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https://doi.org/10.15017/1449024

出版情報：九州大学教養部数学雑誌. 12 (2), pp.93-102, 1980-12. 九州大学教養部数学教室

Math. Rep.
XII-2, 1980.

On a Certain Holomorphic Line Bundle over a Compact Non-Kähler Complex Manifold

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(Received August 30, 1980)

We consider a compact complex manifold X of complex dimension n and a holomorphic line bundle F over X whose refined Chern class is trivial. We denote by F^* the associated principal bundle over X with F. We shall show that there exist exhausting plurisubharmonic and strongly $(n+1)$-pseudoconvex functions Ψ and Ψ^*, respectively, on F and F^* in the part 3. Thus F and F^* are weakly 1-complete in the sense of [6] and strongly $(n+1)$-complete in the sense of [1]. Furthermore, we shall prove that

$$H^0(F, \mathcal{O}) \simeq H^0(C, \mathcal{O}) \text{ or } C$$

$$H^0(F^*, \mathcal{O}) \simeq H^0(C^*, \mathcal{O}) \text{ or } C$$

where the notations $\simeq$ imply isomorphisms as algebra. This is a generalization of the result [3] which was obtained in the case that X is a Kähler manifold. The Kähler condition is unnecessary in our proof. In the part 4 we shall construct an example in the non-Kähler case, using Hopf manifolds.

1. We recall refined Chern classes for holomorphic line bundles ([2] and [7]). We denote by X a complex manifold of complex dimension n throughout this paper. Let $\mathscr{E}^q$ be the sheaf of germs of real C^∞ q-forms on X and $\mathscr{E}^{p,q}$ the sheaf of germs of real C^∞ (p, q)-forms on X: Let $\mathscr{F}^{1,1}$: $:=\{f\in \mathscr{E}^{1,1}; df=0\}$. Let F be a holomorphic line bundle over X and $\{f_{ij}\}$ a holomorphic 1-cocycle defining F on the coordinate covering $\{U_i\}$ of X. For a metric $\{h_i\}$ along the fibres of F, we define a real (1, 1)-form

$$\omega(F):=\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log h_i \text{ on each } U_i.$$

Then $\omega(F)$ is well-defined on X. Thus

$$\omega(F)\in H^0(X, \mathscr{F}^{1,1}).$$

We have the homomorphism

$$\tilde{c}: H^0(X, \mathscr{O}^*)\longrightarrow H^0(X, \mathscr{F}^{1,1})/\sqrt{-1}\,\partial\bar{\partial}H^0(X, \mathscr{E}^0)$$

$$F\longrightarrow \tilde{c}(F) := [\omega(F)].$$

We call $\tilde{c}(F)$ the refined Chern class of F. Let $c(F)$ and $c(F)_c$ denote, respectively, the Chern class of F and the de Rham cohomology class of F. That is

$$H^1(X, \mathscr{O}^*)\longrightarrow H^2(X, \boldsymbol{Z})\longrightarrow H^2(X, \boldsymbol{C})$$

$$F\longrightarrow c(F) \longrightarrow c(F)_c,$$

where the homomorphisms $H^1(X, \mathscr{O}^*)\longrightarrow H^2(X, \boldsymbol{Z})$ and $H^2(X, \boldsymbol{Z})\longrightarrow H^2(X, \boldsymbol{C})$ are induced, respectively, by $0\longrightarrow \boldsymbol{Z}\longrightarrow \mathscr{O}\longrightarrow \mathscr{O}^*\longrightarrow 0$ and the natural inclusion $\boldsymbol{Z}\rightarrow\boldsymbol{C}$.

We have the following propositions.

PROPOSITION 1. $\tilde{c}(F)=0\Longrightarrow c(F)_c=0.$

PROOF. We need only remark

$$c(F)_c=[\omega(F)]\in H^2(X, \boldsymbol{C})=H^0(X, d\,\mathscr{E}^1)/dH^0(X, \mathscr{E}^1)$$

where $\mathscr{E}^p$ is the sheaf of germs of complex C^∞ p-forms on X.

PROPOSITION 2. *If X is a compact Kähler manifold. Then*

$$c(F)=0\Longrightarrow \tilde{c}(F)=0.$$

PROOF. Since $c(F)_c=0$, there exists $h\in H^0(X, \mathscr{E}^1)$ such that

$$\omega(F)=dh \quad \text{on } X.$$

From a theorem by Kodaira [5], we can find $f\in H^0(X, \mathscr{E}^0)$ such that

$$\omega(F)=\frac{\sqrt{-1}}{2}\partial\bar{\partial}f \quad \text{on } X.$$

Thus

$$\omega(F)=\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\Big(\frac{f+\bar{f}}{2}\Big)\in\sqrt{-1}\,\partial\bar{\partial}H^0(X, \mathscr{E}^0).$$

We have the following proposition by Propositions 1 and 2.

PROPOSITION 3. *If X is a compact Kähler manifold such that the homomorphism*

$$H^2(X, \boldsymbol{Z}) \longrightarrow H^2(X, \boldsymbol{C})$$

is injective. Then

$$c(F)=0 \Longleftrightarrow c(F)_c=0 \Longleftrightarrow \tilde{c}(F)=0.$$

2. Let U be an open set of X and P_U the additive group of C^∞ pluriharmonic functions in U. That is

$$P_U := \{f : U \longrightarrow \boldsymbol{R};\ f \in C^\infty(U) \text{ and } \partial\bar{\partial} f = 0\}.$$

The presheaf $\{U, P_U\}_{U \subset X}$ induces the sheaf $\mathscr{P}$ of C^∞ pluriharmonic functions on X. We have the exact sequence

$$0 \longrightarrow \boldsymbol{T} \longrightarrow \mathscr{O}^* \stackrel{\mu}{\longrightarrow} \mathscr{P} \longrightarrow 0,$$

where $\mu(f) := \log|f|$ and $\boldsymbol{T}$ denotes the sheaf of germs of constant functions with values in $\{z \in \boldsymbol{C}; |z|=1\}$. We denote by

$$\mu_1 : H^1 : (X, \mathscr{O}^*) \longrightarrow H^1(X, \mathscr{P})$$

the homomorphism induced by μ: $\mathscr{O}^* \longrightarrow \mathscr{P}$.

We have the following

PROPOSITION 4. $\operatorname{Ker} \mu_1 = \{F \in H^1(X, \mathscr{O}^*); \tilde{c}(F)=0\}$.

PROOF. Let $\{f_{ij}\}$ be a 1-cocycle defining $F \in \operatorname{Ker} \mu_1$ on the covering $\{U_i\}$ of X. We may assume that there exists a 0-cochain $\{g_i\} \in C^0(\{U_i\}, \mathscr{P})$ such that

$$\log|f_{ij}| = g_j - g_i \text{ on } U_i \cap U_j.$$

Thus $\{\exp 2g_i\}$ is a metric along the fibres of F. Then

$$\tilde{c}(F) = \left[\frac{\sqrt{-1}}{\pi} \partial\bar{\partial} g_i\right] = 0.$$

We take a holomorphic line bundle

$$L \in H^1(X, \mathscr{O}^*) \text{ with } \tilde{c}(L)=0.$$

Let $\{h_{ij}\}$ be a 1-cocycle defining L. We have a metric $\{h_i\}$ along the fibres of L and $f \in H^0(X, \mathscr{E}^0)$ such that

$$\partial\bar{\partial}\log h_i = \partial\bar{\partial} f \quad \text{on each } U_i.$$

Then we have

$$\log|h_{ij}| = \frac{1}{2}(\log h_j - f) - \frac{1}{2}(\log h_i - f)$$

and

$$\log h_i - f \in H^0(U_i, \mathscr{P}).$$

This implies $\mu_1(L)=0$.

COROLLARY 1. *Let X be compact and $F \in H^1(X, \mathcal{O}^*)$. Then F satisfies $\tilde{c}(F)=0$ if and only if there exists a 1-cocyle $\{f_{ij}\}$ defining F such that*

$$|f_{ij}|=1 \quad \textit{on each } U_i \cap U_j.$$

PROOF. We have an exact sequence

$$\cdots\cdots \longrightarrow H^0(X, \mathcal{O}^*) \xrightarrow{\mu_0} H^0(X, \mathscr{P}) \longrightarrow H^1(X, \boldsymbol{T}) \longrightarrow$$

$$H^1(X, \mathcal{O}^*) \xrightarrow{\mu_1} H^1(X, \mathscr{P}) \longrightarrow \cdots\cdots.$$

Since X is compact, $H^0(X, \mathcal{O}^*) \simeq \boldsymbol{C}^*$ and $H^0(X, \mathscr{P}) \simeq \boldsymbol{R}$. Then μ_0 is surjective. We get the exact sequence

$$0 \longrightarrow H^1(X, \boldsymbol{T}) \longrightarrow H^1(X, \mathcal{O}^*) \xrightarrow{\mu_1} H^1(X, \mathscr{P}).$$

Thus

$$H^1(X, \boldsymbol{T}) \simeq \operatorname{Ker}\mu_1 \subset H^1(X, \mathcal{O}^*).$$

From Proposition 2 and Corollary 1 we have the following

COROLLARY 2. *Let X be a compact Kähler manifold and $F \in H^1(X, \mathcal{O}^*)$ satisfying $c(F)=0$, then there exists a 1-cocycle $\{f_{ij}\}$ defining F such that*

$$|f_{ij}|=1 \textit{ on each } U_i \cap U_j.$$

3. A complex manifold X of complex dimension n is called strongly (resp. weakly) q-complete if there exists a real C^∞ exhausting function Ψ such that the Levi form of Ψ has at least $n-q+1$ positive (resp. nonnegative) eigenvalues at every point of X (cf. [1] and [6]).

Let X be a compact complex manifold and $F\in H^1(X, \mathcal{O}^*)$ with $\tilde{c}(F)=0$. We denote by F^* the associated principal bundle with F. Then $F^*\simeq F-\boldsymbol{0}$, where $\boldsymbol{0}$ denotes the zero section of F. By Corollary 1, we have a 1-cocycle $\{f_{ij}\}$ defining F with $|f_{ij}|=1$ on each $U_i\cap U_j$. Let $z_i(p)$ denotes the fibre coordinate of $p\in\pi^{-1}(U_i)\simeq U_i\times \boldsymbol{C}$ where π is the projection of F onto X. We set

$$\Psi:\pi^{-1}(U_i)\ni p\longmapsto |z_i(p)|^2\in \boldsymbol{R}$$

and

$$\Psi^*:\pi_*^{-1}(U_i)\ni p\longmapsto |z_i(p)|^2+\frac{1}{|z_i(p)|^2}\in \boldsymbol{R},$$

for each U_i where π_* denotes the projection of F^* onto X. Since $|f_{ij}|=1$, Ψ and Ψ^* are well-defined, respectively, on F and F^*.

LEMMA 1. *Let X be a compact complex manifold, $F\in H^1(X, \mathcal{O}^*)$ with $\tilde{c}(F)=0$ and F^* the principal $\boldsymbol{C}^*$ bundle associated with F. Then F and F^* are weakly 1-complete and strongly $(n+1)$-complete.*

PROOF. The Levi forms of Ψ and Ψ^* are given by

$$L(\Psi)=dz_i d\bar{z}_i \quad\text{and}\quad L(\Psi^*)=\left(\frac{1}{|z_i|^4}+1\right)dz_i d\bar{z}_i$$

for each U_i, respectively. Clearly we have

$$\{p\in F; \Psi(p)<c\}\Subset F$$

and

$$\{p^*\in F^*; \Psi^*(p)<c\}\Subset F^*$$

for any $c\in \boldsymbol{R}$.

LEMMA 2. *Let X be a connected compact complex manifold and $F\in H^1(X, \mathcal{O}^*)$ with $\tilde{c}(F)=0$. Then there exists a non-constant holomorphic function on*

$$\Delta_{r_1,r_2}:=\bigcup_i \{p\in\pi^{-1}(U_i); r_1<|z_i(p)|<r_2\}$$

for some $r_1, r_2(-\infty\leqq r_1<r_2\leqq+\infty)$, if and only if F^l is holomorphically trivial for some positive integer l.

PROOF. Let h be a non-constant holomorphic function on Δ_{r_1,r_2}.

We have the Laurent expansion of $h|\pi^{-1}(U_i)\cap\Delta_{r_1,r_2}$:

$$h(p)=\sum_{\nu=-\infty}^{\infty}a_\nu^i(\pi(p))z_i^\nu(p),\ p\in\pi^{-1}(U_i)\cap\Delta_{r_1,r_2}$$

where a_ν^i are holomorphic in U_i. Since

$$z_i(p)=f_{ij}(\pi(p))z_j(p),\ p\in\pi^{-1}(U_i\cap U_j),$$

we have

$$\begin{aligned}h(p)&=\sum_{\nu=-\infty}^{\infty}a_\nu^i(\pi(p))z_i^\nu(p)\\&=\sum_{\nu=-\infty}^{\infty}a_\nu^i(\pi(p))f_{ij}^\nu(\pi(p))z_j^\nu(p)\\&=\sum_{\nu=-\infty}^{\infty}a_\nu^j(\pi(p))z_j^\nu(p),\quad p\in\pi^{-1}(U_i\cap U_j).\end{aligned}$$

Then

$$a_\nu^i(\pi(p))=f_{ij}^{-\nu}(\pi(p))a_\nu^j(\pi(p)),\ p\in\pi^{-1}(U_i\cap U_j)$$

and hence

$$\{a_\nu^i(x)\}\in H^0(X,\underline{F}^{-\nu}),\quad x\in X,$$

where $\underline{F}$ denotes the sheaf of germs of holomorphic sections of F over X. We put

$$\phi_\nu: U_i\ni x\longmapsto|a_\nu^i(x)|\in\boldsymbol{R}$$

on each U_i. Since $|f_{ij}|=1$, ϕ_ν are continuous plurisubharmonic functions on X. Since X is compact and connected, there exist $r_\nu, \theta_\nu^i\in\boldsymbol{R}$ such that

$$a_\nu^i(x)\equiv r_\nu\exp(\sqrt{-1}\,\theta_\nu^i),\ x\in U_i$$

for any i. Now h is non-constant, so there exists $\nu_0\neq0$ such that $r_{\nu_0}\neq0$. This means that $F^{-\nu_0}$ has a nowhere vanishing global section $\{a_{\nu_0}^i\}$ on X.

Suppose that F^l is holomorphically trivial for some $l>0$. Since

$$H^0(F^l,\ \mathcal{O})\simeq H^0(X\times\boldsymbol{C},\ \mathcal{O})\simeq H^0(\boldsymbol{C},\ \mathcal{O}).$$

we have a non-constant holomorphic function $g\in H^0(F^l,\ \mathcal{O})$. We take a holomorphic and surjective mapping

$$\begin{aligned}&\alpha: F\longrightarrow F^l\\&(\pi(p), z_i(p))\longmapsto(\pi(p), z_i^l(p)).\end{aligned}$$

Then $g\circ\alpha$ is non-constant and holomorphic on F.

THEOREM 1. *Let $\pi: F\longrightarrow X$ be a holomorphic line bundle with $\tilde{c}(F)$*

$=0$ *on a compact complex manifold* X *of complex dimension* n. *Let* $(\{U_i\}, \{f_{ij}\})$ *be a* 1*-cocycle with* $|f_{ij}|=1$ *defining* F. *Then* F *and the associated principal* C^* *bundle* $\pi_*: F^* \longrightarrow X$ *admit* C^∞ *plurisubharmonic and strongly* $(n+1)$*-pseudoconvex exhaustion functions*

$$\Psi: \pi^{-1}(U_i) \ni p \longmapsto |z_i(p)|^2 \in \boldsymbol{R}$$

and

$$\Psi^*: \pi_*^{-1}(U_i) \ni p \longmapsto |z_i(p)|^2 + \frac{1}{|z_i(p)|^2} \in \boldsymbol{R},$$

respectively. Moreover the following conditions are equivalent:

(a) F^l *is holomorphically trivial for some non-zero integer* l.

(b) *There exists a non-constant holomorphic function in*

$$F_c := \{p \in F;\ \Psi(p) < c\} \text{ for some } c > 0.$$

(c) *There exists a non-constant holomorphic function in*

$$F_c^* := \{p \in F^*;\ \Psi^*(p) < c\} \text{ for some } c > \min_{F^*} \Psi^*.$$

PROOF. For any c, we have $r_1 > r_2$ such that

$$\Delta_{r_1, r_2} \Subset F_c^* \subset F_c.$$

Hence by Lemma 1 and Lemma 2, we get the proof of Theorem 1.

PROPOSITION 3. *Let* X, F *be as Theorem* 1. *Assume that* F^l *is not holomorphically trivial for any non-zero integer* l. *If* D *is an open set in* F *and*

$$\{p \in F;\ \Psi(p) = c_0\} \subset D$$

for some $c_0 > 0$, *then*

$$H^0(D, \mathcal{O}) = \boldsymbol{C}.$$

PROOF. We have $r_1 < \sqrt{c_0} < r_2$ such that

$$\Delta_{r_1, r_2} \subset D.$$

Then the result follows by Lemma 2.

THEOREM 2. *Let* X, F *be as in Theorem* 1. *Then one of the* *followings occurs.*

(1) $H^0(F, \mathcal{O})\simeq H^0(\boldsymbol{C}, \mathcal{O})$ *(isomorphic as algebra), and* F^l *is holomorphically trivial for some non-zero integer* l.

(2) $H^0(F, \mathcal{O})\simeq \boldsymbol{C}$ *(isomorphic as algebra), and* F^l *is not holomorphically trivial for any non-zero integer* l.

PROOF. Suppose that for some positive integer l, F^k is not holomorphically trivial for $0<k<l$ and that F^l is holomorphically trivial. Let

$$\alpha: F \longrightarrow F^l$$
$$(\pi(p), z_i(p)) \longmapsto (\pi(p), z_i^l(p)).$$

Then α is holomorphic and surjective. α induces the homomorphism

$$\alpha: H^0(F^l, \mathcal{O}) \longrightarrow H^0(F, \mathcal{O})$$
$$\{\sum_{\nu=0}^{\infty} a_\nu^i w_i^\nu\} \longmapsto \{\sum_{\nu=0}^{\infty} a_\nu^i z_i^{l\nu}\},$$

where w_i denotes the fibre coordinate in $F^l|U_i$ and a_ν^i are holomorphic in U_i. Clearly α is injective. We take a holomorphic function $f\in H^0(F, \mathcal{O})$. We have the Taylor expansion

$$f|\pi^{-1}(U_i)(p)=\sum_{\mu=0}^{\infty} b_\mu^i(\pi(p))z_i^\mu(p), \quad p\in\pi^{-1}(U_i).$$

Similary as the proof of Lemma 2, it follows that $|b_\mu^i|$ is constant on U_i for any (i, μ) and $b_\mu^i=0$ as $\mu\not\equiv 0$ mod l. Then

$$f|\pi^{-1}(U_i)(p)=\sum_{\nu=0}^{\infty} b_{k\nu}^i z_i^{l\nu}(p), \quad p\in\pi^{-1}(U_i).$$

Setting

$$g:=\{\sum_{\nu=0}^{\infty} b_{l\nu}^i w_i^\nu\},$$

we have $g\in H^0(F^l, \mathcal{O})$ and $\alpha(g)=f$. Since $F^l\simeq X\times \boldsymbol{C}$,

$$H^0(F, \mathcal{O})\simeq H^0(F^l, \mathcal{O})\simeq H^0(\boldsymbol{C}, \mathcal{O}).$$

If F^l is not holomorphically trivial for any non-zero integer l, we get $H^0(F, \mathcal{O})\simeq \boldsymbol{C}$ by Lemma 2.

Similary as the proof of Theorem 2 we have the following:

COROLLARY 3. *Let* X, F *be as in Theorem* 1 *and* F^* *denote the associated principal bundle with* F. *Then* $H^0(F^*, \mathcal{O})$ *is isomorphic onto* $H^0(\boldsymbol{C}^*, \mathcal{O})$ *or* $\boldsymbol{C}$ *as algebra.*

4. It is well-known that there exists a compact non-Kähler complex manifold of complex dimension $q \geqq 2$ which is homeomorphic onto $S^1 \times S^{2p+1}, p = q-1$. Such manifolds are called Hopf manifolds. We shall construct an example in the non-Kähler case, using Hopf manifolds. Let M be a Hopf manifold of complex dimension $p+1, p \geqq 1$. We have the exact sequence

$$0 \longrightarrow H^1(M, \boldsymbol{T}) \longrightarrow H^1(M, \mathcal{O}^*) \xrightarrow{\mu_1} H^1(M, \mathcal{P}) \rightarrow \cdots\cdots$$

and

$$\{F \in H^1(M, \mathcal{O}^*); \tilde{c}(F) = 0\} \simeq H^1(M, \boldsymbol{T})$$

as in the part 2. Since M is homeomorphic onto $S^1 \times S^{2p+1}$, we have the isomorphism

$$\beta: \boldsymbol{T} \simeq H^1(M, \boldsymbol{T}).$$

Let $\exp \sqrt{-1}\, 2\pi\theta \in \boldsymbol{T}$ and $F(\theta)$ be the line bundle defined by the cocycle $\beta(\exp \sqrt{-1}\, 2\pi\theta) \in H^1(M, \boldsymbol{T}) \subset H^1(M, \mathcal{O}^*)$. Then we have the following

THEOREM 3. *One of the following occurs.*

(1) *θ is rational and* $H^0(F(\theta), \mathcal{O}) \simeq H^0(\boldsymbol{C}, \mathcal{O})$

(2) *θ is irrational and* $H^0(F(\theta), \mathcal{O}) \simeq \boldsymbol{C}$.

REMARK 1. Theorem 3 shows a non-trivial example in the non-Kähler case with respect to the argument in the part 3.

REMARK 2. From the work of Ise [4], we have $H^1(M, \mathcal{O}^*) \simeq \boldsymbol{C}^*$. Since $H^2(M, \boldsymbol{Z}) \simeq 0$, we obtain

$$\{F \in H^1(M, \mathcal{O}^*); c(F) = 0\} = H^1(M, \mathcal{O}^*) \simeq \boldsymbol{C}^*$$
$$\supsetneq \{F \in H^1(M, \mathcal{O}^*); \tilde{c}(F) = 0\} \simeq \boldsymbol{T}.$$

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