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On global classical solutions of a quasi-linear wave equation with a viscoelastic term

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§0. Introduction

In this paper we are concerned with the existence and decay of global classical solutions to the problem

$$(0.1) \qquad u_{tt}-(\rho(x,t)u_{xt}+\sigma(u_x))_x=0$$

on $[0,1]\times R^+$ with the initial-boundary conditions

$$(0.2) \qquad u(x,0)=u_0(x),\ \ u_t(x,0)=u_1(x) \ \text{ and } \ u(0,t)=u(1,t)=0$$

where $\rho(x,t)$ and $\sigma(s)$ are smoothe functions satisfying appropriate conditions.

If $\rho(x,t)\equiv 0$ the equation (0.1) is a usual quasi-linear wave equation. In this case, however, the problem (0.1)-(0.2) may not admit global smooth solution (Lax [8], MacCamy and Mizel [101). It is expected though, if $\rho(x,t)>0$ the problem (0.1)-(0.2) has global classical solutions to a large class of smooth initial data. This problem is a simple model of a material whose stress depends on the history of the motion, and first treated by Greenberg, MacCamy & Mizel [7]. They proved the existence, uniqueness and stability of smooth solution when $\rho(x,t)\equiv$const.>0. Dafermos [3] generalized their result to a wider class of equations. Subsequently Greenberg and MacCamy [6] showed the exponential decay of smooth solutions also in the case $\rho(x,t)\equiv$const.>0. The result of [6] reads as follows:

$$(0.3) \qquad \max_{x\in[0,1]}\{|(u(x,t)|+|u_x(x,t)|+|u_t(x,t)|+|u_{xx}(x,t)|\} \\ +\int_0^1|u_{xt}(x,t)|^2dx\leq c(u_0,u_1)\exp\{-\lambda t\}$$

for some $\lambda>0$ depending on u_0 and u_1.

In [6, 7] the explicite formula of the Green function of the linearized equation is utilized. This method, however, heavily depends on the fact $\rho(x,t)=$const. and seems to be difficult to apply to the case of variable coefficient. Also the estimate (0.3) does not include the norms of second and third order derivatives of $u(x,t)$.

The object of this paper is to show the exponential decay in the sense of strictly classical norms as well as the existence of global classical solutions when $\rho(x,t)$ is not necessarily constant.

The dependence of ρ on the time t makes the problem of asymptotics more delicate. Indeed, as is already known (see Nakao [14]), the solutions may not decay as $t\to\infty$ if $0\leqq\rho(x,t)\leqq$const. $(1+t)^{-\theta}$ or $\rho(x,t)\geqq$const. $(1+t)^{\theta}$ with $\theta>1$. Thus, for our purpose we assume $c_0(1+t)^{-\theta}\leqq\rho(x,t)\leqq c_1(1+t)^{\theta}$ with $\theta<1$. Concerning σ we assume $\sigma'(s)\geqq$const.>0, which is essential for our argument for the existence and decay of global classical solutions.

Roughly speaking, our result reads as follows: In addition to the above assumptions let u_0 and u_1 be sufficiently smooth and satisfy a certain (strong) compatibility conditions. Then the problem (0.1)−(0.2) admits a unique global classical solution u, satisfying

$$\sum_{k=0}^{3}\sum_{i=0}^{k}\left|\frac{\partial^k}{\partial x^i\partial t^{k-i}}u(x,t)\right|\leqq c(u_0,u_1)\exp\{-\lambda t^{1-|\theta|}\}$$

for some $\lambda>0$.

We employ a Galerkin's method and hence our solution is constructive. Greenberg [5] also gave an constructive existence theorem, but his solutions are generalized ones and not classical. For the decay estimates of solutions we use an energy method which is used systematically in our previous papers [11-14].

Recently the existence theories for related problems have been estiblished by several authors; Clements [2], Pecher [15], Andrews [1], Ebihara [4] and others. Above them [2, 15, 4] treated the higher dimensional equations. It seems to be interesting problem to extend our results to higher dimensional cases.

§1. Preliminaries

We use standard function spaces, and the precise definitions of them are omitted (see e.g. Lions [9]). The following Lemmas are used frequently.

LEMMA 1.1.

The Sobolev space $H_1\equiv H_1([0,1])$ *is imbedded continuously into* $C^{1/2}\equiv C^{1/2}([0,1])$ *and it holds that*

$$|u|_{C^{1/2}} \leqq S_1 \|u\|_{H_1} \quad \textit{for } u \in H_1$$
$$(\textit{or}\ |u|_{C^{1/2}} \leqq S_1' \|u\|_{\mathring{H}_1} \quad \textit{for } u \in \mathring{H}_1)$$

and $\|u\|_{L^\infty} \leqq S_2 \|u\|^{1/2}_{\mathring{H}_1} \|u\|^{1/2}_{L^2}$ *for* $u \in \mathring{H}_1$ *where* S_1, S_1' *and* S_2 *are positive constants and we define* $\|u\|_{\mathring{H}_1} = \|\dot{u}\|_{L^2}$, *which is equivalent to* $\|u\|_{H_1}$ *if* $u \in \mathring{H}_1$.

Hereafter we write $\|u\|$ for $\|u\|_{L^2}$.

LEMMA 1.2.

Let $\phi(t)$ *be a nonnegative function on* $R^+ = [0, \infty)$ *satisfying*

$$\sup_{t \leqq s \leqq t+1} \phi(s) \leqq c_0 (1+t)^r (\phi(t) - \phi(t+1)) + k(t)$$

for $t \in R^+$, *wher* $c_0 (>0)$ *and* $r (|r|<1)$ *are constant and* $k(t)$ *is a function such that*

$$0 \leqq k(t) \leqq c_1 \exp\{-\lambda t^{1-|r|}\}$$

with some c_1, $\lambda > 0$. *Then we have*

$$\phi(t) \leqq c_1' \exp\{-\lambda' t^{1-|r|}\}$$

for some $c_1', \lambda' > 0$ *depending on* $\phi(0)$ *and other known constants.* (λ' *does not depend on* $\phi(0)$)

For a proof of Lemma 1.2 see Nakao [13], where more general inequalities are treated.

Now we state our assumptions on ρ and σ.

A_1. $\rho \in C^3([0,1] \times R^+)$ and it holds that

(i) $c_0 (1+t)^{\theta_0} \leqq \rho(x,t) \leqq c_1 (1+t)^{|\theta_0|}$ and $\sum_{k=1}^{3} \sum_{i=1}^{k} \left| \frac{\partial^k}{\partial x^i \partial t^{k-i}} \rho(x,t) \right| \leqq c_2 (1+t)^{\theta_1}$

with $c_0, c_1, \theta_1 > 0$ and $|\theta_0| < 1$,

(ii) $\rho_x(0,t) = \rho_x(1,t) = \rho_{tx}(0,t) = \rho_{tx}(1,t) = 0$ and

$$\sup_{(x,t) \in [0,1] \times R+} \max(\rho_x(x,t),\ \rho_t(x,t)) \leqq 2 \inf_{s \in R} \sigma(s).$$

A_2. $\sigma \in C^3(R)$, and $c_2 |s|^2 \leqq \sigma(s) s \leqq c_3 (1+|s|)^{\theta_2}$, $0 < k_0 \leqq \sigma'(s) \leqq c_4 (1+|s|)^{\theta_2}$ and $|\sigma''(s)| \leqq c_5 (1+|s|)^{\theta_2}$ with $c_i (i=2,3,4,5) > 0$ and $\theta_2 > 0$.

Remark 1.1. If $\rho(x,t)$ is independent of x and t the assumptions on the growth of $\sigma(s)$, $\sigma'(s)$ and $\sigma''(s)$ are not need.

§ 2. Approximate solutions and their estimation

we employ Galerkin's method and construct approximate solutions $\{u_n(t)\}$ as follows. Let $\{\phi_k\}_{k=1}^{\infty}$ be a basis of $L^2([0,1])$ consisting of the eigen functions of the operator $-\frac{d}{dx^2}$ with the 0-boundary condition, i.e.,

$$\{\phi_k\}_{k=1}=\{\sin(2\pi kx)\}_{k=1}^{\infty}$$

and let V_i be the closure of linear combinations of $\{\phi_k\}_{k=1}^{\infty}$ in H^i.
We assume first

$$(u_0, u_1)\in V_6\times V_6$$

and consider the following system of ordinary differential equations

$$(2.1)\quad \begin{cases} (u_n''(t), \phi_k)+(\rho(x,t)\mathrm{D}_x u_n'', D_x\phi_k)+(\sigma(D_x u_n), D_x\phi_k)=0 \\ \qquad\qquad\qquad\qquad\qquad\qquad (k=1, 2, \cdots\cdots, n) \\ u_n(0)=u_{0n} \text{ and } u_n'(0)=u_{1n} \end{cases}$$

where

$$'=\frac{d}{dt},\quad D_x=\frac{\partial}{\partial x}\left(\text{or}\frac{d}{dx}\right),\quad (\ ,\ ) \text{ denotes } L^2\text{-inner product}$$

and we set $u_n(t)=\sum_{j=1}^{n}\alpha_{j,n}(t)\phi_j(x)$. The initial data u_{0n} are chosen in such a way

$$(2.2)\quad \begin{cases} u_{0n}\to u_0 & in\ H^6 \\ u_{1n}\to u_1 & in\ H^6. \end{cases}$$

The system (2.1) is equivalent to

$$(2.1)'\qquad u_n''(t)-P_n\{(\rho(\cdot,t)D_x u_n'+\sigma(D_x u_n))_x\}=0$$

where P_n is the projection of the space L^2 to the n dimensional subspace $W_n=[\phi_1, \cdots\cdots, \phi_n]$.

For simplicity of notation we write often $u(t)$ and $\rho(t)$ for $u(x, t)$ and $\rho(x,t)$, respectively. Also we use $D_x u$ and $D_t u$ (*or'*) for $\frac{\partial}{\partial x}u$ and $\frac{\partial}{\partial t}u$, respectively.

Multiplying (2.1) by $\alpha_{k,n}(t)$, summing up from $k=1$ to n and integrating over $[0,1]$ we have

$$(2.3)\qquad E_0(u_n(t))\leqq E_0(u_n(0))$$

as long as the solution $u_n(t)$ exists, where we set

$$E_0(u_n(t)) \equiv \frac{1}{2}\|u_n'(t)|^2 + \int_0^1\int_0^{D_x u_n(x,t)} \sigma(s)ds\ dx.$$

Thus (2.1) has a unique solution $u_n(t)$ on R^+ for each n.

Now we carry out the first two steps of the estimations of the approximate solutions $u_n(t)$.

LEMMA 2.1.

There exist positive constants $c, \lambda > 0$ independent of n such that

$$E_0(u_n(t)) + \int_t^{t+1} \|\sqrt{\rho}\, D_x u_n'(s)^2 ds \leqq c(\|u_0\|_{H_2}, \|u_1\|) \exp\{-\lambda t^{1-|\theta_0|}\}$$

PROOF.

The proof is essentially included in [13] and [14] and we sketch it briefly. Multiplying (2.1)′ by $u_n'(t)$ and integrating we have

$$(2.4) \qquad \int_t^{t+1} \|\sqrt{\rho}\, D_x u_n'(s)\|^2 ds = E_0(u_n(t)) - E_0(u_n(t+1)) \equiv Q_0^2(t)$$

and by the assumption A_1

$$(2.5) \qquad \int_t^{t+1} \|D_x u_n'(s)\|^2 ds \leqq \text{const.}\ (1+t)^{-\theta_0} Q_0^2(t).$$

By (2.5) there exist two points $t_1 \in [t, t+\frac{1}{4}]$ and $t_2 \in [t+\frac{3}{4}, t+1]$ such that

$$(2.6) \qquad \|u_n'(t_i)\| \leqq \text{const.}\ \|D_x u_n'(t_i)\| \leqq \text{const.}\ (1+t)^{-\theta_0/2}\ Q_0(t) \quad (i=1.2).$$

Next, multiplying (2.1)′ by $u_n(t)$ and integrating over $[0,1]\times[t_1, t_2]$,

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_0^1 \sigma(D_x u_n) D_x u_n dxds \\
&\quad \leqq \int_{t_1}^{t_2} \|u_n'(s)\|^2 ds + \sum_{i=1}^{2} |(u_n'(t_i), u_n(t_i))| \\
&\quad + \int_{t_1}^{t_2}\int_0^1 |\rho(x,s) D_x u_n' D_x u_n| dxds,
\end{aligned}$$

by (2.5) and (2.6)

$$\begin{aligned}
(2.7) \quad &\leqq \text{const.}\ \{(1+t)^{-\theta_0} Q_0^2(t) + (1+t)^{-\theta_0/2} \max_{t\leqq s\leqq t+1} \sqrt{E_0(u_n(s))} \\
&\qquad + (1+t)^{|\theta_0|/2} Q_0(t) \max_{t\leqq s\leqq t+1} \sqrt{E_0(u_n(s))}\} \\
&\equiv K_0^2(t).
\end{aligned}$$

From (2.5) and (2.7) there exists $t^* \in [t, t+1]$ such that

$$E_0(u_n(t^*)) \leqq \text{const.}\, K_0^2(t)$$

and hence by a similar equality to (2.4)

$$\max_{t \leqq s \leqq t+1} \mathrm{E}_0(u_n(s)) \leqq E_0(u_n(t^*)) + \int_t^{t+1} \|\sqrt{\rho}\, D_x u_n'(s)\|^2 ds$$

$$\leqq \text{const.}\, (1+t)^{|\theta_0|} Q_0(t)^2 + \frac{1}{2} \max_{t \leqq s \leqq t+1} E_0(u_n(s)),$$

that is,

$$(2.8) \quad \max_{t \leqq s \leqq t+1} E_0(u_n(s)) \leqq \text{const.}\, (1+t)^{|\theta_0|} (\{E_0(u_n(t)) - \mathrm{E}_0(u_n(t+1))\}.$$

Applying Lemma 1.2 to the above we obtain the desired result. q. e. d.

Hereafter we denote by c, λ and α various positive constants independent of n which may be different at each appearence. In particular we write $c(\mathrm{a})$ etc. if the dependence of c on a is important.

LEMMA 2.2.

We have

$$\|D_x^2 u_n(t)\| \leqq c(\|u_0\|_{H_2}, \|u_1\|)(1+t)^{(1-\theta_0)/2}.$$

PROOF.

Multiplying (2.1)′ by $-D_x^2 u_n(t) (\in W_n)$, we get

$$(2.9) \quad (u_n''(t), -D_x^2 u_n(t)) - (\rho(t) D_x u_n'(t) + \sigma(D_x u_n(t)), D_x^3 u_n(t)) = 0$$

and integrating by parts

$$(2.10) \quad \frac{1}{2} \frac{d}{dt} \int_0^1 \rho(x, \mathrm{t}) |D_x^2 u_n(x, t)|^2 dx$$

$$= \int_0^1 \rho(t) D_x^2 u_n D_x^2 u_n' dx + \frac{1}{2} \int_0^1 \rho_t |D_x^2 u_n|^2 dx$$

$$= -\frac{d}{dt} \int_0^1 D_x u_n' D_x u_n dx + \int_0^1 |D_x u_n'|^2 dx$$

$$- \int_0^1 \sigma'(D_x u_n)^2 (D_x^2 u_n)^2 dx - \int_0^1 \rho_x D_x u_n' D_x^2 u_n dx$$

$$+ \frac{1}{2} \int_0^1 \rho_t (D_x^2 u_n)^2 dx$$

and

$$\frac{d}{dt}\{\frac{1}{2}\int_0^1(\rho(D_x^2u_n)^2+D_xu_n'D_xu_n)dx\}$$

$$\leqq\int_0^1(\frac{1}{2}\rho_t-\sigma'(D_xu_n))(D_x^2u_n)^2dx+c(1+t)^{-\theta_0}\int_0^1(D_xu_n')^2dx$$

$$+c(1+t)^{\alpha}(\int_0^1\rho|D_xu_n'|^2dx)^{1/2}\|D_x^2u_n\|,\quad \alpha>0,$$

by virture of A_1 -(ii),

$$\leqq c\{(1+t)^{-\theta_0}\|\sqrt{\rho}\,D_xu_n'(t)\|^2 + (1+t)^{\alpha}\|\sqrt{\rho}\,D_xu_n'(t)\|\|\sqrt{\rho}\,D_x^2u_n\|\}. \tag{2.11}$$

Intergrating (2.11) over $[0, t]$,

$$\frac{1}{2}\|\sqrt{\rho}\,D_x^2u_n(t)\|^2\leqq c(\|u_0\|_{H_2}, \|u_1\|)+(u_n'(t),\ D_x^2u_n(t)) +c\int_0^t(1+s)^{-\theta_0}\|\sqrt{\rho}\,D_xu_n'(s)\|^2ds +c\{\int_0^t(1+s)^{2\alpha}\|\sqrt{\rho}\,D_xu_n'(s)\|^2ds\}^{1/2}(\int_0^t\|\sqrt{\rho}\,D_x^2u_n\|^2ds)^{1/2}. \tag{2.12}$$

Using Lemma 2.1 the right hand side of (2.12) is estimated by

$$c(\|u_0\|_{H_2}, \|u_1\|)\{1+(\int_0^t\|\sqrt{\rho}\,D_x^2u_n(s)\|^2ds)^{1/2}\}+\frac{1}{4}\|\sqrt{\rho}\,D_x^2u_n(t)\|^2$$

and hence we can conclude

$$\|\sqrt{\rho}\,D_x^2u_n(t)\|\leqq c(\|u_0\|_{H_2}, \|u_1\|)(1+t)^{1/2}$$

which proves Lemma 2.2 q. e. d.

REMARK 2.1. As is easily seen from the proof of Lemma 2.2, if $|\rho_x|+\rho_t<\varepsilon_0$ for sufficiently small $\varepsilon_0>0$ then it follows from (2.10)

$$\|D_x^2u_n(t)\|\leqq c(\|u_0\|_{H_2}, \|u_1\|)$$

and subsequently we obtain

$$\|\sigma^{(i)}(D_xu_n)\|\leqq c(\|u_0\|_{H_2}, \|u_1\|)\qquad (i=0, 1, 2).$$

Thus the growth condition on $\sigma(s)$ is needless.

§ 3. Further estimations of approximate solutions (I)

Starting from the estimates in the previous section we proceed to further estimations of the approximate solutions. As is stated in §2 c, λ and α denotes various positive constants independent of n.

Now differentiating (2.1)′ in t we get

$$(3.1)\quad D_t^3 u_n(t) - P_n\{\rho(t) D_x D_t^2 u_n(t) + \rho_t(t) D_x u_n'(t) + \sigma'(D_x u_n) D_x u_n'(t)\}_x = 0.$$

By (3.1) and a similar argument as in the proof of Lemma 2.1 we can estimate the energy

$$E_1(u_n(t)) \equiv \|D_t^2 u_n(t)\|^2 + \|\sqrt{\sigma'(D_x u_n)}\, D_x u_n'(t)\|^2.$$

Indeed, multiplying (3.1) by $D_t^2 u_n(t)$ and integrating over $[0,1]\times[t,t+1]$

$$(3.2)\quad \int_t^{t+1} \|\sqrt{\rho}\; D_x^2 u_n(s)\|^2 ds = \frac{1}{2}\{E_1(u_n(t)) - E_1(u_n(t+1))\} + \int_t^{t+1}\int_0^1 \{\frac{1}{2}\sigma''(D_x u_n)(D_x u_n'(s))^3 - \rho_t D_x u_n' D_x u_n''\} dxds.$$

The integrations of the right hand side of (3.2) are estimated as follows.

$$\int_t^{t+1}\int_0^1 |\sigma''(D_x u_n)|\,|D_x u_n'(s)|^3 dxds$$
$$\leq c\|\sigma''(D_x u_n)\|_{L^\infty([0,1]\times[t,t+1])} \int_t^{t+1} \|D_x u_n'(s)\|^2 ds \times$$
$$\times \sup_{t\leq s\leq t+1} \{\|D_x^2 u_n'(s)\| + \|u_n'(s)\|,$$

by the assumption A_2 and Lemmas 2.1, 2.2,

$$\leq c(\|u_0\|_{H_2}, \|u_1\|)(1+t)^\alpha \exp\{-\lambda t^{1-|\theta_0|}\}\times$$
$$\times\{\sup_{t\leq s\leq t+1} \|D_x^2 u_n'(s)\| + \exp\{-\lambda t^{1-|\theta_0|}\}$$
$$\leq c(\|u_0\|_{H_2}, \|u_1\|)\exp\{-\lambda t^{1-|\theta_0|}\}\{1+\sup_{t\leq s\leq t+1} \|D_x^2 u_n'(s)\|\}$$

and

$$\int_t^{t+1}\int_0^1 |\rho_t\cdot D_x u_n' \cdot D_x D_t^2 u_n|\, dxds$$
$$\leq c\|\frac{\rho_x}{\rho}\|_{L^\infty([0,1]\times[t,t+1])}\times$$
$$\times(\int_t^{t+1} \|\sqrt{\rho}\; D_x u_n'\|^2 ds)^{1/2} (\int_t^{t+1} \|\sqrt{\rho}\; D_x D_t^2 u_n\|^2 ds)^{1/2}$$
$$\leq c(\|u_0\|_{H_2}, \|u_1\|)\exp\{-\lambda t^{1-|\theta_0|}\} + \frac{1}{2}\int_t^{t+1} \|\sqrt{\rho}\; D_x D_t^2 u_n(s)\|^2 ds.$$

Thus we have

$$(3.3)\qquad \int_t^{t+1}\|\sqrt{\rho}\, D_x D_t^2 u_n(s)\|^2 ds \le E_1(u_n(t)) - E_1(u_n(t+1)) + c\exp\{-\lambda t^{1-|\theta_0|}\}\{1+\sup_{t\le s\le t+1}\|D_x^2 u_n'(s)\|\} \equiv Q_1^2(t)$$

On the other hand, multiplication (3.1) by u_n' and integration yield

$$(3.4)\qquad \int_t^{t+1}\int_0^1 \sigma'(D_x u_n)|D_x u_n'(s)|^2 dxds$$
$$\le \int_t^{t+1}\|D_t^2 u_n(s)\|^2 ds + |(D_t^2 u_n(t), u_n'(t)| + |(D_t^2 u_n(t+1), u_n'(t+1)| + \int_t^{t+1}\int_0^1\{|\rho\cdot D_x D_t^2 u_n\cdot D_x u_n'| + |\rho_t||D_x u_n'(s)|^2\}dxds$$
$$\le c(1+t)^{-\theta_0}Q_1^2(t) + c(\|u_0\|_{H_2}, \|u_1\|)(Q_1(t) + \max_{t\le s\le t+1}\sqrt{E_1(u(s))}+1)\exp\{-\lambda t^{1-|\theta_0|}\} \equiv K_1^2(t).$$

From (3.3) and (3.4) we see that there exists $t^*\in[t, t+1]$ such that

$$E_1(u_n(t^*)) \le cK_1^2(t)$$

and

$$(3.5)\qquad \max_{t\le s\le t+1} E_1(u_n(s)) \le E_1(u_n(t^*)) + \int_t^{t+1}\|\sqrt{\rho}\, D_x D_t^2(u_n(s))\|^2 ds + c\exp(-\lambda t^{1-|\theta_0|}\}\{1+\sup_{t\le s\le t+1}\|D_x^2 u_n'(s)\|\}$$
$$\le cK_1^2(t) + c\exp\{-\lambda t^{1-|\theta_0|}\}\{1+\sup_{t\le s\le t+1}\|D_x^2 u_n'(s)\|\}$$
$$\le c(1+t)^{(\theta_0)}\{E_1(u_n(t)) - E_1(u_n(t+1))\} + c(\|u_0\|_{H_2}, \|u_1\|)\{\sup_{t\le s\le t+1}\|D_x^2 u_n'(s)\|+1\}\times \times\exp\{-\lambda t^{1-|\theta_0|}\}.$$

To estimate $\|D_x^2 u_n'(t)\|$ we return to the equation (2.1)′. Multiplying (2.1)′ by $-D_x^2 u_n'$,

$$\|\sqrt{\rho}\, D_x^2 u_n'(t)\|^2 = -\int_0^1\{D_t^2 u_n\cdot D_x^2 u_n' + \rho_x D_x u_n'\cdot D_x^2 u_n' + \sigma'(D_x u_n)\cdot D_x^2 u_n\cdot D_x^2 u_n'\}dx$$
$$\le c(1+t)^{-\theta_0}\|u_n''(t)\|^2 + c(1+t)^{\alpha}\|D_x u_n'(t)\|^2 + c(1+t)^{\alpha}\|D_x^2 u_n(t)\| + \frac{1}{2}\|\sqrt{\rho}\, D_x^2 u_n'(t)\|^2$$

and by Lemma 2.2

(3.6) $\|D_x^2 u_n'(t)\|^2 \leq c(\|u_0\|_{H_2}, \|u_1\|)(1+t)^{\alpha}(E_1(u_n(t))+1)$

for some $\alpha>0$. From (3.5) and (3.6) we obtain

$$\max_{t\leq s\leq t+1} E_1(u_n(s)) \leq c(1+t)^{|\theta_0|}\{E_1(u_n(t))-E_1(u_n(t+1))\} + c(\|u_0\|_{H_2}, \|u_1\|)\exp\{-\lambda t^{1-|\theta_0|}\}$$

which implies by Lemma 1.2

(3.7) $E_1(u_n(t)) \leq c(E_1(u_n(0)), \|u_0\|_{H_2}, \|u_1\|)\exp\{-\lambda t^{1-|\theta_0|}\}$.

Moreover, through the equation (2.1)′, we see easily

$$E_1(u_n(0)) \leq c(\|u_0\|_{H_2}, \|u_1\|_{H_2})$$

Therefore we have proved the following Lemma.

LEMMA 3.1.

$$\|D_t^2 u_n(t)\|^2 + \|\sqrt{\sigma'(D_x u_n)}D_x u_n'(t)\|^2 + \int_t^{t+1}\|\sqrt{\rho}\, D_x D_t^2 u_n(s)\|^2 ds \leq c(\|u_0\|, \|u_1\|_{H_2})\exp\{-\lambda t^{1-|\theta_0|}\},$$

and

$$\|D_x^2 u_n'(t)\|^2 \leq c(\|u_0\|_{H_2}, \|u_1\|_{H_2})(1+t)^{\alpha} \qquad (\alpha>0).$$

Along the line of the proof of Lemma 3.1 we can obtain the estimates of the higher derivatives of $u_n(t)$. For this, however, we encounter some difficulties and we cannot omit the proofs.

LEMMA 3.2.

$$\|D_t^3 u_n(t)\|^2 + \|\sqrt{\sigma'(D_x u_n)}D_x\, D_t^2 u_n(t)\|^2 + \int_t^{t+1}\|\sqrt{\rho}\, D_x D_t^3 u_n(t)\|^2 ds \leq c(\|u_0\|_{H_4}, \|u_1\|_{H_4})\exp\{-\lambda t^{1-|\theta_0|}\}.$$

PROOF.

Differentiating (3.1) in t we have

(3.8) $$D_t^4 u_n - P_n\frac{\partial}{\partial x}\{\rho\cdot D_x D_t^3 u_n + \rho_{tt} D_x u_n' + \sigma''(D_x u_n)\cdot(D_x u_n')^2 + \sigma'(D_x u_n)\cdot D_x D_t^2 u_n = 0.$$

Multiplying the above by $D_t^3 u_n$,

(3.9) $$\int_t^{t+1}\|\sqrt{\rho}\, D_x D_t^3 u_n(s)\|^2 ds = \frac{1}{2}\{E_2(u_n(t)) - E_2(u_n(t+1))\} + g_2(t)$$

where

$$E_2(u_n(t)) \equiv \|D_t^3 u_n(t)\|^2 + \|\sqrt{\rho(D_x u_n)} D_x D_t^2 u_n(t)\|^2$$

and

$$g_2(t) = \int_t^{t+1}\int_0^1 \{\frac{1}{2}\sigma''(D_x u_n) D_x u_n'(D_x u_n'')^2 + \rho_{tt} D_x u_n' D_t^3 u_n$$
$$+ 2\rho_t \cdot D_x u_n'' \cdot D_x D_t^3 u_n + \sigma''(D_x u_n) \cdot (D_x u_n')^2 \cdot D_x D_t^3 u_n\} dxds.$$

Noting $\|D_x u\|_{L^\infty} \leqq c(\|u\| + \|D_x^2 u\|)$ for $u \in H_2$, $g_2(t)$ is estimated as follows.

$$|g_2(t)|$$
$$\leqq c \sup_{t \leq s \leq t+1} (1 + \|u_n\| + \|D_x^2 u_n\|)^{\theta_2} (\|D_x^2 u_n'(s)\| + \|u_n'(s)\|\} \times$$
$$\times \int_t^{t+1} \|D_x u_n''(s)\|^2 ds$$
$$+ c(1+t)^\alpha (\int_t^{t+1} \|\sqrt{\rho} D_x u_n'(s)\|^2 ds)^{1/2} (\int_t^{t+1} \|\sqrt{\rho} D_x D_t^3 u_n\|^2 ds)^{1/2}$$
$$+ c(1+t)^\alpha (\int_t^{t+1} \|\sqrt{\rho} D_x D_t^2 u_n(s)\|^2 ds)^{1/2} (\int_t^{t+1} \|\sqrt{\rho} D_x D_t^3 u_n\|^2 ds)^{1/2}$$
$$+ c \sup_{t \leq s \leq t+1} (1 + \|u_n(s)\| + \|D_x^2 u_n(s)\|)^{\theta_2} (\|D_x^2 u_n'\| + \|u_n'(s)\|) \times$$
$$\times (\int_t^{t+1} \|D_x u_n'(s)\|^2 ds)^{1/2} (\int_t^{t+1} \|\sqrt{\rho} D_x D_t^3 u_n(s)\|^2 ds)^{1/2},$$

with the aid of earlier Lemmas,

$$\leqq c(\|u_0\|_{H_2}, \|u_1\|_{H_2}) \exp\{-\lambda t^{1-|\theta_0|}\} + \frac{1}{2}\int_t^{t+1} \|\sqrt{\rho}\, D_x D_t^3 u_n\|^2 ds. \tag{3.10}$$

From (3.9) and (3.10)

$$\int_t^{t+1} \|\sqrt{\rho} D_x D_t^3 u_n(s)\|^2 ds \leqq E_2(u_n(t)) - E_2(u_n(t+1))$$
$$+ c(\|u_0\|_{H_2}, \|u_1\|_{H_2}) \exp\{-\lambda t^{1-|\theta_0|}\} \tag{3.11}$$

On the other hand, by Lemma 3.1,

$$\int_t^{t+1}\int_0^1 \sigma'(D_x u_n) |D_x D_t^2 u_n(s)|^2 dxds \leqq c(\|u_0\|_{H_2}, \|u_1\|_{H_2}) \exp\{-\lambda t^{1-|\theta_0|}\}. \tag{3.12}$$

Combining (3.11) and (3.12) we obtain the difference inequality

$$\max_{t \leq s \leq t+1} E_2(u_n(s)) \leqq c(1+t)^{|\theta_0|} \{E_2(u_n(t)) - E_2(u_n(t+1))\}$$
$$+ c(\|u_0\|_{H_2}, \|u_1\|_{H_2}) \exp\{-\lambda t^{1-|\theta_0|}\}$$

which implies

$$E_2(u_n(t)) \leqq c(E_2(u_n(0)), \|u_0\|_{H_2}, \|u_1\|_{H_2}) \exp\{-\lambda t^{1-|\theta_0|}\}. \tag{3.13}$$

It remains to show the boundedness of $E_2(u_n(0))$. Noting that the operators $-\frac{d^2}{dx^2}$ and P_n are commutative on $H_2\cap \mathring{H}_1$ and $\rho_x(0,t)=\rho_x(1,t)=0$, we see

$$\|D_x^2D_t^2u_n(0)\|=\|D_x^3\{\rho(x,0)\frac{\partial}{\partial x}u_n'(0)+\sigma'(D_xu_n(0))D_x^2u_n(0)\}\|$$
$$\leq c(\|u_0\|_{H_4},\|u_1\|_{H_4})$$

and hence

(3.14) $\|D_xu_n''(0)\|\leq c(\|D_x^2D_t^2u_n(0)\|+\|D_t^2u_n(0)\|)\leq c(\|u_0\|_{H_4},\|u_1\|_{H_4})$.

Moreover

(3.15) $\|D_t^3u_n(0)\|=\|P_n\{D_xD_t(\rho D_xu_n'+\sigma(D_xu_n))\}_{t=0}\|\leq c(\|u_0\|_{H_4},\|u_1\|_{H_4})$.

(3.13)-(3.15) proves the Lemma. q. e. d.

To show the decay of $\|D_x^2u_n(t)\|$ we prepare the following estimate for $\|D_x^3u_n(t)\|$.

LEMMA 3.3.

$$\|D_x^3u_n(t)\|\leq c(\|u_0\|_{H_3},\|u_1\|_{H_2})(1+t)^\alpha$$

for some $\alpha>0$.

PROOF.

Multiplying the equation (2.1)$'$ by $D_x^4u_n$ $(\in W_n)$,

(3.16) $(D_t^2u_n-\rho D_x^2u_n'-\rho_xD_xu_n'-\sigma'(D_xu_n)\cdot D_x^2u_n,D_x^4u_n)=0$.

Now

$$(D_t^2u_n,D_x^4u_n)=\frac{d}{dt}(D_x^2u_n',D_x^2u_n)-\|D_x^2u_n'(t)\|^2,$$

$$-(\rho D_n^2u_n',D_x^4u_n)=(\rho_xD_x^2u_n',D_n^3u_n)+\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}\,D_x^3u_n\|^2$$
$$-\frac{1}{2}\int_0^1\rho_x(D_x^3u_n)^2dx,$$

$$-(\rho_xD_xu_n',D_x^4u_n)=(\rho_{xx}D_xu_n'+\rho_xD_x^2u_n',D_x^3u_n)$$
(note that $\rho_x(0,t)=\rho_x(1,t)=0$)

and

(3.17) $-(\sigma'(D_xu_n)D_x^2u_n,D_x^4u_n)$
$$=\int_0^1\{\sigma'(D_xu_n)\cdot(D_x^3u_n)^2+\sigma''(D_xu_n)\cdot(D_x^2u_n)^2\cdot D_x^2u_n\}dx.$$

It follows that

$$(3,18)\quad \frac{d}{dt}\{\frac{1}{2}\|\sqrt{\rho}\, D_x^3 u_n(t)\|^2-(D_x u_n', D_x^3 u_n)\}$$
$$\leqq \|D_x^2 u_n'(t)\|^2+\int_0^1\{\frac{1}{2}\rho_x-\sigma'(D_x u_n)\}(D_x^3 u_n)^2 dx$$
$$-\int_0^1 (2\rho_x D_x^2 u_n'+\rho_{xx} D_x u_n'+\sigma''(D_x u_n)\cdot(D_x^2 u_n)^2\} D_x^3 u_n \mathrm{d}x$$
$$\leqq c(\|u_0\|_{H_2}, \|u_1\|_{H_2})(1+t)^{\alpha}+c(1+t)^{\theta_1-\theta_0}\|D_x^2 u_n'(t)\|\times$$
$$\times\|\sqrt{\rho}\, D_x^3 u_n(t)\|$$
$$+c(1+t)^{\alpha}\|D_x u_n'(t)\|\ \|\sqrt{\rho}\, D_x^3 u_n(t)\|$$
$$+c(\|u_0\|_{H_2}, \|u_1\|_{H_2})\|D_x^2 u_n(t)\|_{L^\infty}\|D_x^2 u_n(t)\|\ \|D_x^3 u_n\|$$
$$\leqq c(\|u_0\|_{H_2}, \|u_1\|_{H_2})(1+t)^{\alpha}+\|\sqrt{\rho}\, D_x^3 u_n(t)\|^{4/3},$$

where we have used Lemmas 3.1, 2.2, 1.1 and Young's inequality. The inequality (3.18) together with the estimate

$$|(D_x u_n', D_x^3 u_n)|\leqq c(\|u_0\|_{H_2}, \|u_1\|_{H_2})\exp\{-\lambda t^{1-|\theta_0|}\}\|\sqrt{\rho}\, D_x^3\ u_n(t)\|$$

implies easily

$$\|D_x^3 u_n(t)\|\leqq c(\|D_x^3 u_n(0)\|, \|u_0\|_{H_2}, \|u_1\|_{H_2})\|(1+t)^{\alpha}$$
$$\leqq c(\|u_0\|_{H_3}, \|u_1\|_{H_2}(1+t)^{\alpha}$$

for some $\alpha>0$. q. e. d.

With the aid of Lemma 3.3 we can show the decay property of $\|D_x^2 u_n(t)\|$ and $\int_t^{t+1}\|D_x^2 u_n'(s)\|^2\, ds$. Hereafter we often use the notation

$$e_\lambda(t)\equiv\exp\{-\lambda t^{1-|\theta_0|}\}$$

for simplicity.

LEMMA 3.4.

$$\|D_x^2 u_n(t)\|^2+\int_t^{t+1}\|D_x^2 u_n'(s)\|^2 ds\leqq c(\|u_0\|_{H_3}, \|u_1\|_{H_3})e_\lambda(t)$$

for some $\lambda>0$.

PROOF.

The proof is not difficult and we sketch it briefly. Multiplication (2.1)′ by $-D_x^2 u_n'(t)$ $(\in W_n)$ and integration yield

$$(3.19)\quad \int_t^{t+1}\|\sqrt{\rho}\, D_x^2 u_n'(s)\|^2 ds=\frac{1}{2}\{E_3(u_n(t))-E_3(u_n(t+1))\}+g_3(t),$$

where

$$E_3(u_n(t)) = \|D_x u'(t)\|^2 + \|\sqrt{\sigma'(D_x u_n)}\, D^2 u_n(t)\|^2$$

and

$$g_3(t) = \int_t^{t+1}\int_0^1 \{-\rho_x D_x u_n' \cdot D_x^2 u_n' + \frac{1}{2}\sigma''(D_x u_n)\cdot D_x u_n' \cdot (D_x^2 u_n)^2\} dx ds.$$

Then, by Lemmas 3.1 and 3.3, we have

$$|g(t)| \leqq c(\|u_0|_{H_3}, \|_{H_2}) e_\lambda(t) + \frac{1}{2}\int_t^{t+1} \|\sqrt{\rho}\, D_x^2 u_n'(s)\|^2 ds$$

and by (3.19)

$$\text{(3.20)} \quad \int_t^{t+1} \|\sqrt{\rho}\, D_x^2 u_n'(s)\|^2 ds$$
$$\leqq E_3(u_n(t)) - E_3(u_n(t+1)) + c(\|u_0\|_{H_3}, \|u_1\|_{H_2}) e_\lambda(t)$$
$$\equiv Q_3^2(t).$$

On the other hand, Multiplying (2.1)′ by $-D_x^2 u_n\ (\in W_n)$ easy calculations give

$$\text{(3.21)} \quad \int_t^{t+1} \|\sqrt{\sigma'(D_x u_n)}\, D_x^2 u_n(s)\|^2 ds$$
$$\leqq c\int_t^{t+1} \|u_n''(s)\|^2 ds + c(1+t)^{|\theta_0|}\int_t^{t+1} \|\sqrt{\rho}\, D_x^2 u_n'(s)\|^2 ds$$
$$+ c(1+t)^{\theta_1}\int_t^{t+1} \|D_x u_n'(s)\|^2 ds$$
$$\leqq c(1+t)^{|\theta_0|} Q_3^2(t) + c(\|u_0\|_{H_2}, \|u_1\|_{H_2}) e_\lambda(t).$$

Combining (3.20) and (3.21) we obtain

$$\max_{t \leq s \leq t+1} E_3(u_n(s)) \leqq c(1+t)^{|\theta_0|}\{E_3(u_n(t)) - E_3(u_n(t+1))\}$$
$$+ c(\|u_0\|_{H_2}, \|u_1\|_{H_2}) e_\lambda(t)$$

which together with Lemma 1.2 implies Lemma 3.4 immediately. q.e.d.

LEMMA 3.5.

$$\|D_x^2 u_n'(t)\|^2 + \|D_x^3 u_n(t)\| + \int_t^{t+1} \|\sqrt{\rho} D_x^3 u_n'(t)\|^2 ds$$
$$\leqq c(\|u_0\|_{H_3}, \|u_1\|_{H_3}) e_\lambda(t) \qquad (\lambda > 0).$$

PROOF.

Multiplying (2.1)′ by $D_x^4(t)$ $(\in W_n)$ and integrating we obtain

$$\int_t^{t+1}\|\sqrt{\rho}\, D_x^3 u_n'(s)\|^2 ds = \frac{1}{2}\{E_4(u_n(t)) - E_4(u_n(t+1))\} + g_4(t)$$

where

$$E_4(u_n(t)) = \|D_x^2 u_n'(t)\|^2 + \|\sqrt{\sigma'(D_x u_n)}\, D_x^3 u_n(t)\|^2$$

and

$$\begin{aligned} g_4(t) = \int_t^{t+1}\int_0^1 \{ & -2\rho_x \cdot D_x^2 u_n' \cdot D_x^3 u_n' - \rho_{xx} \cdot D_x u_n' \cdot D_x^3 u_n' \\ & + \sigma''(D_x u_n) \cdot (D_x^2 u_n)^2 \cdot D_x^3 u_n' \\ & + \frac{1}{2}\sigma''(D_x u_n) \cdot D_x u_n' \cdot (D_x^3 u_n)^2\} dx ds. \end{aligned}$$

Using earlier Lemmas we can estimate $g_4(t)$ as follows.

$$\begin{aligned} |g_4(t)| \leqq\, & c(1+t)^{2\theta_1 - \theta_0}\int_t^{t+1}(\|D_x^2 u_n'(s)\|^2 + \|D_x u_n'(s)\|^2) ds \\ & + c(\|u_0\|_{H^3}, \|u_1\|_{H^3}) \sup_{t \leq s \leq t+1} \|D_x^3 u_n(s)\|^2 \int_t^{t+1}\|D_x^2 u_n(s)\|^2 ds \\ & + c(\|u_0\|_{H^3}, \|u_1\|_{H^3}) \sup_{t \leq s \leq t+1} \|D_x^3 u_n(s)\|^2 \int_t^{t+1}(\|D_x^2 u_n'(s)\| \\ & + \|u_n'(s)\| ds + \frac{1}{2}\int_t^{t+1}\|\sqrt{\rho}\, D_x^3 u_n'(s)^2 ds \\ \leqq\, & c(\|u_0\|_{H^3}, \|u_1\|_{H^3}) e_\lambda(t) + \frac{1}{2}\int_t^{t+1}\|\sqrt{\rho}\, D_x^3 u_n'(s)\|^2 ds. \end{aligned}$$

Thus

$$\begin{aligned} (3.22) \quad \int_t^{t+1}\|\sqrt{\rho}\, D_x^3 u_n'(s)\|^2 ds \leqq\, & E_4(u_n(t)) - E_4(u_n(t+1)) \\ & + c(\|u_0\|_{H^3}, \|u_1\|_{H^3}) e_\lambda(\mathrm{t}) \\ \equiv\, & Q_4^2(t). \end{aligned}$$

Next, multiplying (2.1)$'$ by $D_x^4 u_n$,

$$\begin{aligned} & \int_t^{t+1}\|\sqrt{\sigma'(D_x u_n)}\, D_x^3 u_n(s)\|^2 ds \\ & = \int_t^{t+1}\int_0^1 \{-\sigma''(D_x u_n) \cdot (D_x^2 u_n)^2 \cdot D_x^3 u_n + (D_x u_n''(s) + \rho D_x^3 u_n' \\ & \quad -2\rho_x D_x^2 u_n' - \rho_{xx} D_x u_n') D_x^3 u_n\} dx ds \\ & \leqq c(\|u_0\|_{H^3}, \|u_1\|_{H^3}) \sup_{t \leq s \leq t+1} \|D_x^3 u_n(s)\| \|D_x^2 u_n(s)\| \; \|D_x^3 u_n(s)\| \\ & \quad + \sup_{t \leq s \leq t+1} \|D_x u_n''(s)\| \; \|D_x^3 u_n(s)\| + c(1+t)^{|\theta_0|/2} \times \\ & \qquad \times \left(\int_t^{t+1}\|\sqrt{\rho}\, D_x^3 u_n'\|^2 ds\right)^{1/2} \int_t^{t+1}\|D_x^3 u_n\|^2 ds \\ & \quad + c(1+t)^{\theta_1} \sup_{t \leq s \leq t+1} (\|D_x u_n'(s)\| \|D_x^3 u_n(s)\|), \end{aligned}$$

by Lemmas 3.1-3.4,

$$\leqq c(\|u_0\|_{H^3}, \|u_1\|_{H^3})e_\lambda(t)+c(1+t)^{|\theta_0|}Q_4^2(t)$$
$$+\frac{1}{2}\int_t^{t+1}\|\sqrt{\sigma'(D_xu_n)}D_x^3u_n(s)\|^2ds$$

and

$$(3.23)\quad \int_t^{t+1}\|\sqrt{\sigma'(D_xu_n)}D_x^3u_n(s)\|^2ds$$
$$\leqq c(1+t)^{|\theta_0|}Q_4^2(t)+c(\|u_0\|_{H^3}, \|u_1\|_{H^3})e_\lambda(t)$$

From (3.22) and (3.23) we can conclude as is usual

$$E_4(u_n(t))\leqq c(\mathrm{E}_4(u_n(0)), \|u_0\|_{H^3}, \|u_1\|_{H^3})e_\lambda(t)$$
$$\leqq c(\|u_0\|_{H^3}, \|u_1\|_{H^3})e_\lambda(t).$$

q. e. d.

The following is an immediate consequence of the above Lemma.

LEMMA 3.6.

$$\|D_x^2D_t^2u_n(t)\|\leqq c(\|u_0\|_{H^3}, \|u_1\|_{H^3})e_\lambda(t).$$

PROOF.

Multiplying (3.1) by $D_x^2\mathrm{D}_t^2u_n$ ($\in W_n$) we have

$$\|\sqrt{\rho}\,D_x^2D_t^2u_n(t)\|\leqq c(1+t)^{-\theta_0/2}\{\|D_t^3u_n(t)\|+\|\rho_xD_xD_t^2u_n$$
$$+\rho_{tx}D_tu_n'+\rho_tD_x^2u_n'+\sigma''(D_xu_n)\cdot D_x^2u_n\cdot D_xu_n'$$
$$+\sigma'(D_xu_n)\cdot D_x^2u_n'\|\}$$
$$\leqq c(\|u_0\|_{H^3}, \|u_1\|_{H^3})e_\lambda(t).$$

q. e. d.

We continue the estimation of $\{u_n(t)\}$.

LEMMA 3.7.

$$\|D_t^4u_n(t)\|^2+\|D_xD_t^3u_n(t)\|^2+\int_t^{t+1}\|D_xD_t^4u_n(s)\|^2ds$$
$$\leqq c(\|u_0\|_{H^6}, \|\mathrm{u}_1\|_{H^6})e_\lambda(t),\qquad \lambda>0.$$

PROOF.

Differentiating (3.1) two times with respect to t,

$$(3.24)\quad D_t^5u_n-P_n[\frac{\partial}{\partial x}\{3\rho_{tt}D_xD_t^2u_n+3\rho_tD_xD_t^3u_n+\rho D_xD_t^4u_n+\rho_{ttt}D_xu_n'$$
$$+\sigma''(D_xu_n)\cdot(D_xu_n')^3+3\sigma''(D_xu_n)\cdot D_xu_n'\cdot D_xD_t^2u_n$$
$$+\sigma'(D_xu_n)\cdot D_xD_t^3u_n\}=0$$

Multiplying the above by $D_t^4 u_n$ we have

$$(3.25)\quad \int_t^{t+1} \|\sqrt{\rho}\, D_x D_t^4 u_n(s)\|^2 ds = \frac{1}{2}\{E_5(u_n(t)) - E_5(u_n(t+1))\} + g_5(t)$$

where

$$E_5(u_n(t)) = \|D_t^4 u_n(t)\|^2 + \|\sqrt{\sigma'(D_x u_n)}\, D_x D_t^3 u_n(t)\|^2$$

and

$$g_5(t) = -\int_t^{t+1}\int_0^1 \{3\rho_{tt} D_x D_t^2 u_n + \rho_{ttt} D_x u_n' + \sigma''(D_x u_n)\cdot(D_x u_n')^3 + 3\sigma''(D_x u_n)\cdot D_x u_n' \cdot D_x D_t^2 u_n\} D_x D_t^4 u_n \, dxds + \frac{1}{2}\int_t^{t+1}\int_0^1 \sigma''(D_x u_n)\cdot D_x u_n' \cdot (D_x D_t^3 u_n)^2 dxds.$$

By Lemmas 3.2, 3.4 and 3.5 we see easily

$$|g_5(t)| \leq c(\|u_0\|_{H3}, \|u_1\|_{H_3}) e_\lambda(t) + \frac{1}{2}\int_t^{t+1} \|\sqrt{\rho}\, \mathrm{D}_x D_t^4 u_n\|^2 ds$$

and hence by 3.35

$$(3.26)\quad \int_t^{t+1} \|\sqrt{\rho}\, D_x D_t^4 u_n(s)\|^2 ds \leq E_5(u_n(t)) - E_5(u_n(t+1)) + c(\|u_0\|_{H3}, \|u_1\|_{H_3}) e_\lambda(t) \equiv Q_5^2(t).$$

We know already by Lemmas 3.2 and 3.4

$$(3.27)\quad \int_t^{t+1} \|\sqrt{\sigma'(D_x u_n)}\, D_x D_t^3 u_n(s)\|^2 ds \leq c(\|u_0\|_{H3}, \|u_1\|_{H_3}) e_\lambda(t).$$

From (3.26) and (3.27) we can conclude as is usual

$$E_5(u_n(t)) + \int_t^{t+1} \|\sqrt{\rho}\, D_x D_t^4 u_n(s)\|^2 ds \leq c(E_5(u_n(0)), \|u_0\|_{H3}, \|u_1\|_{H_3}) e_\lambda(t).$$

Moreover

$$\|D_x D_t^3 u_n(0)\|^2 \leq \mathrm{c}(\|D_x^2 D_t^3 u_n(0)\|^2 + \|D_t^3 u_n(0)\|^2) \leq c(\|u_0\|_{H6}, \|u_1\|_{H_6}) \text{ (cf. Lemma 3.2)}$$

and by (3.8),

$$\|D_t^4 u_n(0)\| \leq c(\|D_t^2 u_n(0)\|_{H_2}, \|u_n'(0)\|_{H_2}, \|D_t^3 u_n(0)\|_{H_2}, \|u_n(0)\|_{H_2}) \leq c(\|u_0\|\|_{H6}, \|u_1\|_{H_6}).$$

Therefore

$$E_5(u_n(0)) \leq c(\|u_0\|_{H_6}, \|u_1\|_{H_6})$$

which proves the Lemma.

q. e. d.

LEMMA 3.8.

$$\|D_x^3 u_n'(t)\|^2 + \int_t^{t+1} \|D_x^3 D_t^2 u_n(s)\|^2 ds \leq c(\|u_0\|\|_{H_3}, \|u_1\|_{H_3}) e_\lambda(t).$$

PROOF.

Multiplying (3.1) by $D_x^4 D_t^2 u_n(t)$,

$$(3.28) \quad (D_t^3 u_n(t), D_x^4 D_t^2 u_n(t)) + (\rho(t) D_x D_t^2 u_n, D_x^5 D_t^2 u_n) + (\rho_t D_x u_n', D_x^5 D_t^2 u_n) + (\sigma'(D_x u_n) \cdot D_x u_n', D_x^5 u_n) = 0.$$

Using the integration by parts we can arrange the equation as follows.

$$(3.29) \quad \frac{1}{2} \frac{d}{dt} \{\|D_x^2 D_t^2 u_n\|^2 + \|\sqrt{\sigma'(D_x u_n)} D_x^3 u_n'(t)\|^2\} + \|\sqrt{\rho}\, D_x^3 D_t^2 u_n(t)\|^2$$

$$= \frac{1}{2} \int_0^1 \sigma''(D_x u_n) \cdot (D_x^3 u_k')^2 dx + \int_0^1 \rho_{xx} \cdot (D_x^2 D_t^2 u_n)^2 dx$$

$$- \int_0^1 \rho_t \cdot D_x^3 u_n' \cdot D_x^3 D_t^2 u_n dx + g_6(t)$$

where

$$g_6(t) = -\int_0^1 \{\rho_{txx} D_x u_n' + 2\rho_{tx} D_x^2 u_n' + \rho_{xx} D_x D_x^2 u_n$$
$$+ \sigma'''(D_x u_n) \cdot (D_x^2 u_n)^2 \cdot D_x u_n' + \sigma'''(D_x u_n) \cdot D_x^3 u_n \cdot D_x u_n'$$
$$+ 2\sigma''(D_x u_n) \cdot D_x^2 u_n \cdot D_x^2 u_n'\} D_x^3 D_t^2 u_n dx.$$

Note that we have used the assumption

$$\rho_x(0, 1) = \rho_x(1, t) = \rho_{tx}(0, t) = \rho_{tx}(1, t) = 0.$$

From Lemmas 3.1, 3.2 and 3.5 it is easy to see

$$|g_6(t)| \leq c(\|u_0\|_{H_3}, \|u_1\|_{H_3}) e_\lambda(t) + \frac{1}{2} \|\sqrt{\rho}\, D_x^3 D_t^2 u_n(t)\|^2.$$

Moreover, by Lemmas 3.4-3.6, the integration over $[t, t+1]$ of the first three terms of the right hand side of (3.29) are bounded by $c(\|u_0\|_{H_3}, \|u_1\|_{H_3}) \times e_\lambda(t)$.

Thus we have from (3.29)

$$(3.30)\quad \int_t^{t+1} \|\sqrt{\rho}\, D_x^3 D_t^2 u_n(s)\|^2 ds \leq E_6(u_n(t)) - E_6(u_n(t+1)) + c(\|u_0\|_{H_3}, \|u_1\|_{H_3}) e_\lambda(t) \equiv Q_6^2(t),$$

where

$$E_6(u_n(t)) = \|D_x^2 D_t^2 u_n(t)\|^2 + \sqrt{\sigma'(D_x u_n)}\, D_x^3 u_n'(t)\|^2.$$

On the other hand we know by Lemma 3.5

$$(3.31)\quad \int_t^{t+1} \|\sqrt{\sigma'(D_x u_n)}\, D_x^3 u_n'(s)\|^2 ds \leq c(\|u_0\|_{H_3}, \|u_1\|_{H_3}) e_\lambda(t).$$

Combinig (3.30) and (3.31) we can obtain

$$E_6(u_n(t)) + \int_t^{t+1} \|\sqrt{\rho}\, D_x^3 D_t^2 u_n(s)\|^2 ds \leq c(E_6(u_n(0)), \|u_0\|_{H_3}, \|u_1\|_{H_3}) e_\lambda(t) \leq c(\|u_0\|_{H_3}, \|u_1\|_{H_3}) e_\lambda(t)$$

q. e. d.

Next, we want to show the decay of $\|u_n(t)\|_{H_4}$. For this we need the following auxiliary Lemma.

LEMMA 3.9.

$$\|D_x^4 u_n(t)\| \leq c(\|u_0\|_{H_4}, \|u_1\|_{H_3}) (1+t)^{(1-\theta_0)/2}.$$

PROOF.

At this time we multiply (2.1)′ by $-D_x^6 u_n$ ($\in W_n$) to obtain (cf. Lemma 3.3)

$$(3.32)\quad \frac{d}{dt}\{\frac{1}{2}\|\sqrt{\rho}\, D_x^4 u_n(t)\|^2 - (D_x^3 u_n', D_x^4 u_n)\}$$
$$= \|D_x^3 u_n'(t)\|^2 + \int_0^1 (\frac{1}{2}\rho_x - \sigma'(D_x u_n))(D_x^4 u_n)^2 dx$$
$$-(\rho_{xxx} D_x u_n' + 3\rho_{xx} D_n^2 u_x' + 3\rho_x D_x^3 u_n' + \sigma'''(D_x u_n)\cdot(D_x^2 u_n)^3$$
$$+3\sigma''(D_x u_n)\cdot D_x^2 u_n \cdot D_x^3 u_n, D_x^4 u_n)$$
$$\leq c(\|u_0\|_{H_3}, \|u_1\|_{H_3}) e_\lambda(t)(1 + \|\sqrt{\rho}\, D_x^4 u_n(t)\|)$$

which yield (cf. Lemma 2.2)

$$\|D_x^4 u_n(t)\| \leq c(\|D_x^4 u_n(0)\|, \|u_0\|_{H_3}, \|u_1\|_{H_3}) (1+t)^{(1-\theta_0)/2}$$

q. e. d.

LEMMA 3.10.

$$\|D_x^4 u_n(t)\|^2+\int_t^{t+1}\|D_x^4 D_t u_n(s)\|^2 ds \leqq c(\|u_0\|_{H_6}, \|u_1\|_{H_6}) e_\lambda(t).$$

PROOF.

Multiplying (2.1)′ by $-D_x^6 u_n'$,

$$\int_t^{t+1}\|\sqrt{\rho}\, D_x^4 u_n'(s)\|^2 ds=\frac{1}{2}\{E_7(u_n(t))-E_7(u_n(t+1))\}+g_7(t) \tag{3.33}$$

where

$$E_7(u_n(t))=\|D_x^3 u_n'(t)\|^2+\|\sqrt{\sigma'(D_x u_n)}\,D_x^4 u_n(t)\|^2$$

and

$$\begin{aligned} g_7(t)=&-\int_t^{t+1}\int_0^1\{3\rho_{xx}D_x^2 u_n'+3\rho_x D_x^3 u_n'+\rho_{xxx}D_x u_n'+\sigma'''(D_x u_n)\cdot(D_x^2 u_n)^3 \\ &+3\sigma''(D_x u_n)\cdot D_x^2 u_n\cdot D_x^3 u_n\}D_x^4 u_n' dx ds \\ &+\int_t^{t+1}\int_0^1 \sigma''(D_x u_n)\cdot D_x^2 u_n (D_x^4 u_n)^2 dx ds. \end{aligned}$$

By Lemmas 3.5 and 3.9 we get easily

$$\begin{aligned} |g_7(t)|\leqq & c(\|u_0\|_{H_3}, \|u_1\|_{H_3})e_\lambda(t)+\frac{1}{2}\int_t^{t+1}\|\sqrt{\rho}\, D_x^4 u_n'(s)\|^2 ds \\ &+c(\|u_0\|_{H_4}, \|u_1\|_{H_3})e_\lambda(t) \end{aligned}$$

and hence

$$\begin{aligned} \int_t^{t+1}\|\sqrt{\rho}\, D_x^4 u_n'(s)\|^2 ds \leqq & E_7(u_n(t))-E_7(u_n(t+1)) \\ &+c(\|u_0\|_{H_4}, \|u_1\|_{H_3})e_\lambda(t) \\ \equiv & Q_7^2(t). \end{aligned}$$

Multiplying (2.1)′ again by $-D_x^6 u_n$,

$$\begin{aligned} &\int_t^{t+1}\|\sigma'(D_x u_n)D_x^4 u_n(s)\|^2 ds \\ &=\int_t^{t+1}\int_0^1\{D_x^2 u_n''-\rho D_x^4 u_n'-3\rho_{xx}D_x^2 u_n'-3\rho_x D_x^3 u_n'-\rho_{xxx}D_x u_n' \\ &\quad -\sigma'''(D_x u_n)\cdot(D_x^2 u_n)^3-3\sigma''(D_x u_n)\cdot D_x^2 u_n\cdot D_x^3 u_n\}D_x^4 u_n dx ds \\ &\leqq c(\|u_0\|_{H_4}, \|u_1\|_{H3})e_\lambda(t)+c(1+t)^{|\theta_0|}Q_7^2(t) \\ &\quad +\frac{1}{2}\int_t^{t+1}\|\sigma'(D_x u_n)\cdot D_x^4 u_n(s)\|^2 ds \end{aligned}$$

and

$$(3.35)\quad \int_t^{t+1}\|\sigma'(D_x u_n)D_x^4 u_n(s)\|^2 ds \leqq c(1+t)^{|\theta_0|}Q_7^2(t) \\ +c(\|u_0\|_{H_4}, \|u_1\|_{H_3})e_\lambda(t).$$

From (3.34) and (3.35) we can conclude

$$E_7(u_n(t))+\int_t^{t+1}\|\sqrt{\rho}\, D_x^4 u_n'(s)\|^2 ds \\ \leqq c(E_7(u_n(0)), \|u_0\|_{H_4}, \|u_1\|_{H_3})e_\lambda(t) \\ \leqq c(\|u_0\|_{H_4}, \|u_1\|_{H_3})e_\lambda(t).$$

q. e. d.

Finally in this section we summarize the estimates obtained.

PROPOSITION 3.1.

Let us assume A.1 *and* A.2, *and let* $(u_0, u_1) \in V_6\times V_6$. *Then, the approximate solutions* $\{u_n(t)\}$ *constructed through* (2.1)–(2.2) *satisfy the following estimate.*

$$\sum_{i=1}^{4}\|D_t^i u_n(t)\|^2+\sum_{i=0}^{3}\|D_t^i u_n(t)\|_{H_1}^2+\sum_{i=0}^{2}\|D_t^i u_n(t)\|_{H_2}^2 \\ +\sum_{i=0}^{1}\|D_t^i u_n(t)\|_{H_3}^2+\|u_n(t)\|_{H_4}^2 \\ +\int_t^{t+1}\{\|D_x D_t^4 u_n(s)\|^2+\|D_x^3 D_t^2 u_n(s)\|^2+D_x^4 D_t u_n(s)\|^2\}ds \\ \leqq c\|u_0\|_{H_6}, \|u_1\|_{H_6})\exp\{-\lambda t^{1-|\theta_0|}\}$$

for some $\lambda>0$ *independent of* n *and* (u_0, u_1).

§ 4. Further estimations of approximate solutions (II)

Here we make some additional assumptions on ρ, σ and the initial data u_0, u_1 to obtain the estimates for higher order derivatives of $\{u_n(t)\}$.

A.3. $\sigma(s)\in C^4(R)$, $\rho\in C^4([0,1]\times R^+)$

and

$$\rho_{xxx}(0,t)=\rho_{xxx}(1,t)=\rho_{ttx}(0,1)=\rho_{ttx}(1,t)=0 \text{ and } |\rho_{ttxx}|\leqq c_3(1+t)^{\theta_3}$$

for some θ_3.

A.4. $(u_0, u_1)\in V_8\times V_8$ and we assume for (2.2)

(2.2)′ $u_{0n}\to u_0$ in H_8 and $u_{1n}\to u_1$ in H_8.

Under the assumptions (A.1)-(A.4) we derive further estimates for $\{u_n(t)\}$

on the basis of proposition 3.1. The proofs are given in a similar manner as in the previous section and we state them as briefly as possible.

LEMMA 4.1.

$$\|D_t^5 u_n(t)\|^2+\|\sqrt{\sigma'(D_x u_n)}D_x D_t^5 u_n(t)\|^2+\int_t^{t+1}\|\sqrt{\rho}\, D_x D_t^5 u_n(s)\|^2 ds$$
$$\leq c\|u_0\|_{H_8}, \|u_1\|_{H_8})e_\lambda(t) \qquad (\lambda>0).$$

PROOF.

Differentiating (2.1)′ four times with respect to t and multiplying it by $D_t^5 u_n$, easy caluculations with the aid of prop. 3.1 yield

$$(4.1) \quad \int_t^{t+1}\|\sqrt{\rho}\, D_x{}_t^5 u_n(s)\|^2 ds\leq E_8(u_n(t))-E_8(u_n(t+1))$$
$$+c(\|u_0\|_{H_6}, \|u_1\|_{H_6})e_\lambda(t),$$

where

$$E_8=\|D_t^5 u_n(t)\|^2+\|\sqrt{\sigma'(D_x u_n)}\, D_x D_t^4 u_n(t)\|^2.$$

On the other hand we know already (see Prop. 3.1)

$$(4.2) \quad \int_t^{t+1}\|\sqrt{\sigma'(D_x u_n)}D_x D_t^4 u_n(s)\|^2 ds\leq c(\|u_0\|_{H_6})e_\lambda(t).$$

From (4.1) and (4.2) we can obtain by the argument used in §3

$$E_8(u_n(t))+\int_t^{t+1}\|\sqrt{\rho}\, D_x{}_t^5 u_n(s)\|^2 ds$$
$$\leq c(E_7(u_n(0)), \|u_0\|_{H6}, \|u_1\|_{H_6})e_\lambda(t).$$

Using the equations (3.24), (3.8), (3.1) and (2.1)′ we obtain also

$$\mathrm{E}_8(u_n(0))\leq c(\|u_0\|_{H_8}, \|u_1\|_{H_8})$$

q. e. d.

LEMMA 4.2.

$$\|D_x^2 D_t^3 u_n(t)\|^2+\|D_x^3 D_t^2 u_n(t)\|^2+\int_t^{t+1}\|D_x^3 D_t^3 u_n(s)\|^2 ds$$
$$\leq c(\|(u_0\|_{H_6}, \|u_1\|_{H_6})e_\lambda(t).$$

PROOF.

Multiplying (3.8) by $D_x^4 D_t^3 u_n$ we can obtain with the aid of Prop. 3.1

$$(4.3) \quad \int_t^{t+1}\|\sqrt{\rho}\, D_x^3 D_t^3 u_n(s)\|^2 ds\leq E_9(u_n(t))-E_9(u_n(t+1))$$
$$+c(\|u_0\|_{H_6}, \|u_1\|_{H_6})e_\lambda(t)$$

where

$$E_9(u_n(t)) = \|D_x^2 D_t^3 u_n(t)\|^2 + \|\sqrt{\sigma'(D_x u_n)} D_x^3 D_t^2 u_n(t)\|^2.$$

In the above procedure we need the assumption

$$\rho_{ttx}(0, t) = \rho_{ttx}(1, t) = 0.$$

Now we know

$$(4.4) \quad \int_t^{t+1} \|\sqrt{\sigma'(D_x u_n)} D_x^3 D_t^2 u_n(s)\|^2 ds \leqq c(\|u_0\|_{H_3}, \|u_1\|_{H_3}) \int_t^{t+1} \|D_x^3 D_t^2 u_n\|^2 ds$$
$$\leqq c(\|u_0\|_{H_6}, \|u_1\|_{H_6}) e_\lambda(t).$$

The inequalities (4.3) and (4.4) yield the Lemma.

q. e. d.

Finally we want to estimate $\|D_x^4 u_n'(t)\| + \|D_x^5 u_n(t)\|$. For this we prepare the following.

LEMMA 4.3.

$$\|D_x^5 u_n(t)\|^2 \leqq c(\|u_0\|_{H_6} : \|u_1\|_{H_6})(1+t)^{(1-\theta_0)/2}$$

PROOF.

Multiplying (2.1)$'$ by $D_x^8 u_n$ $(\in W_n)$ and calculating we have

$$(4.5) \quad \frac{d}{dt}\{\frac{1}{2}\|\sqrt{\rho}\, D_x^5 u_n(t)\|^2 - (D_x^3 u_n' D_x^5 u_n)\}$$
$$= \|D_x^4 u_n'(t)\|^2 + \int_0^1 (\frac{1}{2}\rho_t - \sigma'(D_x u_n))(D_x^5 u_n)^2 dx$$
$$-4(\rho_x D_x^4 u_n', u_n) - (g_8(t), D_x^5 u_n)$$

where

$$g_8(t) = \rho_{xxxx} D_x u_n' + 4\rho_{xxx} D_x^2 u_n' + 6\rho_{xx} D_x^3 u_n' + \sigma''''(D_x u_n) \cdot (D_x^2 u_n)^4$$
$$+ 6\sigma'''(D_x u_n) \cdot (D_x^2 u_n)^2 \cdot D_x^3 u_n + 3\sigma''(D_x u_n) \cdot (D_x^3 u_n)^2$$
$$+ 4\sigma''(D_x u_n) \cdot D_x^2 u_n \cdot D_x^4 u_n.$$

In the above we have used

$$\rho_{xxx}(0, t) = \rho_{xxx}(1, t) = 0.$$

Noting that the second term of the right hand side of (4.5) is nonnegative and integrating over $[0, t]$ of (4.5) we obtain from Proposition 3.1

$$\|\sqrt{\rho}\, D_x^5 u_n(t)\|^2 \leqq c(\|u_0\|_{H_6}, \|u_1\|_{H_6})(1 + \int_0^{t+1} \|\sqrt{\rho}\, D_x^5 u_n(s)\|^2 ds)^{1/2}$$

which implies (cf. Lemma 2.2)

$$\|D_x^5 u_n(t)\| \leq c(\|u_0\|_{H_6}, \|u_1\|_{H_6})(1+t)^{(1-\theta_0)/2}.$$

q. e. d.

Now we are in a position to prove our final lemma.

LEMMA 4.4.

$$\|D_x^5 u_n(t)\|^2 + \|D_x^4 u_n'(t)\|^2 + \int_t^{t+1} \|D_x^5 u_n'(s)\|^2 ds \leq c(\|u_0\|_{H_6}, \|u_1\|_{H_6}) e_\lambda(t),$$

$$\lambda > 0.$$

PROOF.

At this time we multiply (2.1)′ by $D_x^8 u_n'$ and arrange the equation to obtain (cf. Lemma 4.3)

$$(4.6) \quad \frac{1}{2}\frac{d}{dt}\{\|D_x^4 u_n'(t)\|^2 + \|\sqrt{\sigma'(D_x u_n)} D_x^5 u_n(t)\|^2\} + \|\sqrt{\rho}\, D_x^5 u_n'(t)\|^2$$

$$= \frac{1}{2}\int_0^1 \sigma''(D_x u_n) \cdot D_x^2 u_n \cdot (D_x^5 u_n)^2 dx - 4(\rho_x D_x^4 u_n', D_x^5 u_n')$$

$$- (g_8(t), D_x^5 u_n').$$

By Prop. 3.1 and Lemma 4.3 we see easily

$$\int_t^{t+1} |\{\text{the right hand side of } (4.6)\}| ds$$

$$\leq c(\|u_0\|_{H_6}, \|u_1\|_{H_6}) e_\lambda(t) + \frac{1}{2}\int_t^{t+1} \|\sqrt{\rho}\, D_x^5 u_n'(s)\|^2 ds$$

and hence

$$(4.7) \quad \int_t^{t+1} \|\sqrt{\rho}\, D_x^5 u_n'(s)^2 ds \leq E_{10}(u_n(t)) - E_{10}(u_n(t+1))$$

$$+ c(\|u_0\|_{H_6}, \|u_1\|_{H6}) e_\lambda(t)$$

$$\equiv Q_{10}^2(t)$$

where

$$E_{10}(u_n(t)) = \|D_x^4 u_k'(t)\|^2 + \|\sqrt{\sigma'(D_x u_n)} D_x^5 u_n(t)\|^2.$$

On the other hand, by (4.5) we have

$$\int_t^{t+1} \|\sqrt{\sigma'(D_x u_n)} D_x^5 u_n(s)\|^2 ds$$

$$= \int_t^{t+1} \|D_x^4 u_n(s)\|^2 ds - (D_x^4 D_t u_n(t+1)), D_x^4 u_n(t+1))$$

$$+ (D_x^4 D_t u_n(t), D_x^4 D_t u_n(t))$$

$$-\int_t^{t+1}\int_0^1 \rho D_x^5 u_n' \cdot D_x^5 u_n \, dxds - \int_t^{t+1} (g_8(t), D_x^5 u_n) ds$$
$$\leqq c(\|u_0\|_{H_6}, \|u_1\|_{H_6})(1+\max_{t\leqq s\leqq t+1} \sqrt{E_{10}(u_n(s))})e_\lambda(t)$$
$$+c(1+t)^{|\theta_0|}Q_{10}^2(t)+\frac{1}{2}\int_t^{t+1}\|\sqrt{\sigma'(D_x u_n)}D_x^5 u_n(s))\|^2 ds$$

and

$$\int_t^{t+1}\|\sqrt{\sigma'(D_x u_n)}D_x^5 u_n(s)\|^2 ds \tag{4.8}$$
$$\leqq c(\|u_0|_{H_6}, \|u_1\|_{H_6})(1+\max_{t\leqq s\leqq t+1} \sqrt{E_{10}(u_n(s))})e_\lambda(t)$$
$$+c(1+t)^{|\theta_0|}Q_{10}^2(t)$$

From (4.7) and (4.8) we obtain (see Lemma 2.1)

$$E_{10}(u_n(t))+\int_t^{t+1}\|\sqrt{\rho}\, D_x^5 u_n'(s)\|^2 ds$$
$$\leqq c(E_{10}(u_n(0)), \|u_0\|_{H_6}, \|u_1\|_{H_6})e_\lambda(t)$$

q. e. d.

Summarizing above estimates we have the following Proposition.

PROPOSITION 4.1

Under the assumption (A.1)-(A.4) *the approximate solutions* $\{u_n(t)\}$ *constructed through* (2.1)-(2.2)′ *satisfy*:

$$\sum_{i=1}^{5}\|D_t^i u_n(t)\|^2+\sum_{i=0}^{4}\|D_t^i u_n(t)\|_{H_1}^2+\sum_{i=0}^{3}\|D_t^i u_n(t)_{H_2}+\sum_{i=0}^{2}\|D_t^i u_n(t)_{H_3}^2$$
$$+\sum_{i=0}^{1}\|D_t^i u_n(t)\|_{H_4}^2+\|u_n(t)\|_{H_5}^2+\int_t^{t+1}\{\|D_x D_t^5 u_n(s)\|^2+\|D_x^3 D_t^3 u_n(s)\|^2$$
$$+\|D_x^5 u_n'(s)\|^2\}ds$$
$$\leqq c(\|u_0\|_{H_8}(\exp\{-\lambda t^{1-|\theta_0|}\}$$

for some $\lambda>0$ *independent of* n *and* (u_0, u_1).

§ 5. Existence and decay of solution

On the basis of Propositions 3.1 and 4.1 we shall show that a subsequence of $\{u_n(t)\}$ (in fact $\{u_n(t)\}$ itself) converges to a smooth function in various topologies and that the limit function $u(x, t)$ is the required classical solution of the original problem (0.1)-(0.2).

First we assume A_1 and A_2, and let $(u_0, u_1)\in V_6\times V_6$. We can use Proposition 3.1. Since H_i is compactly imbedded in C^{i-1} ([0.1]) ($i=1, 2, \cdots$) we see by Ascoli-Alzera's Lemma:

(5.1) $D_t^i u_n(x,t) \longrightarrow D_t^i u(x,t)$ uniformly in $[0,1]\times[0,T]$ $(T>0)$
for $i=0,1,2$,
$D_x D_t^i u_n(x,t) \longrightarrow D_x D_t^i u(x,t)$ uniformly in $[0,1]\times[0,\ T](T>0)$
for $i=0,1$,

and

$$D_x^2 u_n(x,t) \longrightarrow D_x^2 u(x,t) \text{ uniformly in } [0,1]\times[0,T]\ (T>0),$$

for a function $u(x,t)$.

Also, by Rellich's Lemma,

(5.2) $D_x^2 D_t u_n \longrightarrow D_x^2 D_t u_n$ strongly in $L^2{}_{loc}(R^+;L^2)$.

Thus taking the limit of $\{u_n(t)\}$ along the subsequence, we obtain from (2.1)

(5.3) $u_{tt}+(\rho(x,t)u_{xt})+(\sigma(u_x))_x=0$ a.e. on $[0,1]\times R^+$.

By (5.1) we see $u\in C(R^+;C([0,1]))\cap C^1(R^+;C([0,1]))$ and

(5.4) $u(x,0)=u_0(x)$ and $u_t(x,t)=u_1(x)$.

Moreover, by Prop. 3.1, we may assume

$$\begin{aligned} D_t^i u_n \longrightarrow D_t^i u \qquad & \text{weakly* in } L^\infty{}_{loc}(R^+;L^2),\ i=1,2,3,4,\\ & \text{weakly* in } L^\infty{}_{loc}(R^+;H_1),\ i=0,1,2,3,\\ & \text{weakly* in } L^\infty{}_{loc}(R^+;H_2),\ i=0,1,2,\\ & \text{weakly* in } L^\infty{}_{loc}(R^+;H_3),\ i=0,1,\\ & \text{weakly* in } L^\infty{}_{loc}(R^+;H_4),\ i=0 \end{aligned}$$

and

$$D_x^3 D_t^2 u_n \longrightarrow D_x^3 D_t^2 u_n \text{ weakly in } L^2{}_{loc}(R^+;L^2)$$

The function u satisfies of course the estimates of Prop. 3.1 with $u_n=u$ and hence we have by Sobolev's Lemma

(5.5) $u\in \bigcap_{i=0}^{3} C^{i+1/2}(R^+;H_{3-i})\subset \bigcap_{i=0}^{2} C^{i+1/2}(R^+;C^{2-i+1/2})\subset C^2([0,1]\times R^+)$

and

(5.6) $$\sup_{x\in[0,1]}\{|u_{xx}(x,t)|+|u_{xt}(x,t)|+|u_{tt}(x,t)|\} \leq c(\|u_0\|_{H_6}, \|u_1\|_{H_6})\exp\{-\lambda t^{1-|\theta_0|}\}.$$

Returning to the equation (5.3) and using the above facts we see

$$u_{xxt}\in C([0,1]\times R^+),\ \sup_{x\in[0,1]}|u_{xxt}(x,t)|\leq ce_\lambda(t)$$

and

$$D_t^i u \in L^\infty(R^+; H_{5-i}) \qquad (i=1,2,3).$$

The uniqueness of smooth solution of (0.1) – (0.2) follows from the usual way. Thus we have proved the following theorem.

THEOREM 5.1.

Let $(u_0, u_1) \in V_6 \times V_6$. *Then, under the assumptions* (A.1) *and* (A.2) *the problem* (0.1) – (0.2) *admits a unique solution* $u(x,t)$ *such that*

$$D_t^i u \in L^\infty(R^+; H_{4-i}) \ (i=0,4), \ D_t^i u \in L^\infty(R^+; H_{5-i}) \quad (i=1,2,3)$$

and

$$\sum_{i=0,4} \|D_t^i u(t)\|_{H_{4-i}} + \sum_{i=1}^{3} \|D_t^i u(t)\|_{H_{5-i}}$$
$$\leq c(\|u_0\|_{H_6}, \|u_1\|_{H_6}) \exp\{\lambda t^{1-|\theta_0|}\}$$

for some $\lambda > 0$. *In particular we see*

$$u \in C^2([0,1] \times R^+), \ u_{xxt} \in C([0,1] \times R^+)$$

and

$$\sup_{x \in [0,1]} \{|u_{xx}(x,t)| + |u_{xt}(x,t)| + x_{tt}(x,t) + u_{xxt}(x,t)|\}$$
$$\leq c(\|u_0\|_{H_6}, \|u_1\|_{H_6}) \exp\{-\lambda t^{1-|\theta_0|}\}, \quad \lambda > 0.$$

If we assume, in addition, (A.3) and $(u_0, u_1) \in V_8 \times V_8$ we can use propotion (4.1) to obtain the following result.

THEOREM 5.2.

Let $(u_0, u_1) \in V_8 \times V_8$. *Then, under* (A.1)-(A.3) *the solution* $u(x,t)$ *of* (0.1)-(0.2) *satisfies*:

$$\sum_{i=0,5} \|D_t^i u(t)\|_{H_{5-i}} + \sum_{i=1}^{4} \|D_t^i u(t)\|_{H_{6-i}}$$
$$\leq c(\|u_0\|_{H_8}, \|u_1\|_{H_8}) \exp\{-\lambda t^{1-|\theta_0|}\}$$

for some $\lambda > 0$. *In particular,*

$$u \in C^2([0,1] \times R^+), \ u_{xxt} \in C^1([0,1] \times R^+)$$

and

$$\sup_{x \in [0,1]} \{|u_{xxxt}(x,t)| + |u_{xxtt}(x,t)| + \sum_{k=0}^{3} \sum_{i+j=k} |\frac{\partial^k}{\partial x^i \partial t^j} u(x,t)|\}$$
$$\leq c(\|u_0\|_{H_8}, \|u_1\|_{H_8}) \exp\{-\lambda t^{1-|\theta_0|}\}, \quad \lambda > 0.$$

REMARK 5.1

By the proofs of above theorems it is clear that we can obtain a solution as smooth as wanted if we take (u_0, u_1) as elements of the closure of linear combinations of $\{\phi_n\}$ in H^N (N: large) and if ρ, σ are sufficiently smooth (ρ must satisfy in addition certain specific property at $x=0$ and $x=1$).

After completing this paper I knew the work by Y. Yamada [16] which proves the existence of golbal classical solutions under weaker assumption $\sigma'(s)\geqq 0$, and moreover, in the case $\sigma'(s)>0$, the exponential decay in the classical norm. However, he considers the case $\rho(x,t)\equiv$const.>0 and uses the abstract theory of semi-groups. Thus his method is quitely different from ours.

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