\phi(0)) \exp\{-\lambda' t\}$$

for some $\lambda' > 0$.

For a proof of the above lemma see [18] or [19].

§ 4. Blowing up of solutions for the case $\alpha > p$

In preceding two sections we have proved that if $\alpha > p$ and $v_0^{p+1} \in W$ the problem (0.1) admits a global solution $u(t)$ which decays to 0 at a certain rate. Here we shall show that if $J(u_0^{p+1}) \leqq 0$, $u_0 \neq 0$, (and consequently $u_0^{p+1} \notin W$, we cannot expect the global existence of solution.

THEOREM 4.1. *Let $u(t)$ be a solution of (0.1) ((1.1)') on $[0, T]$ such that*

$$\frac{\partial}{\partial t} u^{\frac{p}{2}+1} \in L^2([0, T]; L^2(\Omega)) \text{ and } u^{p+1} \in L^\infty([0, T]; \mathring{H}_1).$$

We suppose that $J(u^{p+1}(t)) \leqq 0$ for $t \in [0, T]$. Then T must satisfy

$$T < T_0 \equiv \frac{p+2+\alpha}{\alpha(\alpha-p)} \|u_0\|_{L^{p+2}}^{-\alpha}$$

and we have

(4. 1) $$\|u(t)\|_{L^{p+2}} \geqq c(T_0-t)^{-1/\alpha}$$

for a certain $c>0$.

REMARK 4. 1. Multiplying (0. 1) by $U=u^{p+1}$ and integrating we can obtain formally

(4. 2) $$\frac{4(p+1)}{(p+2)^2}\int_0^t \|\frac{\partial}{\partial t}u^{\frac{p}{2}+1}(s)\|^2_{L^2}ds+J(u^{p+1}(t))=J(u_0^{p+1}),$$

and hence the assumption $J(u^{p+1}(t))\leqq 0$ in the Theorem follows from $J(u_0^{p+1}) \leqq 0$. However, at this time, we do not know whether our solutions in the Theorem satisfy the energy eqality (4. 2) or not. See Theorem 4. 2 below.

PROOF of Theorem 4. 1. The proof is given by a standard way as in [28]. We note first

(4. 3) $$\int_0^t \langle u^{p+1}, \frac{\partial}{\partial t}u\rangle_{\mathring{H}_1\times H^{-1}}ds=\frac{2}{p+2}\int_0^t\int_\Omega u^{\frac{p}{2}+1}\frac{\partial}{\partial t}(u^{\frac{p}{2}+1})dxds$$
$$=\frac{1}{p+2}(\|u(t)\|^{p+2}_{L^{p+2}}-|u_0|^{p+2}_{L^{p+2}})$$

which follows from a mollifier argument. Multiplying (1. 1)$'$ by $u^{p+1}(t) \in \mathring{H}_1$ and integrating, we get

(4. 4) $$\frac{1}{p+2}(\|u(t)\|^{p+2}_{L^{p+2}}-\|u_0\|^{p+2}_{L^{p+2}})+\int_0^t \|\nabla(u^{p+2}(s))\|^2_{L^2}ds$$
$$-\int_0^t\int_\Omega u^{p+\alpha+2}dxds=0.$$

Now by our assumption we see

$$\|\nabla(u^{p+1}(t))\|^2_{L^2}\leqq\frac{2(p+1)}{p+\alpha+2}\int_\Omega u^{p+\alpha+2}(t)dx,$$

and by (4. 4)

(4. 5) $$\|u(t)\|^{p+2}_{L^{p+2}}\geqq\|u_0\|^{p+2}_{L^{p+2}}+\frac{(p+2)(\alpha-p)}{p+2+\alpha}\int_0^t\|u(s)\|^{p+\alpha+2}_{L^{p+2}}ds.$$

Since we may assume $\|u(t)\|^{p+2}_{L^{p+2}}$ is continuous (cf. (4. 3)) we obtain from (4. 5)

$$\|u(t)\|^{p+2}_{L^{p+2}}\geqq(\|u_0\|^{-\alpha}_{L^{p+2}}-\frac{\alpha(\alpha-p)}{p+2+\alpha}t)^{-(p+2)/\alpha}$$

and this proves the Theorem. q. e. d.

As was mentioned in Remark 4.1, we see $J(u^{p+1}(t))\leqq J(u_0^{p+1})$ if $u(t)$ is a sufficiently smooth solution. Finally we shall give a local existence theorem for solutions which satisfy such an inequality.

THEOREM 4.2. (*local existence*) *Let us assume*

$$0\leqq\alpha\leqq\frac{(n+2)p+4}{2(n-2)} \qquad if \quad n\geqq 3$$

$$0\leqq\alpha<\infty \qquad if \quad n=1,2.$$

Then, for u_0 *with* $u_0\geqq 0$ *and* $u_0^{p+1}\in\mathring{H}_1(\Omega)$, *there exists* $T_1=T_1(u_0)>0$ *such that* (0.1) *admits a solution* $u(t)$ *on the interval* $[0, T_1]$ *in the sense of* Theorem 1.1, *satisfying*

$$\frac{4(p+1)}{(p+2)^2}\int_0^t\|\frac{\partial}{\partial t}u^{\frac{p}{2}+1}(s)\|^2_{L^2}ds+J(u^{p+1}(t))\leqq J(u_0^{p+1}) \tag{4.6}$$

a. e. $t\in[0, T_1]$.

REMARK 4.2. It is easy to see that the solutions $u(t)$ in the sense of Theorem 1.1 are continuous in t with respect to the L^{p+2}-norm. Therefore it holds in fact for (1.1)

$$(1.1)'' \qquad \int_0^T\int_\Omega\{-u(x,t)\phi_t(x,t)+\nabla(u^{p+1})\nabla\phi(x,t)-u^{\alpha+1}\phi(x,t)\}\,dxdt$$
$$-\int_\Omega u_0(x)\phi(x,0)\,dx+\int_\Omega(x,T)\phi(x,T)dx=0$$

for $\forall\phi\in C^1([0T];\mathring{H}_1)$.

Thus, combining the above Theorem with the usual continuation procedure we can take a maximal interval $[0, T_{\max}]$ where the solution of Theorem exists.

PROOF of Theorem 4.2. Let $U_{m,\varepsilon}(t)$ be the approximate solutions of (1.4). Multiplying (1.4) by $\frac{\partial}{\partial t}(U_{m,\varepsilon}(t))$ and integrating we have

$$\int_\Omega(\beta'\varepsilon(U_{m,\varepsilon})+\varepsilon)\,|\frac{\partial}{\partial t}U_{m,\varepsilon}(t)|^2dx+\frac{1}{2}\frac{d}{dt}\|\nabla U_{m,\varepsilon}(t)\|^2_{L^2}$$

$$=\int_\Omega\Psi(U_{m,\varepsilon})\frac{\partial}{\partial t}U_{m,\varepsilon}dx$$

$$\leqq 2\int_\Omega\frac{|U_{m,\varepsilon}|^{2(\alpha+1)/(p+1)}}{\beta'\varepsilon(U_{m,\varepsilon})+\varepsilon}dx+\frac{1}{2}\int_\Omega\beta'\varepsilon(U_{m,\varepsilon})+\varepsilon)\,|\frac{\partial}{\partial t}U_{m,\varepsilon}|^2dx \tag{4.7}$$

and

$$\frac{d}{dt}\|\nabla U_{m,\varepsilon}(t)\|^2_{L^2} \leqq c\int_\Omega |U_{m,\varepsilon}|^{2(\alpha+1)/(p+1)}(|U_{m,\varepsilon}|+\varepsilon^{p/(p+1)}dx \tag{4.8}$$

$$\leqq c(\int_\Omega |U_{m,\varepsilon}|^{(p+2\alpha+2)/(p+1)}dx+\varepsilon^{(p+2\alpha+2)/(p+1)})$$

$$\leqq c(\|\nabla U_{m,\varepsilon}(t)\|^2_{L^2}+\varepsilon^2)^{(p+2\alpha+2)/2(p+1)}$$

where we have used our assumpion on α at the last step. From (4.8) we get

$$\|\nabla U_{m,\varepsilon}(t)\|^2_{L^2} \leqq \left\{\frac{1}{(|\nabla U_{m,\varepsilon}(0)|^2+\varepsilon^2)-\dfrac{c(2\alpha-p)t}{2(p+1)}}\right\}^{2(1+p)/(2\alpha-p)}-\varepsilon^2$$

which implies that there exists $T_1=T_1(\|\nabla u_0\|_{L^2})$ independent of m and ε such that $U_{m,\varepsilon}(t)$ exists on $[0, T_1]$ for sufficiently large m and small $\varepsilon>0$, and

$$\|\nabla U_{m,\varepsilon}(t)\|^2_{L^2} \leqq c \quad \text{for} \quad t\in[0, T_1] \tag{4.9}$$

and also (see (4.7))

$$\int_0^{T_1}\|\sqrt{\beta'_\varepsilon(U_{m,\varepsilon}+\varepsilon}\frac{\partial}{\partial t}U_{m,\varepsilon}(t)\|^2_{L^2}dt \leqq c. \tag{4.10}$$

From above estimates we can get the modified solution $U_\varepsilon(t)$ and subsequently the desired solution $U(t)$ of (1.2) by the same procedure as in §1. It remains to show the inequality (4.6). For this we first note that

$$\int_0^t\|\sqrt{\beta'_\varepsilon(U_\varepsilon(t))+\varepsilon}\frac{\partial}{\partial t}U_\varepsilon\|^2_{L^2}ds+J(U_\varepsilon(t)) \leqq J(U_0) \tag{4.11}$$

for a. e. $t\in[0, T]$ which follows from (4.9), (4.10) and the fact that $\mathring{H}_1(\Omega)$ is compactly imbedded in $L^{(p+\alpha+2)/(p+1)}$. $U_\varepsilon(t)$ satisfies the estimates (4.9) and (4.10) with $U_{m,\varepsilon}$ replaced by U_ε and the equation

$$\begin{aligned}&\frac{d}{dt}(\beta_\varepsilon(U_\varepsilon(t))+\varepsilon U_\varepsilon)-\Delta U_\varepsilon(t)-U_\varepsilon^{(\alpha+1)/(p+1)}=0 \text{ in } H^{-1}\\ &U_\varepsilon(0)=u_0\in\mathring{H}_1\end{aligned} \tag{4.12}$$

and hence

$$\int_0^T\|\frac{d}{dt}\beta_\varepsilon(U_\varepsilon(t))\|_{H^{-1}}dt=\int_0^T\|\varepsilon\frac{d}{dt}U_\varepsilon-\Delta U_\varepsilon(t)-U_\varepsilon^{(\alpha+1)/(p+1)}\|_{H^{-1}}dt \tag{4.13}$$

$$\leqq\int_0^T\{\varepsilon\|\frac{d}{dt}U_\varepsilon\|_{L^2}+\|\nabla U_\varepsilon\|_{L^2}$$

$$+c\|U_\varepsilon\|_{L^{(p+\alpha+2)/(p+1)}}^{(\alpha+1)/(p+1)}\} dt$$

$$\leqq \text{Const.} \quad <\infty$$

On the other hand, in a similar way as in (1.20), we can get

(4.14) $$\|\beta_\varepsilon(U_\varepsilon(t))\|_{\mathrm{Lip}(1/(p+1),2(p+1))} \leqq c\|U_\varepsilon(t)\|_{\overset{\circ}{H}_1}^{1/(p+1)} < c < \infty.$$

Since

$$\mathrm{Lip}(1/(p+1), 2p+1)) \subset \mathrm{Lip}(1/(p+1)+n/q-n/2(p+1), q)$$

if $q \geqq 2(p+1)$ and $1/(p+1)+n/q-n/2(p+1)>0$,
we can take $q=p+\alpha+2$ to obtain

(4.15) $$\mathrm{Lip}(1/(p+1), 2(p+1)) \subset \mathrm{Lip}(\theta, p+\alpha+2)$$

with $\theta=\frac{n}{p+1}\left(\frac{p+1}{p+\alpha+2}+\frac{1}{n}-\frac{1}{2}\right)(>0)$.

From (4.14) and (4.15) we have

(4.16) $$\|\beta_\varepsilon(U_\varepsilon(t))\|_{\mathrm{Lip}(\theta,p+\alpha+2)} \leqq c < \infty.$$

Since $\mathrm{Lip}(\theta, p+\alpha+2)$ $(\theta>0)$ is compactly imbedded in $L^{p+\alpha+2}$ we can conclude from (4.13) and (4.16) (Aubin's compactness Theorem) that

(4.17) $$\beta_\varepsilon(U_\varepsilon(t)) \to \beta(U(t)) = U^{1/(p+1)}(t) \text{ as } \varepsilon \to 0$$

strongly in $L^q([0, T_1]; L^{p+\alpha+2})$ for $q \geqq 1$ or equivalently

(4.17)′ $$U_\varepsilon(t)^{1/(p+1)} \to U^{1/(p+1)}(t) \text{ as } \varepsilon \to 0$$

strongly in L^q $([0, T_1]; L^{p+\alpha+2})$.
Thus we may assume

(4.18) $$\int_\Omega \Psi(U_\varepsilon(t))U_\varepsilon(t)dx = \|U_\varepsilon(t)^{1/(p+1)}\|_{L^{p+\alpha+2}}$$

$$\to \|U(t)^{1/(p+1)}\|_{L^{p+\alpha+2}} = \int_\Omega \Psi(U)Udx$$

for a.e. $t \in [0, T_1]$. Combining (4.18) with the trivial estimate

$$\|\nabla U(t)\|_{L^2}^2 \geqq \varliminf_{\varepsilon\to 0}\|\nabla U_\varepsilon(t)\|_{L^2}^2$$

we obtain

(4.19) $$J(U(t)) \leqq \varliminf_{\varepsilon\to 0} J(U_\varepsilon(t)) \qquad \text{a.e. } t \in [0, T_1].$$

Applying (4.19) to (4.11) we obtain (4.6). q.e.d.

REMARK 4.3. The assumption on α in Theorem 4.2 is used to get the estimates (4.9) and (4.10). After these estimates are established the restriction

$$0 \leqq \alpha < \frac{p(n+2)+4}{n-2} \qquad \textit{if } n \geqq 3$$

$$0 \leqq \alpha < \infty \qquad \text{if } n=1, 2$$

is sufficient for the inequality (4.6). Multiplying (1.4) by $U_{m,\varepsilon}$ and using the Gagliardo-Nirenberg inequality we can show (4.9), (4.10) are valid if $0 \leqq \alpha \leq p + \frac{2(p+2)}{n}$ (cf. [28]). Thus the assertion of Theorem holds also for such α.

§ 5. Final remarks

As in earlier papers we considered the nonnegative solutions. It is clear, however, that all results of this paper, apart from nonnegativity, are valid to the problem

$$\frac{\partial}{\partial t} u - \Delta(|u|^p u) - |u|^\alpha u = 0 \text{ on } \Omega \times [0, T]$$

$$u|_{\partial\Omega} = 0, \quad u(x, 0) = u_0.$$

Needless to say, the results on global existence and decay in § 1-3 are valid to the problem

$$\text{(5.1)} \qquad \frac{\partial}{\partial t} u - \Delta(|u|^p u) + |u|^\alpha u = 0 \qquad \text{on } \Omega \times [0, T]$$

$$u|_{\partial\Omega} = 0, \quad u(x, 0) = u_0$$

for any initial data u_0 with $|u_0|^p u_0 \in \mathring{H}_1$. In this case uniqueness of solution is also derived easily.

After completing our work we have learned the paper [29] by Galaktionv, where the same equation (0.1) is treated, and our Theorems 1.1 and 2.1 are included there. His proofs, however, seem to be somewhat ambiguous, because the singularity of (1.2) is ignored there. A nonexietence theorem is also given in [29], but our Theorem 4.1 is more general. Local exietence and decay property are not discussed there. We have learned also a recent work [31] by Levine & Sacks, where the same problem mith $\alpha < $m are treated in a different way. A detailed asymptotic behaviour for (5.1) in L^∞-norm has been investigated by Bertsch, Nanbu & Peletier [30].

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