Generalized Boolean-like rings

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1. Introduction

In this paper we introduce the concept of generalized Boolean-like rings which is a generalization of the concept of Boolean-like rings. It is the purpose of this paper to initiate a study of generalized Boolean-like rings.

Boolean-like rings were introduced by A.L. Foster in [2]. Many properties of these rings have been studied (also see [3], [5], [6], [7] and [8]). The following properties of Boolean-like rings are well known:

(a) Each element is weakly idempotent;
(b) The nilpotent elements form an ideal;
(c) The idempotent elements form a subring;
(d) Each element can be uniquely written as the sum of an idempotent element and a nilpotent element.

Now, in Section 2, we introduce generalized Boolean-like rings and give an example of a generalized Boolean-like ring which is noncommutative.

In Section 3 and Section 4, we extend the above properties (a) and (b) to generalized Boolean-like rings.

In generalized Boolean-like rings, the properties (c) and (d) do not hold in general. We characterize generalized Boolean-like rings with the property (c) or (d) in Section 5 and Section 6, respectively.

2. Definition and example

A Boolean-like ring introduced by Foster [2] is a commutative ring with identity of characteristic 2 in which \((1-a)a(1-b)b=0\) holds for all elements \(a, b\) of the ring. Omitting the commutativity and the existence of identity in Boolean-like rings, we get the following concept:

A ring \(R\) is called a generalized Boolean-like ring if \(R\) is of characteristic 2 and \((a-a^2)(b-b^2)=0\) holds for all \(a, b\) of \(R\).

Every Boolean ring is a generalized Boolean-like ring. Of course,
every Boolean-like ring is a generalized Boolean-like ring. These rings are commutative. We have noncommutative one as follows:

Let $B$ be a Boolean ring with identity, $M$ a unitary left $B$-module and $S=B\oplus M$ the direct sum of $B$, $M$ as additive groups. Define a multiplication in $S$ by

$$(a, \alpha)(b, \beta) = (ab, a\beta)$$

for all $a, b$ of $B$ and $\alpha, \beta$ of $M$. Then $S$ is a generalized Boolean-like ring, and $S$ is commutative if and only if $M=\{0\}$.

In fact, it can be easily seen that $S$ is a ring. Also

$$(a, \alpha) + (a, \alpha) = (a+a, \alpha+\alpha) = (0, 0),$$

for $a+a=(1+1)a=0a=0$. Further

$$\{(a, \alpha) - (a, \alpha)^2\} \{b, \beta\} = \{b, \beta\} - (a, \alpha)\{a, \alpha\},$$

$$= (0, \alpha-\alpha\alpha)(0, \beta-\beta\beta) = (0, 0),$$

which imply that $S$ is a generalized Boolean-like ring.

Finally, if $M\neq\{0\}$, then there exists an element $\alpha\neq0$ in $M$, and we have

$$(1, 0)(0, \alpha) = (0, \alpha) \neq (0, 0),$$

and

$$(0, \alpha)(1, 0) = (0, 0),$$

which imply that $S$ is noncommutative.

3. Weak idempotency

We recall that each element of a Boolean-like ring is weakly idempotent. This is extended to generalized Boolean-like rings. Namely, we have

**Theorem 1.** Each element $a$ of a generalized Boolean-like ring satisfies

$$a^4 = a^2.$$

**Proof.** This follows from the expansion of $(a-a^2)^2$, for the characteristic of a generalized Boolean-like ring is 2, and $(a-a^2)^2=0$.

From this we immediately have
COROLLARY. For each element \( a \) of a generalized Boolean-like ring, and for all nonnegative integer \( n \)
\[
a^{n+4} = a^{n+2}.
\]

That is, there are at most 3 powers \( a, a^2, a^3 \) of \( a \) which are distinct.

4. Nilpotency

We recall that, in a Boolean-like ring \( H \), the set \( N \) of all nilpotent elements of \( H \) is an ideal of \( H \), and that the factor ring \( H/N \) is a Boolean ring.

In this section, we show that these properties can be extended to generalized Boolean-like rings. To do so we need a preliminary result.

**Lemma 1.** In a generalized Boolean-like ring, an element \( a \) is nilpotent only if \( a^2 = 0 \).

**Proof.** If \( a \) is nilpotent, then the least integer \( n \) such that \( a^n = 0 \) must either be 1, 2 or 3 by the corollary to Theorem 1. But \( n \neq 3 \), for \( a^3 = 0 \) implies \( a^2 (= a^4) = 0 \) by Theorem 1, and 3 would not be least. Hence if \( a \neq 0 \), then \( n = 2 \), and in any case \( a^2 = 0 \).

**Lemma 2.** Let \( R \) be a generalized Boolean-like ring and \( N \) the set of all nilpotent elements of \( R \). Then
\[
N = \{a - a^2 | a \in R \}.
\]

**Proof.** We have \( (a - a^2)^2 = 0 \) by definition of generalized Boolean-like ring, whence \( a - a^2 \) is nilpotent.

Conversely if \( b \) is nilpotent, then \( b^2 = 0 \) by Lemma 1. Hence \( b = b - b^2 \), which completes the proof.

We have the immediate corollary, which is not needed in the sequel.

**Corollary.** A generalized Boolean-like ring is Boolean if and only if 0 is its sole nilpotent element.
LEMMA 3. In a generalized Boolean-like ring, if \( a, b \) are any nilpotent elements, then \( ab=0 \).

PROOF. This is an immediate consequence of Lemma 2 and the definition of a generalized Boolean-like ring.

We now are able to show

THEOREM 2. Let \( R \) be a generalized Boolean-like ring and \( N \) the set of all nilpotent elements of \( R \). Then

1. \( N \) is an ideal of \( R \);
2. \( R/N \) is a Boolean ring.

PROOF. (1): Since \( R \) is periodic by Theorem 1, and since nilpotent elements of \( R \) commute with each other by Lemma 3, this follows from Theorem 4.3 in [1]; however, the full complexity of the proofs in [1] is not required here, so we include a more elementary proof.

For any element \( a, b \) of \( N \), we have
\[
(a-b)^2 = 0,
\]
by Lemma 1 and Lemma 3.

For any element \( a \) of \( N \) and \( r \) of \( R \), \( e=(ar)^2 \) is idempotent by Theorem 1, and therefore \( re-e-re \) is nilpotent. Hence we have
\[
a(re-e-re)=0,
\]
by Lemma 3; that is, \( (ar)^3=0 \), so we have
\[
(ar)^2=(ar)^4=0.
\]
Since \( a \) and \( ar \) are nilpotent, we have \( a(ra)=0 \) by Lemma 3, so \( (ra)^2=0 \) as well.

(2): For any element \( r \) of \( R \), \( r-r^2 \) is nilpotent by Lemma 2.

Hence we have
\[
r^2=r \ (N),
\]
which implies that the factor ring \( R/N \) is Boolean.

5. Idempotency

We recall that, in a Boolean-like ring, the idempotent elements form its subring. However, in the case of generalized Boolean-like rings, this
does not hold in general.

For instance, in the generalized Boolean-like ring \( S \) constructed in Section 2, if \( M \neq \{0\} \), then there exists an element \( \alpha \neq 0 \) in \( M \). Then \((1, \alpha), (1, 0)\) are idempotent, but \((1, \alpha) - (1, 0)\) is not idempotent, for \((1, \alpha) - (1, 0) = (0, \alpha)\) and \((0, \alpha)^2 = (0, 0) \neq (0, \alpha)\).

In this section, we characterize generalized Boolean-like rings in which the idempotent elements form a subring. We begin with the following lemmata.

**Lemma 4.** Let \( R \) be a generalized Boolean-like ring and \( J \) the set of all idempotent elements of \( R \). Then
\[
J = \{a^2 \mid a \in R\}.
\]

**Proof.** For any element \( a \) of \( R \), \( a^2 \) is idempotent by Theorem 1. Conversely if \( b \) is idempotent, then \( b = b^2 \).

**Lemma 5.** In a generalized Boolean-like ring \( R \), each element can be written as the sum of an idempotent element and a nilpotent element.

**Proof.** For any element \( a \) of \( R \), we have
\[
a = a^2 + (a - a^2),
\]
which is a demanded decomposition by Lemma 2 and Lemma 4.

We now have

**Theorem 3.** Let \( R \) be a generalized Boolean-like ring, \( J \) the set of all idempotent elements of \( R \) and \( N \) the set of all nilpotent elements of \( R \). Then the following conditions are equivalent:

1. \( J \) is a subring of \( R \);
2. Each element of \( J \) commutes with each element of \( N \);
3. \( N \) is contained in the center of \( R \);
4. \( R \) is commutative.

**Proof.** \((1) \Rightarrow (2)\): For any element \( a \) of \( J \) and \( b \) of \( N \), we have
\[
(a + b)^2 = a + ab + ba,
\]
where \((a+b)^2\) and \(a\) are elements of \(J\). Hence \(ab+ba\) is an element of \(J\), for \(J\) is a subring of \(R\). On the other hand, \(ab+ba\) is an element of \(N\) by Theorem 2. Therefore
\[ab+ba \in J \cap N = \{0\},\]
which implies \(ab=ba\), for \(R\) is of characteristic 2.

(2) \(\Rightarrow\) (3): For any element \(x\) of \(R\), by Lemma 5 we can write
\[x = a + b,\]
with some \(a\) of \(J\) and \(b\) of \(N\). Then, for any element \(c\) of \(N\), we have
\[cx = ca + cb = ac + bc = xc.\]

(3) \(\Rightarrow\) (4): \(R\) is periodic, and \(N\) is contained in the center of \(R\). Then this follows from Herstein's result in [4].

(4) \(\Rightarrow\) (1): This is easily seen.

6. Uniqueness of additive decomposition

We recall that, in a Boolean-like ring, each additive decomposition mentioned in Lemma 5 is unique. However, in the case of generalized Boolean-like ring, this does not hold in general.

For instance, in the generalized Boolean-like ring \(S\) constructed in Section 2, if \(M \neq \{0\}\), then there exists an element \(a \neq 0\) in \(M\). Then \((1, a)\) can be written in two ways as follows:
\[(1, a) = (1, 0) + (0, a) = (1, a) + (0, 0),\]
where \((1, 0), (1, a)\) are idempotent, and \((0, a), (0, 0)\) are nilpotent.

In this section, we characterize generalized Boolean-like rings in which each additive decomposition is unique. We begin with

**Lemma 6.** Suppose that each element of a generalized Boolean-like ring \(R\) can be uniquely written as the sum of an idempotent element and a nilpotent element.

If \(a, b\) are idempotent elements of \(R\) and \(a-b\) is a nilpotent element of \(R\), then \(a=b\).

**Proof.** Put \(a-b=c\), then we have
\[a = a + 0 = b + c,\]
where \(a, b\) are idempotent, and \(0, c\) are nilpotent. Hence the assumption shows that \(a=b\) and \(c=0\).
We now are able to show

**Theorem 4.** Let $R$ be a generalized Boolean-like ring, $J$ the set of all idempotent elements of $R$ and $N$ the set of all nilpotent elements of $R$.

Then each element of $R$ can be uniquely written as the sum of an idempotent element and a nilpotent if and only if $R$ is commutative.

**Proof.** Necessity: For any element $a$ of $J$ and $b$ of $N$, we have 

$$(a+b)^2 = a + ab + ba,$$

where $(a+b)^2$, $a$ are elements of $J$ and $ab+ba$ is an element of $N$. Hence Lemma 6 shows that $ab+ba=0$. Therefore we have $ab=ba$.

Since each element of $J$ commutes with each element of $N$, Theorem 3 shows that $R$ is commutative.

Sufficience: If 

$$a+b=a'+b' \ (a, a' \in J, b, b' \in N),$$

then $a+a'=b+b'$. By Theorem 3 and Lemma 1 together with Theorem 2, we have 

$$(a+a')^2 = a + a' = (b+b')^2 = 0,$$

which implies that $a=a'$, and therefore $b=b'$.

**References**