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濱地, 敏弘
九州大学教養部数学教室

押川, 元重
九州大学教養部数学教室

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Anzai transformation and its centralizer

Toshihiro HAMACHI*) and Motosige OSIKAWA

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1. Introduction

For a set F of non-singular transformations on a Lebesgue measure space we denote by C(F) the set of all non-singular transformations on the space commuting with every element in F. We say that for an ergodic non-singular transformation T the set $C(T)$ is the *centralizer* and that the set $CC(T)(=C(C(T)))$ is the second centralizer.

We say that a transformation $\phi_{\sigma,\lambda,\eta}:(x,y)\longrightarrow(x+\lambda,y+\sigma x+\eta)$ on $T^n\times T^n$ is an *Anzai transformation,* if $\{1,\lambda_1,\lambda_2,\cdots,\lambda_n\}$ is rationally independent, where $\lambda=(\lambda_1,\lambda_2,\cdots,\lambda_n)$ and η are in T^n and σ is a regular matrix with integer entries. In section 2 we determine the sets $C(\phi_{\sigma,\lambda,\eta})$ and $CC(\phi_{\sigma,\lambda,\eta})$ and show that the set $\{\phi^j_{\sigma,\lambda,\eta};j\in Z\}$ is closed with respect to the weak topology, and that $\{\phi^j_{\sigma,\lambda,\eta};j\in Z\}\subsetneq CC(\phi_{\sigma,\lambda,\eta})\subsetneq C(\phi_{\sigma,\lambda,\eta})$. We note that in general for an ergodic non-singular transformation T, $\{T^j;j\in Z\}\subset\overline{\{T^j;j\in Z\}}^{(weak\ closue)}\subset CC(T)\subset C(T)$, and that $C(T)$ is commutative if and only if $C(T)=CC(T)$. D. Ornstein [**6**] gave an example of an ergodic measure preserving transformation T with $C(T)=\{T^j;j\in Z\}$. If T is a Bernoulli transformation then $CC(T)\subsetneq C(T)$, and $CC(T)=\{T^j;\ j\in Z\}$ which was proved by D. Rudolph [**11**]. For an ergodic non-singular transformation T it has pure point spectrum if and only if $C(T)$ is compact [**5**]. In this case $\{T^j;j\in Z\}\subsetneq\overline{\{T^j;\in Z\}}=C(T)$.

H. Anzai [**2**] discussed the conjugacy problem for the 1-dimensional Anzai transformations. The n-dimensional Anzai transformations are in the class of transformations with quasidiscrete spectrum whose conjugacy problem was solved by L. M. Abramov [**1**], and by F. Hahn and W. Parry [**3**]. In section 3 we give an explicite condition for the conjugacy of n-dimensional Anzai transformations. For this we introduce an invariant for ergodic measure preserving transformations which is, roughly speeking, the

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family of point spectra of elements of the centralizer of an ergodic transformation, and show that it is a complete invariant for the 1-dimensional Anzai transformations.

In section 4 we introduce a new equivalence relation of ergodic measure preserving transformations and discuss it.

2. Anzai transformation

We denote by $M(n, Z)$ the set of all $n\times n$ matrices with integer entries and by $M^1(n, Z)$ the set of all matrices in $M(n, Z)$ whose determinants are 1 or -1. For an element λ in the n-dimensional torus T^n and an element k in the n-dimensional lattice Z^n we write $<k, \lambda>=\exp 2\pi i(k_1\lambda_1+k_2\lambda_2+\cdots+k_n\lambda_n)$, where λ_i and k_i are the i-th coordinates of λ and k respectively for $i=1, 2, \cdots, n$. For elements λ and η in T^n and a matrix σ in $M(n, Z)$ we define a transformation $\phi_{\sigma,\lambda,\eta}$ on the direct product space $T^n\times T^n$ by $\phi_{\sigma,\lambda,\eta}(x, y)=(x+\lambda, y+\sigma x+\eta)$ for (x, y) in $T^n\times T^n$. The $\phi_{\sigma,\lambda,\eta}$ is measure preserving with respect to the Haar measure $dxdy$ on $T^n\times T^n$. We say that the transformation $\phi_{\sigma,\lambda,\eta}$ is an *n-dimensional Anzai transformation* if $\{1, \lambda_1, \lambda_2, \cdots, \lambda_n\}$ is rationally independent and if the determinant of σ is not zero.

THEOREM 1. *Every n-dimensional Anzai transformation* $\phi_{\alpha,\lambda,\eta}$ *has the following properties*:

(1) $\phi_{\sigma,\lambda,\eta}$ *is ergodic.*

(2) $S_p(\phi_{\sigma,\lambda,\eta})=\{<k, \lambda>; k\in Z^n\}$.

(3) $C(\phi_{\sigma,\lambda,\eta})=\{\phi_{\tau,\alpha,\beta}; \alpha, \beta\in T^n, \tau\in M(n, Z)$ *such that* $\sigma(\alpha)=\tau(\lambda)\}$.

(4) *A transformation* $\phi_{\tau,\alpha,\beta}$ *in* $C(\phi_{\sigma,\lambda,\eta})$ *is ergodic if and only if it is an Anzai transformation.*

(5) *Let* p_0 *be the greatest common divisor of all entries of* σ, *and let* $\sigma=p_0\sigma_0$ *with* σ_0 *in* $M(n, Z)$, $\lambda=p_0\lambda_0$ *with* λ_0 *in* T^n *and* $\eta-\frac{p_0(p_0-1)}{2}\sigma_0(\lambda_0)=p_0\eta_0$ *with* η_0 *in* T^n, *then* $CC(\phi_{\sigma,\lambda,\eta})=\{\phi^j_{\sigma_0,\lambda_0,\eta_0}\cdot\phi_{0,0,\beta}; j\in Z, \beta\in T^n\}$.

(6) $\{\phi^j_{\sigma,\lambda,\eta}; j\in Z\}$ *is a closed and proper subst of* $CC(\phi_{\sigma,\lambda,\eta})$, *and* $CC(\phi_{\sigma,\lambda,\eta})=C(\phi_{\sigma,\lambda,\eta})$ *if and only if* $n=1$.

PROOF. (1), (2) and (3) follows from Theorem 9 of [**9**]. We note that (3) is a special case of Theorem 3 if we replace $\phi_{\sigma',\lambda',\eta'}$ by $\phi_{\sigma,\lambda,\eta}$. In this case every isomorphism between $\phi_{\sigma,\lambda,\eta}$ and itself is in $C(\phi_{\sigma,\lambda,\eta})$.

(4) Assume that a $\phi_{\tau,\alpha,\beta}$ in $C(\phi_{\sigma,\lambda,\eta})$ is ergodic, then it is easy to see that $\{1, \alpha_1, \alpha_2, \cdots, \alpha_n\}$ is rationally independent. Since the determinant of σ is not zero, $\{1, \sigma(\alpha)_1, \sigma(\alpha)_2, \cdots, \sigma(\alpha)_n\}$ is rationally independent. Hence, we have from $\sigma(\alpha)=\tau(\lambda)$ that the determinant of τ is not zero, i.e., $\varphi_{\tau,\alpha,\beta}$ is an Anzai transformation.

(5) By (3) every transformation in $CC(\phi_{\sigma,\lambda,\eta})$ has a form $\phi_{\tau,\alpha,\beta}$ with $\sigma(\alpha)=\tau(\lambda)$, and we have $\tau'(\alpha)=\tau(\alpha')$ for any τ' in $M(n, Z)$ and any α' in T^n such that $\sigma\alpha'=\tau'\lambda$. From $\sigma(\alpha)=\tau(\lambda)$ we have $\tau'\sigma^{-1}\tau(\lambda)=\tau\sigma^{-1}\tau'(\lambda)$, and hence, $\tau'\sigma^{-1}\tau=\tau\sigma^{-1}\tau'$ for any τ' in $M(n, Z)$ follows from that $\{1, \lambda_1, \lambda_2, \cdots, \lambda_n\}$ is rationally independent. Then there exists a real number r with $\tau=r\sigma$. From $\sigma(\alpha)=r\sigma(\lambda)$ we have $\alpha=r\lambda$. Let p_0, σ_0, λ_0 and η_0 be ones mentioned in (4). Since $\tau=rp_0\sigma_0$ is in $M(n, Z)$, rp_0 is an integer which we denote by j. Then we have $\tau=j\sigma_0$ and $\alpha=j\lambda_0$. Conversely, it is obvious that every $\phi_{j\sigma_0,j\lambda_0,\beta}$, j in Z and β in T^n, is in $CC(\phi_{\sigma,\lambda,\eta})$. Here, we note that $\phi_{\sigma_0,\lambda_0,\eta_0}=\phi_{\tau,\lambda,\eta}$.

(6) For a non-zero element k in Z^n let $f(x, y)=\langle k, y\rangle$, (x, y) in $T^n\times T^n$, then we have

$$\iint |f(\phi^j_{\sigma,\lambda,\eta}(x, y))-f(x, y)|\,dxdy=\int |\langle k, j\sigma(x)+\frac{j(j-1)}{2}\sigma(\lambda)+j\eta\rangle -1|\,dx=\int |\langle j^t\sigma(k), x\rangle-\langle k, \frac{j(j-1)}{2}\sigma(\lambda)-j\eta\rangle|\,dx$$

$$>\sqrt{2}\,\mathrm{Prob}(|\langle j^t\sigma(k), x\rangle-\langle k, -\frac{j(j-1)}{2}\sigma(\lambda)-j\eta\rangle|>\sqrt{2})=\frac{\sqrt{2}}{2}$$

if $j\neq 0$. It follows from this that $\{\phi^j_{\sigma,\lambda,\eta}; j\in Z\}$ is closed. The other parts of (5) follows from (3) and (5). q.e.d.

3. Isomorphism problem of Anzai transformations

We discuss the isomorphism problem of Anzai transformations.

LEMMA 2. *Let λ be an element in T^n such that $\{1, \lambda_1, \lambda_2, \cdots, \lambda_n\}$ is rationally independent. Then for an element λ' in T^n, $\{\langle k, \lambda\rangle; k\in Z^n\}$ $=\{\langle k, \lambda\rangle; k\in Z^n\}$ if and only if there exists a matrix δ in $M^1(n, Z)$ with $\lambda'=\delta\lambda$.*

PROOF. If $\lambda'=\delta\lambda$ for some δ in $M^1(n, Z)$, then from $\langle k', \lambda'\rangle = \langle {}^t\delta(k'), \lambda\rangle$ for k' in Z^n we have $\{\langle k, \lambda\rangle; k\in Z^n\} = \{\langle k', \lambda'\rangle; k'\in Z^n\}$, where ${}^t\delta$ is the transposed matrix of δ. Next, assume that $\{\langle k, \lambda\rangle; k\in Z^n\} = \{\langle k', \lambda'\rangle; k'\in Z^n\}$. Then, since $\{1, \lambda_1, \lambda_2, \cdots, \lambda_n\}$ is rationally independent, for every k' in Z^n there exists a unique k in Z^n with $\langle k', \lambda'\rangle = \langle k, \lambda\rangle$. If we write $k=\rho(k')$, ρ is a homomorphism from Z^n into itself. Since for every k in Z^n there exists a k' in Z^n with $\langle k', \lambda'\rangle = \langle k, \lambda\rangle$, ρ is an automorphism of Z^n, and so, it is a matrix in $M^1(n, Z)$. Hence, put $\delta={}^t\rho$, δ is a matrix in $M^1(n, Z)$ and $\lambda'=\delta\lambda$.

THEOREM 3. *Anzai transformations $\phi_{\sigma,\lambda,\eta}$ and $\phi_{\sigma',\lambda',\eta'}$ are mutually isomorphic if and only if there exist matrices δ_0 and δ_1 in $M^1(n, Z)$ such that $\lambda'=\delta_0\lambda$ and such that $\sigma'=\delta_1\sigma\delta_0^{-1}$. In this case every isomorphism ψ between them has a form: $\psi(x, y) = (\delta_0(x)+u, \delta_1(y)+\tau(x)+v)$ for (x, y) in $T^n\times T^n$, where u and v are elements in T^n and τ is a matrix in $M(n, Z)$ such that $\sigma' u=\tau\lambda+\delta_1\eta-\eta'$.*

PROOF. Let δ_0 and δ_1 be matrices in $M^1(n, Z)$ such that $\lambda'=\delta_0\lambda$ and such that $\sigma'=\delta_1\sigma\delta_0^{-1}$. Let u be an element in T^n with $\sigma' u=\delta_1\eta-\eta'$, and put $\psi(x, y) = (\delta_0(x)+u, \delta_1(y))$ for (x, y) in $T^n\times T^n$. Then we have $\psi\phi_{\sigma,\lambda,\eta} = \phi_{\sigma',\lambda',\eta'}\psi$. Conversly, let ψ be an isomorphism between $\phi_{\sigma,\lambda,\eta}$ and $\phi_{\sigma',\lambda',\eta'}$. Since $S_p(\phi_{\sigma,\lambda,\eta}) = S_p(\phi_{\sigma',\lambda',\eta'})$, by (2) of Theorem 1 and Lemma 2 there exists a matrix δ_0 in $M^1(n, Z)$ with $\lambda'=\delta_0\lambda$. For k in Z^n set $f_k(x, y) = \langle k, x\rangle$ for (x, y) in $T^n\times T^n$ and $g_k(x, y) = f_k(\psi(x, y))/f_{{}^t\delta_0(k)}(x, y)$ for (x, y) in $T^n\times T^n$. Then $g_k(x, y)$ is $\phi_{\sigma,\lambda,\eta}$-invariant, and hence it is a constant from the ergodicity of $\phi_{\sigma,\lambda,\eta}$. We write $g_k(x, y)=c_k$. Since $c_{k+k'}=c_k c_{k'}$ for k and k' in Z^n there exists an element u in T^n with $c_k=\langle k, u\rangle$ for k in Z^n. Then we have $f_k(\psi(x, y)) = \langle k, \delta_0(x)+u\rangle$ for (x, y) in $T^n\times T^n$ and k in Z^n, and hence, ψ has a form: $\psi(x, y) = (\delta_0(x)+u, \psi(x, y))$ for some $\psi(x, y)$ in T^n. Set $A(k, k', k'') = \iint \langle k, \psi(x, y)\rangle \overline{\langle k', x\rangle\langle k'', y\rangle}\, dxdy$ for k, k' and k'' in Z^n. Then from $\psi\phi_{\sigma,\lambda,\eta} = \phi_{\sigma',\lambda',\eta'}\psi$ we have $\psi(x+\lambda, y+\sigma(x)+\eta) = \psi(x, y)+\sigma'(\delta_0(x)+u)+\eta'$ for (x, y) in $T^n\times T^n$ and $A(k, k', k'') = \langle k, \sigma'\delta_0(\lambda)-\sigma'(u)-\eta'\rangle\langle k', \lambda\rangle\langle k'', -\sigma(\lambda)+\eta\rangle A(k, k'+{}^t\delta_0{}^t\sigma'(k)-{}^t\sigma(k''), k'')$ for k, k' and k'' in Z^n. If ${}^t\delta_0{}^t\sigma'(k)-{}^t\sigma(k'')\neq 0$, then since $|A(k, k', k'')| = |A(k, k'+j({}^t\delta_0{}^t\sigma'(k)-{}^t\sigma(k'')), k'')|$ for all j in Z^n we have $\sum_{j=1}^{\infty}|A((k, k'+j({}^t\delta_0{}^t\sigma'(k)-{}^t\sigma(k''), k'')|^2 = \sum_{j=1}^{\infty}|A(k, k', k'')|^2<\infty$, and so,

$A(k, k', k'')=0$. Therefore, it follows that ${}^t\delta_0{}^t\sigma'(k)={}^t\sigma(k'')$ if $A(k, k', k'')\neq 0$ for some k' in Z^n. Since for every k in Z^n there exists a (k', k'') in $Z^n\times Z^n$ with $A(k, k', k'')\neq 0$, $({}^t\sigma)^{-1t}\delta_0{}^t\sigma'(k)$ takes a value in Z^n for all k in Z^n. Therefore, $\sigma'\delta_0\sigma^{-1}$ is a matrix in $M(n, Z)$, which we denote by δ_1, and for each k in Z^n there exists a k' in Z^n with $A(k, k', {}^t\delta_1(k))\neq 0$. Such a k' satisfies $<k, \delta_1(\eta)-\sigma'(u)-\eta'><k', \lambda>=1$. From this k' is uniquely determined since $\{1, \lambda_1, \lambda_2, \cdots, \lambda_n\}$ is rationally independent. Let ${}^t\tau$ be the matrix in $M(n, Z): k\in Z^n\longrightarrow k'\in Z^n$. Then we have $\tau(\lambda)=\sigma'(u)-\delta_1(\eta)+\eta'$. Taking the Fourier inverse transform we have $<k, \psi(x, y)>=A(k, {}^t\tau(k), {}^t\delta_1(k))<k, \tau(x)>\times<k, \delta_1(y)>$ for (x, y) in $T^n\times T^n$ and k in Z^n, and it follows from this that there exists an element v in T^n such that $\psi(x, y)=\delta_1(y)+\tau(x)+v$ for (x, y) in $T^n\times T^n$. Thus, we obtain $\psi(x, y)=(\delta_0(x)+u, \delta_1(y)+\tau(x)+v)$ for (x, y) in $T^n\times T^n$. q. e. d.

REMARK 1. In the same way as for the proof of Theorem 3 one can obtain the following coboundary condition for an Anzai transformation $\phi_{\sigma,\lambda,\eta}$: *Let σ' be a matrix in $M(n, Z)$ and u an element in T^n. Then there exists a measurable T^n-valued function $f(x, y)$ on $T^n\times T^n$ such that $\sigma'(x)+u=f(\phi_{\sigma,\lambda,\eta}(x, y))-f(x, y)$ for (x, y) in $T^n\times T^n$ if and only if there exist matrices δ and τ in $M(n, Z)$ such that $\sigma'=\delta\sigma$ and such that $u=\tau\lambda+\delta\eta$. In this case $f(x, y)=\tau x+\delta y+v$ for (x, y) in $T^n\times T^n$, for some v in T^n.*

REMARK 2. A shorter proof of Theorem 3 and a computation of centralizers of Anzai transformations can be also given by using the result of W. Parry [10] which says any isomorphism between unipotent affine transformations of nilmanifolds is necessarily affine.

THEOREM 4. *Let $\phi_{\sigma,\lambda,\eta}$ and $\phi_{\sigma',\lambda',\eta'}$ be Anzai transformations. Then $S_p(\phi_{\sigma,\lambda,\eta})=S_p(\phi_{\sigma',\lambda',\eta'})$ and $\{S_p(U); U\in C(\phi_{\sigma,\lambda,\eta})$ is ergodic$\}=\{S_p(U'); U'\in C(\phi_{\sigma',\lambda',\eta'})$ is ergodic$\}$ if and only if there exist matrices δ_0, δ_1 and δ_2 in $M^1(n, Z)$ such that $\lambda'=\delta_0\lambda$ and such that $\sigma'=\delta_1\sigma\delta_2^{-1}$.*

PROOF. Assume that $S_p(\phi_{\sigma,\lambda,\eta})=S_p(\phi_{\sigma',\lambda',\eta'})$ and that $\{S_p(U); U\in C(\phi_{\sigma,\lambda,\eta})$ is ergodic$\}=\{S_p(U'); U'\in C(\phi_{\sigma',\lambda',\eta'})$ is ergodic$\}$. Applying (2) of Theorem 1 and Lemma 2 to the assumption $S_p(\phi_{\sigma,\lambda,\eta})=S_p(\phi_{\sigma',\lambda',\eta'})$ we have a matrix δ_0 in $M^1(n, Z)$ such that $\lambda'=\delta_0\lambda$. Since $\phi_{E,\sigma^{-1}(\lambda),0}$ is in $C(\phi_{\sigma,\lambda,\eta})$

and ergodic, where E is the unit matrix, by the other assumption we take an ergodiic $\phi_{\tau',\alpha',0}$ in $C(\phi_{\sigma',\lambda',\eta'})$ such that $S_p(\phi_{E,\sigma^{-1}(\lambda),0})=S_p(\phi_{\tau',\alpha',0})$. In the same way as above we have a matrix δ_2 in $M^1(n, z)$ with $\alpha'=\delta_2\sigma^{-1}\lambda$.
Since $\sigma'\alpha'=\tau'\lambda'$ by (3), we have $\sigma'\delta_2\sigma^{-1}\lambda=\sigma'\alpha'=\tau'\lambda'=\tau'\delta_0\lambda$, from which we have $\sigma'\delta_2\sigma^{-1}=\tau'\delta_0$, since $\{1, \lambda_1, \lambda_2, \cdots, \lambda_n\}$ is rationally independent. Replacing σ, λ and η by σ', λ' and η' respectively we have a matrix δ_3 in $M^1(n, Z)$ such that $\sigma\delta_3\sigma'^{-1}=\tau\delta_0^{-1}$. Since $|\tau\tau'|=|\sigma\delta_3\sigma'^{-1}\delta_0\sigma'\delta_2\sigma^{-1}\delta_0^{-1}|=|\sigma\delta_3\sigma'^{-1}|\,|\sigma'\delta_2\sigma^{-1}|=|\sigma\delta_3|\,|\delta_2\sigma^{-1}|$ $=|\sigma|\,|\sigma^{-1}|=1$, τ and τ' are in $M^1(n, Z)$. Put $\delta_1=\tau'\delta_0$ then δ_1 is in $M^1(n, Z)$ and we have $\sigma'=\delta_1\sigma\delta_2^{-1}$. Conversely, assume $\lambda'=\delta_0\lambda$ and $\sigma'=\delta_1\sigma\delta_2^{-1}$ for matrices δ_1 and δ_2 in $M^1(n, Z)$. Then by Theorem 3 $\phi_{\sigma',\lambda',\eta'}$ and $\phi_{\sigma,\delta_2^{-1}\delta_0(\lambda),\eta}$ are mutually isomorphic, and so, it is enough to show that $\{S_p(U); U \in C(\phi_{\delta,\lambda,\eta})$ is ergodic$\}=\{S_p(U'); U'\in C(\phi_{\sigma,\delta_2^{-1}\delta_0(\lambda),\eta})$ is ergodic$\}$. Let $\phi_{\tau,\alpha,\beta}$ be an Anzai transformation in $C(\phi_{\sigma,\delta_2^{-1}\delta_0(\lambda),\eta})$, then $\phi_{\tau\delta_0^{-1}\delta_2,\alpha,0}$ is an Anzai transformation in $C(\phi_{\sigma,\delta_2^{-1}\delta_0(\lambda),\eta})$ since $\sigma(\alpha)=\tau\delta_0^{-1}\delta_2\delta_2^{-1}\delta_0(\lambda)$. It is obvious that $S_p(\phi_{\tau,\alpha,\beta})=S_p(\phi_{\tau\delta_0^{-1}\delta_2,\alpha,0})=\{<k, \alpha>; k\in Z^n\}$. In the same way as above one can get for an ergodic U' in $C(\phi_{\sigma,\delta_2^{-1}\delta_0(\lambda),\eta})$ an ergodic U in $C(\phi_{\sigma,\lambda,\eta})$ such that $S_p(U')=S_p(U)$. q. e. d.

Since $M^1(1, Z)=\{1, -1\}$, by Theorem 3 and Theorem 4 we have the following Corollary.

COROLLARY 5. *The pair* $(S_p(T), \{S_p(U);$ *U in* $C(T)$ *is ergodic*$\})$ *is a complete invariant for isomorphism of the* 1-*dimensional Anzai transformations.*

4. A new equivalence relation

Ergodic measure preserving transformations T and T' of Lebesgue spaces $(\Omega, \mathscr{F}, P)$, $P(\Omega)=1$ and $(\Omega', \mathscr{F}', P')$, $P'(\Omega')=1$ respectively, are said to be *C-connected by n-steps* if the there exists a finite sequence $U_0, U_1, \cdots U_n$ of ergodic measure preserving transformations of a Lebesgue space $(\Omega_0, \mathscr{F}_0, P_0)$, $P_0(\Omega_0)=1$ such that $U_{i-1}U_i=U_iU_{i-1}$ for $i=1, 2, \cdots\cdots\cdots, n$, such that U_0 is isomorphic to T, and such that U_n is isomopphic to T'. Ergodic measure preserving transformations T and T' are said to be *C-equivalent* if they are C-connected by finite steps. C-equivalence satisfies the equivalence relation. A C-equivalence class is said to have *index n* if all transformations in the class are mutually C-connected by n steps and if there are transformations in the class which are not mutually C-connected by n-1 steps.

THEOREM 6. *If ergodic measure preserving transformations T and T' are mutually C-equivalent, then $S_p(T)$ and $S_p(T')$ are mutually isomorphic as groups.*

PROOF. It follows from Proposition 2 of [5] that if ergodic measure preserving transformations T and T' satisfy $TT'=T'T$, then $S_p(T)$ and $S_p(T')$ are mutually isomorphic as groups. From this we obtain the theorem. q. e. d.

THEOREM 7. (1) *The set of all pure point spectrum transformations whose point spectra are mutually isomorphic as a group is a C-equivalence class with index* 1.

(2) *For an irrational number λ, the set of all* 1*-dimensional Anzai transformations $\phi_{n,n(\lambda+q),\eta}$, n in Z, q in $Q=\{$rational numbers$\}$, η in R, is a C-equivalence class with index* 2.

(3) *All Bernoulli transformations with finite entropy are mutually C-equivalent.*

PROOF. (1) If an ergodic measure preserving transformation T has pure point spectrum and if the spectrum $S_p(T)$ is isomorphic to a countable subgroup Γ of the 1-dimensional torus, by a theorm of Halmos-von Neumann [4] it is isomorphic to a translation on the character group $\hat{\Gamma}$ of Γ.
Hence, (1) follows from that all translations on the character group $\hat{\Gamma}$ commute with each other.

(2) let $n_i \in Z, q_i \in Q$ and $\eta_i \in R$ for $i=1,2$. Take m, r_1 and r_2 in Z such that $m(q_1-q_2)=r_1/n_1-r_2/n_2$, and put $q=q_1-r_1/n_1m_1=q_2-r_2/n_2m_2$.
Then by (3) of Theorem 1 both $\phi_{n_1,n_1(\lambda+q_1),\eta_1}$ and $\phi_{n_2,n_2(\lambda+q_2),\eta_2}$ commute with $\phi_{m,m(\lambda+q),0}$ and hence, are mutually C-connected by 2 steps. Let $\phi_{m,\alpha,\beta}$ be an Anzai transformation in $C(\phi_{2,2\lambda,0})$. Then $m\times 2\lambda=2\alpha$ mod 1 implies $\alpha=m\lambda$ or $\alpha=m\lambda+1/2$, and hence, $S_p(\phi_{m,\alpha,\beta})=\{\exp 2\pi ikm\lambda; k\in Z\}$ or $\{\exp 2\pi ik(m\lambda+1/2); k\in Z\}$ for some integer m. Since $S_p(\phi_{1,\lambda+1/3,0})=\{\exp 2\pi ik(\lambda+1/3); k\in Z\}$, $\phi_{1,\lambda+1/3,0}$ and $\phi_{2,2\lambda,0}$ are not mutually C-connected by 1-step.

(3) follows from the fact that Bernoulli transformation with same en-

tropy are mutually isomorphic and from the existence of a Bernoulli flow [7] [8].

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