Generalized Boolean-like rings

Yakabe, Iwao
Department of Mathematics, College of General Education, Kyushu University

https://doi.org/10.15017/1449033

出版情報：九州大学教養部数学雑誌. 13 (2), pp.79-85, 1982-12. 九州大学教養部数学教室

Generalized Boolean-like rings

Iwao YAKABE

(Received September 4, 1982)

1. Introduction

In this paper we introduce the concept of generalized Boolean-like rings which is a generalization of the concept of Boolean-like rings. It is the purpose of this paper to initiate a study of generalized Boolean-like rings.

Boolean-like rings were introduced by A. L. Foster in [2]. Many properties of these rings have been studied (also see [3], [5], [6], [7] and [8]). The following properties of Boolean-like rings are well known:

(a) Each element is weakly idempotent;

(b) The nilpotent elements form an ideal;

(c) The idempotent elements form a subring;

(d) Each element can be uniquely written as the sum of an idempotent element and a nilpotent element.

Now, in Section 2, we introduce generalized Boolean-like rings and give an example of a generalized Boolean-like ring which is noncommutative.

In Section 3 and Section 4, we extend the above properties (a) and (b) to generalized Boolean-like rings.

In generalized Boolean-like rings, the properties (c) and (d) do not hold in general. We characterize generalized Boolean-like rings with the property (c) or (d) in Section 5 and Section 6, respectively.

2. Definition and example

A Boolean-like ring introduced by Foster [2] is a commutative ring with identity of characteristic 2 in which $(1-a)a(1-b)b=0$ holds for all elements a, b of the ring. Omitting the commutativity and the existence of identity in Boolean-like rings, we get the following concept:

A ring R is called a *generalized Boolean-like ring* if R is of characteristic 2 and $(a-a^2)(b-b^2)=0$ holds for all a, b of R.

Every Boolean ring is a generalized Boolean-like ring. Of course, every Boolean-like ring is a generalized Boolean-like ring. These rings are commutative. We have noncommutative one as follows:

Let B be a Boolean ring with identity, M a unitary left B-module and $S=B\oplus M$ the direct sum of B, M as additive groups. Define a multiplication in S by

$$(a, \alpha)(b, \beta)=(ab, a\beta)$$

for all a, b of B and α, β of M. Then S is a generalized Boolean-like ring, and S is commutative if and only if $M=\{0\}$.

In fact, it can be easily seen that S is a ring. Also

$$(a, \alpha)+(a, \alpha)=(a+a, \alpha+\alpha)=(0, 0),$$

for $\alpha+\alpha=(1+1)\alpha=0\alpha=0$. Further

$$\begin{aligned}&\{(a, \alpha)-(a, \alpha)^2\}\{(b, \beta)-(b, \beta)^2\}\\ &=\{(a, \alpha)-(a, a\alpha)\}\{(b, \beta)-(b, b\beta)\}\\ &=(0, \alpha-a\alpha)(0,\ \beta-b\beta)=(0, 0),\end{aligned}$$

which imply that S is a generalized Boolean-like ring.

Finally, if $M\neq\{0\}$, then there exists an element $\alpha\neq 0$ in M, and we have

$$(1, 0)(0, \alpha)=(0, \alpha)\neq(0, 0),$$

and

$$(0, \alpha)(1, 0)=(0, 0),$$

which imply that S is noncommutative.

3. Weak idempotency

We recall that each element of a Boolean-like ring is weakly idempotent. This is extended to generalized Boolean-like rings. Namely, we have

THEOREM 1. *Each element a of a generalized Boolean-like ring satisfies*

$$a^4=a^2.$$

PROOF. This follows from the expansion of $(a-a^2)^2$, for the characteristic of a generalized Boolean-like ring is 2, and $(a-a^2)^2=0$.

From this we immediately have

COROLLARY. *For each element a of a generalized Boolean-like ring, and for all nonnegative integer n*

$$a^{n+4}=a^{n+2}.$$

That is, there are at most 3 powers a, a^2, a^3 of a which are distinct.

4. Nilpotency

We recall that, in a Boolean-like ring H, the set N of all nilpotent elements of H is an ideal of H, and that the factor ring H/N is a Boolean ring.

In this section, we show that these properties can be extended to generalized Boolean-like rings. To do so we need a preliminary result.

LEMMA 1. *In a generalized Boolean-like ring, an element a is nilpotent only if $a^2=0$.*

PROOF. If a is nilpotent, then the least integer n such that $a^n=0$ must either be 1, 2 or 3 by the corollary to Theorem 1. But $n\neq 3$, for $a^3=0$ implies $a^2(=a^4)=0$ by Theorem 1, and 3 would not be least. Hence if $a\neq 0$, then $n=2$, and in any case $a^2=0$.

LEMMA 2. *Let R be a generalized Boolean-like ring and N the set of all nilpotent elements of R. Then*

$$N=\{a-a^2\mid a\in R\}.$$

PROOF. We have $(a-a^2)^2=0$ by definition of generalized Boolean-like ring, whence $a-a^2$ is nilpotent.

Conversely if b is nilpotent, then $b^2=0$ by Lemma 1. Hence $b=b-b^2$, which completes the proof.

We have the immediate corollary, which is not needed in the sequel.

COROLLARY. *A generalized Boolean-like ring is Boolean if and only if 0 is its sole nilpotent element.*

LEMMA 3. *In a generalized Boolean-like ring, if a, b are any nilpotent elements, then $ab=0$.*

PROOF. This is an immediate consequence of Lemma 2 and the definition of a generalized Boolean-like ring.

We now are able to show

THEOREM 2. *Let R be a generalized Boolean-like ring and N the set of all nilpotent elements of R. Then*

(1) *N is an ideal of R;*

(2) *R/N is a Boolean ring.*

PROOF. (1): Since R is periodic by Theorem 1, and since nilpotent elements of R commute with each other by Lemma 3, this follows from Theorem 4.3 in [1]; however, the full complexity of the proofs in [1] is not required here, so we include a more elementary proof.

For any element a, b of N, we have

$$(a-b)^2=0,$$

by Lemma 1 and Lemma 3.

For any element a of N and r of R, $e=(ar)^2$ is idempotent by Theorem 1, and therefore $re-ere$ is nilpotent. Hence we have

$$a(re-ere)=0,$$

by Lemma 3; that is, $(ar)^3=0$, so we have

$$(ar)^2=(ar)^4=0.$$

Since a and ar are nilpotent, we have $a(ra)=0$ by Lemma 3, so $(ra)^2=0$ as well.

(2): For any element r of R, $r-r^2$ is nilpotent by Lemma 2.
Hence we have

$$r^2\equiv r \ (N),$$

which implies that the factor ring R/N is Boolean.

5. Idempotency

We recall that, in a Boolean-like ring, the idempotent elements form its subring. However, in the case of generalized Boolean-like rings, this does not hold in general.

For instance, in the generalized Boolean-like ring S constructed in Section 2, if $M\neq\{0\}$, then there exists an element $\alpha\neq 0$ in M. Then $(1,\alpha)$, $(1,0)$ are idempotent, but $(1,\alpha)-(1,0)$ is not idempotent, for $(1,\alpha)-(1,0)=(0,\alpha)$ and $(0,\alpha)^2=(0,0)\neq(0,\alpha)$.

In this section, we characterize generalized Boolean-like rings in which the idempotent elements form a subring. We begin with the following lemmata.

LEMMA 4. *Let R be a generalized Boolean-like ring and J the set of all idempotent elements of R. Then*

$$J=\{a^2 \mid a\in R\}.$$

PROOF. For any element a of R, a^2 is idempotent by Theorem 1. Conversely if b is idempotent, then $b=b^2$.

LEMMA 5. *In a generalized Boolean-like ring R, each element can be written as the sum of an idempotent element and a nilpotent element.*

PROOF. For any element a of R, we have

$$a=a^2+(a-a^2),$$

which is a demanded decomposition by Lemma 2 and Lemma 4.

We now have

THEOREM 3. *Let R be a generalized Boolean-like ring, J the set of all idempotent elements of R and N the set of all nilpotent elements of R. Then the following conditions are equivalent:*

(1) *J is a subring of R;*

(2) *Each element of J commutes with each element of N;*

(3) *N is contained in the center of R;*

(4) *R is commutative.*

PROOF. (1)$\Rightarrow$(2): For any element a of J and b of N, we have

$$(a+b)^2=a+ab+ba,$$

where $(a+b)^2$ and a are elements of J. Hence $ab+ba$ is an element of J, for J is a subring of R. On the other hand, $ab+ba$ is an element of N by Theorem 2. Therefore

$$ab+ba \in J \cap N = \{0\},$$

which implies $ab=ba$, for R is of characteristic 2.

(2)$\Rightarrow$(3): For any element x of R, by Lemma 5 we can write

$$x=a+b,$$

with some a of J and b of N. Then, for any element c of N, we have

$$cx=ca+cb=ac+bc=xc.$$

(3)$\Rightarrow$(4): R is periodic, and N is contained in the center of R. Then this follows from Herstein's result in [4].

(4)$\Rightarrow$(1): This is easily seen.

6. Uniqueness of additive decomposition

We recall that, in a Boolean-like ring, each additive decomposition mentioned in Lemma 5 is unique. However, in the case of generalized Boolean-like ring, this does not hold in general.

For instance, in the generalized Boolean-like ring S constructed in Section 2, if $M \neq \{0\}$, then there exists an element $\alpha \neq 0$ in M. Then $(1, \alpha)$ can be written in two ways as follows:

$$(1, \alpha)=(1, 0)+(0, \alpha)=(1, \alpha)+(0, 0),$$

where $(1, 0), (1, \alpha)$ are idempotent, and $(0, \alpha), (0, 0)$ are nilpotent.

In this section, we characterize generalized Boolean-like rings in which each additive decomposition is unique. We begin with

LEMMA 6. *Suppose that each element of a generalized Boolean-like ring R can be uniquely written as the sum of an idempotent element and a nilpotent element.*

If a, b are idempotent elements of R and $a-b$ is a nilpotent element of R, then $a=b$.

PROOF. Put $a-b=c$, then we have

$$a=a+0=b+c,$$

where a, b are idempotent, and 0, c are nilpotent. Hence the assumption shows that $a=b$ and $c=0$.

We now are able to show

THEOREM 4. *Let R be a generalized Boolean-like ring, J the set of all idempotent elements of R and N the set of all nilpotent elements of R.*

Then each element of R can be uniquely written as the sum of an idempotent element and a nilpotent if and only if R is commutative.

PROOF. Necessity: For any element a of J and b of N, we have

$$(a+b)^2=a+ab+ba,$$

where $(a+b)^2$, a are elements of J and $ab+ba$ is an element of N. Hence Lemma 6 shows that $ab+ba=0$. Therefore we have $ab=ba$.

Since each element of J commutes with each element of N, Theorem 3 shows that R is commutative.

Sufficience: If

$$a+b=a'+b' \quad (a, a' \in J, b, b' \in N),$$

then $a+a'=b+b'$. By Theorem 3 and Lemma 1 together with Theorem 2, we have

$$(a+a')^2=a+a'=(b+b')^2=0,$$

which implies that $a=a'$, and therefore $b=b'$.

References

[1] H. E. BELL: *On quasi-periodic rings*, Arch. Math. **36** (1981), 502-509.

[2] A. L. FOSTER: *The theory of Boolean-like rings*, Trans. Amer. Math. Soc. **59** (1946), 166-187.

[3] F. HARARY: *Atomic Boolean-like rings with finite radical*, Duke Math. J. **17** (1950), 273-276.

[4] I. N. HERSTEIN: *A note on rings with central nilpotent elements*, Proc. Amer. Math. Soc. **5** (1954), 620.

[5] V. SWAMINATHAN: *On Foster's Boolean like rings*, Math. Sem. Notes Kobe Univ. 8 (1980), 347-367.

[6] ――――――: *Geometry of Boolean like rings*, *ibid.* **9** (1981), 195-214.

[7] ――――――: *A Stone's representation theorem for Boolean like rings*, *ibid.* **9** (1981), 215-218.

[8] ――――――: *Structure of Boolean like rings*, *ibid.* **9** (1981), 471-493.