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Pseudovaluations of near-rings

Dedicated to Professor Koichi Yamamoto
on his 60th birthday

Iwao YAKABE

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1. Introduction

The theory of pseudovaluations of commutative rings with identity has been developed by Mahler [4], [5], [6] and Cohn and Mahler [2]. Recently Rhodes [9] has considered pseudovaluations of commutative rings without identity. In this paper we are concerned with pseudovaluations of near-rings.

It is a great difference between rings and near-rings that the zero-symmetric parts and the constant parts of near-rings are in general nonzero. So we are interested in how pseudovaluations of a near-ring are related to those of its zero-symmetric part and its constant part.

The necessary definitions and notations concerning near-rings are given in Section 2. Pseudovaluations of near-rings are introduced in Section 3.

In Section 4 we consider extensions of pseudovaluations of the zero-symmetric part or the constant part to pseudovaluations of the original near-ring. We characterize pseudovaluations of the zero-symmetric part which can be uniquely extended to pseudovaluations of the original near-ring. Applying this result, in Section 5 we obtain a class of near-rings whose all pseudovaluations are determined by those of their zero-symmetric parts.

The final section is devoted to extensions of pseudovaluations under admissible multiplications. It is shown that, given a zero-symmetric near-ring and a constant near-ring, there always exists an admissible multiplication under which all pseudovaluations of their direct sum are determined by those of the zero-symmetric near-ring.

2. Preliminaries on near-rings

A *near-ring* is a set N together with two binary operations "$+$" and "$\cdot$" such that

(1) $(N, +)$ is a group (not necessarily abelian),

(2) $(N, \cdot)$ is a semigroup,

(3) $(n+n')\cdot n''=n\cdot n''+n'\cdot n''$ for all $n, n', n''\in N$.

We often abbreviate $(N, +, \cdot)$ by N. Multiplication will in most cases be indicated by juxtaposition; so we write nn' instead of $n\cdot n'$. In dealing with general near-rings the neutral element of $(N, +)$ will be denoted by 0.

In this section, N will denote a near-ring. Then $0n=0$ and $(-n)n'=-nn'$ hold for all $n, n'\in N$. But $n0=0$ and $n(-n')=-nn'$ do not hold in general. So one defines for N; the set

$$N_0=\{n\in N \mid n0=0\}$$

is called the *zero-symmetric part* of N;

$$\begin{aligned} N_c &= \{n\in N \mid n0=n\} \\ &= \{n\in N \mid nn'=n \quad \text{for all } n'\in N\} \end{aligned}$$

is called the *constant part* of N. N_0 is a normal subgroup of $(N, +)$; N_c is a subgroup of $(N, +)$, but not always normal. N is called *zero-symmetric* if $N=N_0$; N is called *constant* if $N=N_c$.

A subgroup M of $(N, +)$ with $MM\subset M$ is called a *subnear-ring* of N. A subnear-ring M of N is called *invariant* if $MN\subset M$ and $NM\subset M$.

A normal subgroup I of $(N, +)$ is called an *ideal* of N if

(a) $IN\subset I$,

(b) $n(i+n')-nn'\in I$ for all $n, n'\in N$ and $i\in I$.

A normal subgroup of $(N, +)$ with (a) is called a *right ideal* of N, while a normal subgroup of $(N, +)$ with (b) is said to be a *left ideal*.

N_0 is a subnear-ring of N and a left ideal of N, but not generally an ideal. N_c is an invariant subnear-ring of N, but in general neither a right nor a left ideal.

An element $d\in N$ is called *distributive* if $d(n+n')=dn+dn'$ for all $n, n'\in N$. The set of all distributive elements of N will be denoted by N_d. $(N_d, \cdot)$ is a subsemigroup of $(N, \cdot)$. N is said to be *distributively generated* if there is a subsemigroup D of $(N_d, \cdot)$ generating $(N, +)$.

The terminology used is essentially that found in Pilz [8].

3. Pseudovaluations and basic properties

Let N be a near-ring and P the set of all nonnegative real numbers. A map $v:N\to P$ is called a *pseudovaluation* of N provided that

(1) $v(0)=0$,

(2) $v(-n)=v(n)$,

(3) $v(n+m)\leq v(n)+v(m)$,

(4) $v(nm)\leq v(n)v(m)$.

The following inequality is an immediate consequence of the definition of a pseudovaluation v: for all $n, m\in N$

$$|v(n)-v(m)|\leq v(n+m)\leq v(n)+v(m).$$

Considering this inequalty, we have

PROPOSITION 1. *Let v be a pseudovaluation of a near-ring N. Then*

(1) $v(c)=0$ *for all* $c\in N_c$.

(2) $v(n+c)=v(c+n)=v(n)$ *for all* $n\in N$ *and* $c\in N_c$.

(3) $v(c+n+c')=v(n)$ *for all* $n\in N$ and $c, c'\in N_c$.

(4) $v(n(n'+c)-nc)\leq v(n)v(n')$ *for all* $n, n'\in N$ and $c\in N_c$.

PROOF. (1) follows from that

$$0\leq v(c)=v(c0)\leq v(c)v(0)=0.$$

To prove (2), by the above mentioned inequalty we have

$$|v(n)-v(c)|\leq v(n+c)\leq v(n)+v(c),$$
$$|v(c)-v(n)|\leq v(c+n)\leq v(c)+v(n)$$

and $v(c)=0$. Hence

$$v(n)=|v(n)|\leq v(n+c)\leq v(n)$$

and

$$v(n)=|v(n)|\leq v(c+n)\leq v(n).$$

(3) follows from (2). Finally, to prove (4), we first remark that $-nc\in N_c$, since N_c is an invariant subnear-ring of N. Hence

$$\begin{aligned} v(n(n'+c)-nc)&=v(n(n'+c))\\ &\leq v(n)v(n'+c)=v(n)v(n'). \end{aligned}$$

As example of a pseudovaluation of a near-ring N we have the map

$$v_0(n)=0 \qquad \text{for all } n\in N.$$

This is called the *improper pseudovaluation.*

By Proposition 1 (1), we immediately have

PROPOSITION 2. *The improper pseudovaluation is the only pseudovaluation of a constant near-ring.*

As another example of a pseudovaluation of a nonconstant near-ring N, we have the map

$$v_c(n)=\begin{cases}0 & \text{if } n\in N_c\\ 1 & \text{if } n\notin N_c.\end{cases}$$

This is called the *trivial pseudovaluation.*

Now, let v be a pseudovaluation of a near-ring N. Then, the kernel of v, namely

$$\text{Ker } v=\{n\in N\,|\,v(n)=0\}$$

is an invariant subnear-ring of N.

In fact, if $m, m'\in\text{Ker } v$, then

$$0\leq v(m-m')\leq v(m)+v(m')=0$$

and

$$0\leq v(mm')\leq v(m)v(m')=0.$$

Hence $m-m'$, $mm'\in\text{Ker } v$. Therefore Ker v is a subnear-ring of N.

If $m\in\text{Ker } v$ and $n\in N$, then

$$0\leq v(mn)\leq v(m)v(n)=0$$

and

$$0\leq v(nm)\leq v(n)v(m)=0.$$

Hence $mn, nm\in\text{Ker } v$. Therefore Ker v is an invariant subnear-ring of N.

Conversely, let M be an invariant subnear-ring of a near-ring N. Then the map

$$v_M(n)=\begin{cases}0 & \text{if } n\in M\\ 1 & \text{if } n\notin M\end{cases}$$

is a pseudovaluation of N, and $M=\text{Ker } v_M$.

In fact: (1) $v_M(0)=0$ follows from that $0\in M$.

(2) Let $n\in N$. If $n\in M$, then $-n\in M$. Hence

$$v_M(-n)=0=v_M(n).$$

If $n\notin M$, then $-n\notin M$. Hence

$$v_M(-n)=1=v_M(n).$$

(3) Let $n, m\in N$. If $n\notin M$ or $m\notin M$, then $v_M(n)=1$ or $v_M(m)=1$. Hence

$$v_M(n+m)\leq 1\leq v_M(n)+v_M(m).$$

If $n\in M$ and $m\in M$, then $n+m\in M$. Hence

$$v_M(n+m)=0=v_M(n)+v_M(m).$$

(4) Let $n, m\in N$. If $nm\in M$, then

$$v_M(nm)=0\leq v_M(n)v_M(m).$$

If $nm\notin M$, then $n\notin M$ and $m\notin M$. Hence

$$v_M(nm)=1=v_M(n)v_M(m).$$

Finally, it is clear that $M=\mathrm{Ker}\ v_M$.

To sum up, we have

PROPOSITION 3. *The invariant subnear-rings of a near-ring N are exactly the kernels of the pseudovaluations of N.*

4. Extensions of pseudovaluations

In this section, to avoid trivial cases, N will be a near-ring whose zero-symmetric part N_0 and constant part N_c are nonzero.

Let N' be a subnear-ring of N. A pseudovaluation v of N is called an *extension* of a pseudovaluation v' of N' if $v(n')=v'(n')$ for all $n'\in N'$.

We are going to investigate whether pseudovaluations of N_0 and N_c can be extended to those of N.

In the case of N_c, we immediately have

THEOREM 1. *A pseudovaluation of N_c has an extension to a pseudovaluation of N.*

PROOF. Take any invariant subnear-ring M of N, and construct the pseudovaluation v_M of N as stated in Section 3. Since M contains N_c, we have $v_M(c)=0$ for all $c\in N_c$. On the other hand, the only pseudovaluation of N_c is the improper pseudovaluation by Proposion 2. Hence v_M is an extension of the only pseudovaluation of N_c. Of course, each pseudovaluation of N is an extension of the only pseudovaluation of N_c.

We now turn to the case of N_0. In this case the following result is useful for our consideration: Berman and Silverman [1] showed that, for any near-ring N, $(N, +)$ is a semidirect sum of $(N_0, +)$ by $(N_c, +)$. Also, if $n=n_0+n_c$, where $n_0\in N_0$ and $n_c\in N_c$, then this representation is unique up to the order in which the terms appear in the sum and $n_0=n-n0$ and $n_c=n0$.

Let w be a pseudovaluation of N_0. If w has an extension v to a pseudovaluation of N, then by Proposition 1 (2)

$$v(n)=w(n_0) \qquad \text{for all } n=n_0+n_c\in N,$$

where $n_0\in N_0$ and $n_c\in N_c$.

Therefore we have proved

THEOREM 2. *If a pseudovaluation of N_0 has an extension to a pseudovaluation of N, then the extension is uniquely determined.*

However, pseudovaluations of N_0 can not be extended to pseudovaluations of N in general, as the following example shows:

EXAMPLE. Let R be the field of real numbers and $R[x]$ the ring of polynomials in x over R. Then $N=(R[x], +, \circ)$ is a near-ring, where $+$ means usual addition and $\circ$ usual substitution. Also

$$N_0=\{\sum_{i=1}^{n} r_i x^i\} \quad \text{and} \quad N_c=R.$$

Take the map $w: N_0\to P$ defined by

$$w(\sum_{i=1}^{n} r_i x^i)=|r_1|,$$

then w is a pseudovaluation of N_0. But the map $v: N\to P$ extended in the above-mentioned manner, namely, defined by

$$v(r_0+\sum_{i=1}^{n} r_i x^i)=|r_1|$$

is not a pseudovaluation of N.

In fact, for instance, we have $v(x^2)=0$, $v(x+1)=1$ and $v(x^2\circ(x+1))=v(x^2+2x+1)=2$. Hence the inequalty

$$v(x^2\circ(x+1))\leq v(x^2)x(x+1)$$

does not hold.

We will characterize pseudovaluations of N_0 which can be extended to pseudovaluations of N. We begin with

LEMMA 1. *For any $n\in N$, we denote by n_0 the zero-symmetric part of n and n_c the constant part of n. Namely, $n=n_0+n_c$, where $n_0\in N_0$ and $n_c\in N_c$. Then*

(1) $(-n)_0=-n_c-n_0+n_c$ *for all* $n\in N$.

(2) $(n+m)_0=n_0+(n_c+m_0-n_c)$ *for all* $n, m\in N$.

(3) $(nm)_0=n_0(m_0+m_c)-n_0m_c$ *for all* $n, m\in N$.

PROOF. (1) Since $(-n)0=-n0=-n_c$,

$$(-n)_0=(-n)-(-n)0=-n_c-n_0+n_c.$$

(2) Since $(n+m)0=n0+m0=n_c+m_c$,

$$\begin{aligned}(n+m)_0&=(n+m)-(n_c+m_c)\\&=(n_0+n_c+m_0+m_c)-m_c-n_c\\&=n_0+(n_c+m_0-n_c).\end{aligned}$$

(3) Since $(nm)0=(n_0+n_c)m_c=n_0m_c+n_c$,

$$\begin{aligned}(nm)_0&=(n_0+n_c)(m_0+m_c)-(n_0m_c+n_c)\\&=n_0(m_0+m_c)+n_c-n_c-n_0m_c\\&=n_0(m_0+m_c)-n_0m_c.\end{aligned}$$

We are now able to characterize pseudovaluations of N_0 which can be extended to pseudovaluations of N.

THEOREM 3. *A pseudovaluation w of N_0 can be extended to a pseudovaluation of N if and only if the following two conditions hold*:

(1) $w(-c+n+c)=w(n)$ *for all* $n\in N_0$ *and* $c\in N_c$.

(2) $w(n(m+c)-nc)\leq w(n)w(m)$ *for all* $n, m\in N_0$ *and* $c\in N_c$.

PROOF. We first remark that $-c+n+c\in N_0$ in (1) and $n(m+c)-nc\in N_0$ in (2), because N_0 is a left ideal of N.

The necessity is an immediate consequence of Proposition 1.

To prove the sufficiency, suppose that w satisfies two conditions (1) and (2). Then the map v defined by

$$v(n)=w(n_0) \qquad \text{for all } n=n_0+n_c\in N,$$

where $n_0\in N_0$ and $n_c\in N_c$, is an extension of w.

In fact: (a) Since $0\in N_0$, $v(0)=w(0)=0$.

(b) For all $n=n_0+n_c\in N$ where $n_0\in N_0$ and $n_c\in N_c$, we have by Lemma 1

$$\begin{aligned} v(-n)&=w(-n_c-n_0+n_c)\\ &=w(-n_0)=w(n_0)=v(n). \end{aligned}$$

(c) For all $n=n_0+n_c$, $m=m_0+m_c\in N$ where n_0, $m_0\in N_0$ and n_c, $m_c\in N_c$, similarly we have

$$\begin{aligned} v(n+m)&=w(n_0+(n_c+m_0-n_c))\\ &\leq w(n_0)+w(n_c+m_0-n_c)\\ &=w(n_0)+w(m_0)=v(n)+v(m), \end{aligned}$$

and

$$\begin{aligned} v(nm)&=w(n_0(m_0+m_c)-n_0m_c)\\ &\leq w(n_0)w(m_0)=v(n)v(m). \end{aligned}$$

5. Extensions in the case of direct sums

In this section, N also will denote a near-ring whose zero-symmetric part N_0 and constant part N_c are nonzero.

Applying Theorem 3, we will get near-rings whose all pseudovaluations are determined by those of their zero-symmetric parts. We start with

THEOREM 4. *Let $(N, +)$ be a direct sum of $(N_0, +)$ and $(N_c, +)$. Then a pseudovaluation w of N_0 can be extended to a pseudovaluation of N if and only if*

$$w(n(m+c)-nc)\leq w(n)w(m)$$

holds for all $n, m\in N_0$ and $c\in N_c$.

PROOF. Since $(N, +)$ is a direct sum of $(N_0, +)$ and $(N_c, +)$, N_0 and N_c commute elementwise with respect to addition. Hence

$$w(-c+n+c)=w(n-c+c)=w(n)$$

for all $n\in N_0$ and $c\in N_c$. Therefore, this theorem is an immediate consequence of Theorem 3.

To show that, if N_0 is distributively generated, then all pseudovaluations of N are determined by those of N_0, we need

LEMMA 2. *Let $(N, +)$ be a direct sum of $(N_0, +)$ and $(N_c, +)$, and D a subsemigroup of $(N_d, \cdot)$, where N_d is the set of all distributive elements of N. Let N_0 be distributively generated by D. Then*

(1) $(-d)(n+c)=(-d)n+(-d)c$ *for all $d \in D$, $n \in N_0$ and $c \in N_c$.*

(2) $n(n'+c)=nn'+nc$ *for all $n, n' \in N_0$ and $c \in N_c$.*

PROOF. (1) We have

$$(-d)(n+c)=-d(n+c)=-(dn+dc)=-dc-dn,$$

where $-dc \in N_c$ and $-dn \in N_0$, because $D \subset N_0$. Hence

$$(-d)(n+c)=-dn-dc=(-d)n+(-d)c.$$

To prove (2), let $S=D \cup \{-d \mid d \in D\}$. Then each $n \in N_0$ is a finite (ordered) sum $n=\sum_{i=1}^{k} d_i$ with $d_i \in S$. Therefore (2) follows from (1) and from that N_0 and N_c commute elementwise with respect to addition.

In detail, first of all, (2) is true for $k=1$ by (1). For $k \geq 2$ we assume the truth of (2) for $k-1$ instead of k. Then we have

$$\begin{aligned}(\sum_{i=1}^{k} d_i)(n'+c) &= (\sum_{i=1}^{k-1} d_i + d_k)(n'+c) \\ &= (\sum_{i=1}^{k-1} d_k)(n'+c)+d_k(n'+c) \\ &= (\sum_{i=1}^{k-1} d_i)n' + (\sum_{i=1}^{k-1} d_i)c + d_k n' + d_k c,\end{aligned}$$

where $(\sum_{i=1}^{k-1} d_i)c \in N_c$ and $d_k n' \in N_0$. Hence

$$\begin{aligned}(\sum_{i=1}^{k} d_i)(n'+c) &= (\sum_{i=1}^{k-1} d_i)n' + d_k n' + (\sum_{i=1}^{k-1} d_i)c + d_k c \\ &= (\sum_{i=1}^{k} d_i)n' + (\sum_{i=1}^{k} d_i)c,\end{aligned}$$

by which (2) has been proved.

Now we have

THEOREM 5. *Let $(N, +)$ be a direct sum of $(N_0, +)$ and $(N_c, +)$, and D a subsemigroup of $(N_d, \cdot)$. Let N_0 be distributively generated by D. Then each pseudovaluation of N_0 can be uniquely extended to a pseudovaluation of N. All pseudovaluations of N are obtained from those of*

N_0, *and vice versa.*

PROOF. Let w be a pseudovaluation of N_0. By Lemma 2, we have

$$n(m+c)-nc=nm+nc-nc=nm,$$

for $n, m \in N_0$ and $c \in N_c$. Hence

$$w(n(m+c)-nc) \leq w(n)w(m).$$

Therefore, w can be extended to a pseudovaluation of N by Theorem 4, and the extension is uniquely determined by Theorem 2.

Furthermore, for each pseudovaluation v of N, let w be the restriction of v to N_0. Then w is a pseudovaluation of N_0, and v is the unique extension of w by Theorem 2. Therefore, all pseudovaluations of N are obtained from those of N_0, and the converse holds.

N is called an *abstract affine near-ring* provided that

(1) $(N, +)$ is abelian,

(2) $N_0 = N_d$.

Hence an abstract affine near-ring N is a direct sum of $(N_0, +)$ and $(N_c, +)$, and N_0 is distributively generated by N_d. Consequently, we have

COROLLARY. *Let N be an abstract affine near-ring. Then each pseudovaluation of N_0 can be uniquely extended to a pseudovaluation of N. All pseudovaluations of N are obtained from those of N_0, and vice versa.*

Abstract affine near-rings are members of the class of more general near-rings, namely C-Z-decomposable near-rings; N is *C-Z-decomposable* if N_c is an ideal of N (see [3]). In this case, we also have

THEOREM 6. *Let N be a C-Z-decomposable near-ring. Then each pseudovaluation of N_0 can be uniquely extended to a pseudovaluation of N. All pseudovaluations of N are obtained from those of N_0, and vice versa.*

PROOF. $(N, +)$ is a direct sum of $(N_0, +)$ and $(N_c, +)$, since N_c is a normal subgroup of $(N, +)$.

We next remark that we have

$$n(m+c)-nc=nm,$$

for all $n, m \in N_0$ and $c \in N_c$.

In fact, put

$$z=n(m+c)-nc-nm,$$

then $z\in N_0$, since $n(m+c)-nc\in N_0$ and $nm\in N_0$.

On the other hand, since $-nc\in N_c$ and $-nm\in N_0$ commute,

$$z=n(m+c)-nm-nc,$$

where $n(m+c)-nm\in N_c$ and $nc\in N_c$, because N_c is an ideal of N. Hence $z\in N_c$.

$z\in N_0\cap N_c$ means $z=0$. Therefore we have

$$n(m+c)-nc=nm.$$

Now, let w be a pseudovaluation of N_0. Then

$$w(n(m+c)-nc)=w(nm)\leq w(n)w(m),$$

for all $n, m\in N_0$ and $c\in N_c$ by the above remark.

Consequently, this theorem follows from Theorem 4 and Theorem 2.

6. Extensions under admissible multiplications

In the preceding section, it was shown that there is a near-ring such that it is a direct sum of its zero-symmetric part and its constant part, and each pseudovaluation of its zero-symmetric part can be extended to its pseudovaluation.

This section is devoted to the converse problem: given a zero-symmetric near-ring N_1 and a constant near-ring N_2, is it possible to define a multiplication $\circ$ on the direct sum $(N, +)$ of the groups $(N_1, +)$ and $(N_2, +)$ such that N_1 and N_2 turn out to be the zero-symmetric part and the constant part of a near-ring $(N, +, \circ)$, respectively, and each pseudovaluation of N_1 can be extended to a pseudovaluation of N ?

The first half of this problem has been discussed by Pilz [7]. He defines: let N_1 be a zero-symmetric near-ring and N_2 a constant near-ring. Let $(N, +)$ be the direct sum of the groups $(N_1, +)$ and $(N_2, +)$. A multiplication $\circ$ on $(N, +)$ is called ***admissible*** if $(N, +, \circ)$ is a near-ring and N_1 and N_2 are its zero-symmetric part and its constant part, respectively.

Pilz [7] also showed that there always exists an admissible multiplication for any zero-symmetric near-ring N_1 and any constant near-ring N_2. For instance, the multiplication $\circ$ on $(N, +)$ defined by

$$(n_1+n_2)\circ(n'_1+n'_2)=n_1n'_1+n_2$$

is admissible, where $n_1, n'_1\in N_1$ and $n_2, n'_2\in N_2$ and juxtaposition denotes the original multiplication in N_1.

This admissible multiplication solves our problem.

In fact, for all $n, m\in N_1$ and $c\in N_2$

$$n\circ(m+c)-n\circ c=nm-n0=nm.$$

Therefore, let w be any pseudovaluation of N_1, then

$$w(n\circ(m+c)-n\circ c)=w(nm)\leq w(n)w(m)$$

for all $n, m\in N_1$ and $c\in N_2$. Consequently, any pseudovaluation w of N_1 can be extended to a pseudovaluation of N by Theorem 4.

We have proved

THEOREM 7. *Let N_1 be a zero-symmetric near-ring and N_2 a constant near-ring. Let $(N, +)$ be the direct sum of the groups $(N_1, +)$ and $(N_2, +)$. Then there exists an admissible multiplication $\circ$ such that each pseudovaluation of N_1 can be extended in a unique manner to a pseudovaluation of $(N, +, \circ)$, and all pseudovaluations of $(N, +, \circ)$ are obtained from those of N_1 and vice versa.*

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