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On the partial sums of the Fourier coefficients of cuspforms

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1. Introduction

Let $f(z)=\sum_{n=1}^{\infty}a_n e^{2\pi inz}$ be a cuspform of weight k with respect to the congruence subgroup $\Gamma_1(N)$. Then it is well known that the following Dirichlet series associated with $f(z)$

$$L(s, f)=\sum_{n=1}^{\infty}a_n n^{-s}$$

absolutely converges for $\mathrm{Re}(s)>\frac{k+1}{2}$ and is holomorphically continued to the whole s plane.

In this note we shall show that the above Dirichlet series still converges in the region $\mathrm{Re}(s)>\frac{k}{2}$. Therefore the abscissa of convergence of $L(s,f)$ is less than or equal to $\frac{k}{2}$. Let s_n be the partial sum of the sequence of the Fourier coefficients $\{a_n\}$, *i. e.*, $s_n=a_1+a_2+\cdots\cdots\cdots+a_n$. Then our result is the following

$$s_n=O(n^{k/2}\log n), \text{ for } n\to\infty.$$

2. Preliminaries

The congruence subgroup $\Gamma_1(N)$ is defined by

$$\Gamma_1(N)=\left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in SL_2(Z)\,|\,a\equiv d\equiv 1,\ c\equiv 0 \mod N\right\}.$$

Let $S_k(\Gamma_1(N))$ be the vector space of cuspforms of weight k with respect to $\Gamma_1(N)$. The following two lemmas are well known.

LEMMA 1. *If $f(x+iy)\in S_k(\Gamma_1(N))$ with real numbers x and $y>0$,* *then we have* $|f(x+iy)|<K_1 y^{-k/2}$ *with a constant* K_1 *independent of* x.

LEMMA 2. *For an element* $f(z)$ *of* $S_k(\Gamma_1(N))$, $f(z)=\sum_{n=1}^{\infty} a_n e^{2\pi i i n z}$ *be the Fourier expansion of* $f(z)$. *Then we have* $|a_n|<K_2 n^{k/2}$ *with a constant* K_2. *Hence we have* $|a_1+a_2+\cdots\cdots\cdots+a_n|<K_3 n^{k/2+1}$ *with an another constant* K_3.

For the proof of the above lemmas, see shimura [**1**].

LEMMA 3. *If* $f(z)=\sum_{n=1}^{\infty} a_n e^{2\pi i n z} \in S_k(\Gamma_1(N))$, *then we have*

$$f(z)=(1-e^{2\pi i z})\sum_{n=1}^{\infty} s_n e^{2\pi i n z}, \quad \textit{for } \mathrm{Im}(z)>0,$$

where $s_n=a_1+a_2+\cdots\cdots\cdots+a_n$.

PROOF. If we put $g(z)=\sum_{n=1}^{\infty} s_n e^{2\pi i n z}$, then by Lemma 2 we can easily see that $g(z)$ uniformly converges in any compact set of the complex upper half plane $\mathrm{Im}(z)>0$. We set $s_0=0$. Then we get

$$\begin{aligned} f(z) &= \sum_{n=1}^{M} a_n e^{2\pi i n z} + \sum_{n=M+1}^{\infty} a_n e^{2\pi i n z} \\ &= \sum_{n=1}^{M} (s_n - s_{n-1}) e^{2\pi i n z} + \sum_{n=M+1}^{\infty} a_n e^{2\pi i n z} \\ &= \sum_{m=1}^{M} s_n (e^{2\pi i n z} - e^{2\pi i (n+1) z}) + s_M e^{2\pi (M+1) z} + \sum_{n=M+1}^{\infty} a_n e^{2\pi i n z} \\ &= (1-e^{2\pi i z}) \sum_{n=1}^{M} s_n e^{2\pi i n z} + s_M e^{2\pi i (M+1) z} + \sum_{n=M+1}^{\infty} a_n e^{2\pi i n z}. \end{aligned}$$

By Lemma 2 it is easily seen that for any $\varepsilon>0$, we can choose a number M so that

$$|s_M e^{2\pi i (M+1) z}| \varepsilon/3,$$

$$\Big|\sum_{n=M+1}^{\infty} a_n e^{2\pi i n z}\Big|<\varepsilon/3,$$

$$\Big|(1-e^{2\pi i z})\sum_{n=M+1}^{\infty} s_n e^{2\pi i n z}\Big|<\varepsilon/3.$$

Then we get

$$|f(z)-(1-e^{2\pi iz})g(z)|<|s_M e^{2\pi i(M+1)z}|+|\sum_{n=M+1}^{\infty} a_n e^{2\pi inz}|$$

$$+(1-e^{2\pi iz})\sum_{n=M+1}^{\infty} s_n e^{2\pi inz}|$$

$$<\varepsilon/3+\varepsilon/3+\varepsilon/3=\varepsilon,$$

which completes the proof of Lemma.

LEMMA 4. *For a positive number $y<1/4$, we have*

$$I(y)=\int_0^1 \frac{dx}{|1-e^{-y+2\pi ix}|}<K_4 \log\frac{1}{y}$$

with a constant K_4.

PROOF. Dividing the interval $[0, 1]$ into three parts we have

$$I(y)=\int_0^{1/4}\frac{dx}{|1-e^{-y+2\pi ix}|}+\int_{1/4}^{3/4}\frac{dx}{1-e^{-y+2\pi ix}|}$$

$$+\int_{3/4}^{1}\frac{dx}{|1-e^{-y+2\pi ix}|}$$

$$=2\int_0^{1/4}\frac{dx}{|1-e^{-y+2\pi ix}|}+\int_{1/4}^{3/4}\frac{dx}{|1-e^{-y+2\pi ix}|}.$$

Now we first estimate the latter term. In this case we have

$$\int_{1/4}^{3/4}\frac{dx}{|1-e^{-y+2\pi ix}|}<\int_{1/4}^{3/4}\frac{dx}{|1-e^{-y}\cos 2\pi x|}<\int_{1/4}^{3/4}dx=\frac{1}{2}.$$

Next we estimate the former term. In this case we moreover divide the interval $[0, \frac{1}{4}]$ into two parts. Then we have

$$\int_0^{1/4}\frac{dx}{|1-e^{-y+2\pi ix}|}=\int_0^{y}\frac{dx}{|1-e^{-y+2\pi ix}|}+\int_{y}^{1/4}\frac{dx}{|1-e^{-y+2\pi ix}|}$$

$$<\int_0^{y}\frac{dx}{1-e^{-y}\cos 2\pi x}+\int_{y}^{1/4}\frac{dx}{e^{-y}\sin 2\pi x}$$

$$<\int_0^{y}\frac{dx}{1-e^{-y}}+e^{y}\Big[\frac{1}{2\pi}\log\tan\pi x\Big]_{y}^{1/4}$$

$$=\frac{y}{1-e^{-y}}+\frac{e^{y}}{2\pi}\log\frac{1}{\tan\pi y}$$

$$<\frac{y}{1-e^{-y}}+\frac{e^{y}}{2\pi}\log\frac{1}{\pi y}.$$

If we note the condition on y, then we can easily see that $\frac{y}{1-e^{-y}}$ and $\frac{e^{y}}{2\pi}$ are bounded. Hence we have

$$\int_0^{1/4} \frac{dx}{|1-e^{-y+2\pi ix}|} < K_4' \log\frac{1}{y}$$

with a suitable constant K_4'. Summing up the above results we have

$$I(y) < \frac{1}{2} + 2K_4' \log\frac{1}{y} < K_4 \log\frac{1}{y}$$

with a constant K_4. This completes the proof of Lemma.

3. Estimation of s_n

We are now in a position to state our result

THEOREM. *If* $f(z)=\sum_{n=1}^{\infty} a_n e^{2\pi inz} \in S_k(\Gamma_1(N))$, *then we have*

$$|a_1+a_2+\cdots\cdots\cdots+a_n| < K_0 n^{k/2} \log n$$

with a suitable constant K_0.

PROOF. By Lemma 3 we get

$$\begin{aligned} a_1+a_2+\cdots\cdots\cdots+a_n = s_n &= \int_{z_0}^{z_0+1} g(z)e^{-2\pi inz}dz \\ &= \int_{z_0}^{z_0+1} \frac{f(z)e^{-2\pi inz}}{1-e^{2\pi iz}}dz \end{aligned}$$

for any z_0 such that $\mathrm{Im}(z)>0$. If we put $z_0=\frac{yi}{2\pi}$ and $z=x+\frac{yi}{2\pi}$, then we have

$$s_n = \int_0^1 \frac{f(x+\frac{yi}{2\pi})e^{ny-2\pi inx}}{1-e^{-y+2\pi ix}}dx.$$

By Lemma 1 and Lemma 4 we have

$$\begin{aligned} |s_n| &\leqq \int_0^1 \frac{|f(x+\frac{yi}{2\pi})|e^{ny}}{|1-e^{-y+2\pi ix}|}dx \\ &< K_1 (\frac{y}{2\pi})^{k/2} e^{ny} \int_0^1 \frac{dx}{|1-e^{-y+2\pi ix}|} \\ &< K_1K_4 (\frac{y}{2\pi})^{k/2} e^{ny} \log\frac{1}{y}. \end{aligned}$$

Here if we put $y=1/n$, then we have

$$|s_n|<K_1K_2(2\pi)^{-k/2}en^{k/2}\log n.$$

Thus the proof of Theorem is completed.

By the method of Abel transformation we can easily show the following corollary.

COROLLARY. *If* $f(z)=\sum_{n=1}^{\infty}a_n e^{2\pi inz}\in S_k(\Gamma_1(N))$, *then the following Dirichlet series* $L(s,f)$ *associated with* $f(z)$

$$L(s,f)=\sum_{n=1}^{\infty}a_n n^{-s}$$

converges in the region $\mathrm{Re}(s)>k/2$.

Reference

[1] G. SHIMURA: Introduction to the arithmetic theory of automorphic functions, Publ. Math. Soc. of Japan No. **11**, Iwanami Shoten and Princeton Univ. Press, 1971.