Existence of classical periodic solutions of some nonlinear wave equations in one space dimension

Nakao, Mitsuhiro
Department of Mathematics, College of General Education, Kyushu University

https://doi.org/10.15017/1449023

出版情報：九州大学教養部数学雑誌. 12 (2), pp.77-91, 1980-12. 九州大学教養部数学教室

Math. Rep.
XII-2, 1980.

Existence of classical periodic solutions of some nonlinear wave equations in one space dimension

Mitsuhiro NAKAO

(Received August 27, 1980)

§ 0. Introduction

In this paper we are concerned with the existence of periodic solutions of the nonlinear dissipative wave equations of the form

$$(0.1) \qquad u_{tt} - u_{xx} + \rho_0(x, u_t) + \rho_1(x, u, u_t) + \beta(x, u) = f(x, t)$$

on $[0, 1] \times R$ with the boundary and periodicity conditions

$$(0.2) \qquad u(0, t) = u(1, t) = 0, \quad u(x, t+\omega) = u(x, t) \quad (\omega > 0).$$

Here ρ_0, ρ_1 and β are smooth functions with certain monotonicity and growth properties, and $f(x, t)$ is a smooth function with period ω in t.
In our argument the term $\rho_0(x, u_t)$ (dissipative term) plays an essential role, which is the reason why we call the equation (0.1) 'dissipative' wave equation.

Periodic solutions of the nonlinear wave equations have been studied by a number of authors. In particular, as for the dissipative equations, we refer to Ficken-Fleishman [4], Prodi [11], Rabinowitz [12], Amerio-Prouse [1], Biroli [2], Clements [3], Krylova [6], wahl [13], Kakita [5], Yamaguchi [14] and the author [8, 10] etc.. However most of papers have treated the generalized (weak) solutions, and there are not so many works concerning classical solutions. Rabinowitz [12], Wahl [13], Yamaguchi [14] and the author [10] considered the classical periodic solutions. In them, however, it is assumed that the nonlinear term contains small parameter or the forcing term $f(x, t)$ is small.

The object of this paper is to prove the existence of classical solution of the problem (0.1)-(0.2) with no smallness conditions on the data. For this we must make assumptions of certain monotonicity and growth conditions on nonlinear term, which seems to be a natural restriction.

At this time we restrict ourselves to one dimensional equations. But it seems to be very interesting to extend our result to higher dimensional equations.

We employ a Galerkin method. In particular our method is related to [9], where decay property of classical solutions of similar equations was investigated.

§ 1. Preliminaries

The functions considered are all real valued. We use standard notation for familiar function spaces (Lions [7]). In particular, for measurable function f and g on $[0, 1]$, we set

$$(f, g)=\int_0^1 f(x)g(x)dx$$

if the right hand side is convergent. Let X be a (real) Banach space. Then, for $1\leqq p\leqq\infty$, we denote by $L^p(\omega;X)$ the Banach space of X-valued strongly measurable function on R with period ω and with

$$\|f\|_{L^p(\omega:X)}\equiv(\int_\omega\|f(s)\|_X^p ds)^{1/p}<\infty \quad \text{if } 1\leqq p<\infty$$

and

$$\|f\|_{L^\infty(\omega:X)}\equiv\text{ess.}\sup_\omega\|f(s)\|_X<\infty \quad \text{if } p=\infty.$$

Hereafter $\int_\omega$ and $\sup_\omega$ etc. should be taken over any interval with length ω. We denote by $C^i(\omega;x)$ the space of X-valued i-times continuously differentiable functions on R with period ω.

Our argument depends heavily on the following well-known lemma.

LEMMA 1.1. *The sobolev space $H_1\equiv H_1([0, 1))$ is imbedded continuously and compactly into the space $C^{1/2}\equiv C^{1/2}([0, 1])$, and we have*

(1.1) $\quad |u|_{C^{1/2}}\leqq\text{const.}\|u\|_{H_1} \quad$ *for $u\in H_1$.*

Moreover, for ${}^\forall\varepsilon>0$, there exists $C_\varepsilon>0$ such that

(1.2) $\quad \|u\|_{L^\infty}\leqq\varepsilon\|u\|_{\mathring{H}_1}+C_\varepsilon\|u\|_{L^2}$ *for $u\in \mathring{H}_1$*

where $\|u\|_{\mathring{H}_1}=\|\frac{\partial}{\partial x}u\|_{L^2}$, Now, we state our assumptions.

A_1. $\rho_0(x, v)$ belongs to $C^3([0,1]\times R)$ and satisfies the following conditions:

(i) $\rho_0(x, 0)\equiv 0$,

(ii) $\frac{\partial}{\partial v}\rho_0(x, v)\geqq k_0>0$ for some constant $k_0>0$,

(iii) there exists some $K>0$ and $C_K>0$ such that

$$(1.3)\qquad \frac{\partial}{\partial v}\rho_0(x, v)\leqq C_K(|\rho_0(x, v)|+1)|v| \quad \text{if } |v|\geqq K.$$

A_2. $\beta(x, u)$ belongs to $C^3([0,1]\times R)$ and there exists $k_1>0$ such that

$$(1.4)\qquad \beta(x, u)\,u\geqq k_1\int_0^u \beta(x, \eta)d\eta)\geqq 0.$$

A_3. $\rho_1(x, u, v)$ belongs to $C^3([0,1]\times R^2)$ and satisfies

$$(1.5)\qquad \left|\frac{\partial}{\partial u}\rho_1(x, u, v)\right|\leqq C(|u|)(\sqrt{\rho_0(x, v)v}+1)$$

$$(1.6)\qquad 0\leqq\frac{\partial}{\partial v}\rho_1(x, u, v)\leqq C(|u|)(\rho_0(x, v)v+1)$$

where $C(\eta)$ depends on η continuously. Moreover $\rho_1(x, u, v)=\rho_{11}(x, u, v)+$ $+\rho_{12}(x, u, v)$ with ρ_{11}, ρ_{12} satisfying $\frac{\partial}{\partial v}\rho_{11}\geqq 0, \frac{\partial}{\partial v}\rho_{12}\geqq 0$ and

(i) there exist $K>0$ and $C_K>0$ such that

$$(1.7)\qquad \sup_{|v|\leqq K}|\rho_{11}(x, u, v)|\leqq C_K(\beta(x, u)u+|u|^2+1)^{\theta/2}$$

with $0<^{\exists}\theta<1$,

(ii) $\rho_{12}\equiv 0$

or

$$(1.8)\qquad k_2\sum_{i=1}^m |u|^{\alpha_i}|v|^{\gamma_i+1}\leqq\rho_{12}(x, u, v)\,v\leqq k_3\sum_{i=1}^m |u|^{\alpha_i}|v|^{\gamma_i+1}$$

for some $k_2>0, k_3>0, \gamma_i\geqq 0$ and α_i with $0\leqq\alpha_i<\gamma_i+1 (i=1, 2, \cdots\cdots, m)$.

Finally, concerning the forcing term f, we assume:

A_4. $\frac{\partial^i}{\partial t^i}f\in L^2(\omega; L^2([0,1]))$ $(i=0, 1, 2, 3)$

and

$$\frac{\partial^i}{\partial t^i}f\in L^\infty(\omega; H^{2-i}([0,1]))\quad (i=0, 1, 2).$$

We ofen use D_t and D_x for $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x}$, respectively, and set

$\delta_i = \|D_t^i f\|_{L^2(\omega; L^2)}$ and $\nu_i = \|D_t^i f\|_{L^\infty(\omega; H^{2-i})}$
We enclose this section by giving typical examples.

example 1.

$$u_{tt} - u_{xx} + u_t + \{(\sin u)^2 + u^2\} u_t + u^5 = f(x, t).$$

example 2.

$$u_{tt} - u_{xx} + \sum_{k=0}^{m} C_k u_t^{2k+1} + \sum_{k=0}^{m} d_k u^{2k} u_t^{2k+1} = f(x, t)$$

with $C_0 > 0$, $C_1, C_2, \cdots\cdots, C_{m-1} \geqq 0$, $C_m > 0$ and $d_i \geqq 0$ $(i = 0, 1, \cdots\cdots, m)$.

example 3.

$$u_{tt} - u_{xx} + u_t \exp\{u_t^2\} + |u|^{2m} u_t \exp\{u_t\} + u \exp\{u^3\} = f(x, t).$$

§ 2. Approximate solutions

Let $\{\phi\}_{k=1}^{\infty}$ be a basis of $L^2([0, 1])$. We may assume without loss of generality that $\phi_k \in C_0^\infty((0, 1))$ $(k = 1, 2, \cdots\cdots)$. First we shall show that the following equation admits an ω-periodic solution:

$$(2.1) \qquad (D_t^2 u_n(t), \phi_k) + (D_x u_n(t), D_x \phi_k) + (\rho_0(D_t u_n) + \beta(u_n) + \rho_1(u_n, D_t u_n), \phi_k) = (f(t), \phi_k) \quad (k = 1, 2, \cdots\cdots, n)$$

where $u_n(t) = \sum_{k=1}^{n} \alpha_{k,n}(t) \phi_k(x)$ and we use simplified natation $\rho_0(x, D_t u_n(x, t)) \equiv \rho_0(D_t u_n)$, $\beta(x, u_n(x, t)) \equiv \beta(u_n)$ and $\rho_1(x, u_n(x, t), D_t u_n(x, t)) \equiv \rho_1(u_n, D_t u_n)$ etc.. The unknown function $u_n(t)$ should be sought in the space $X_n = C^2(\omega; W_n)$, where W_n is the n-dimensional subspace of L^2 spanned by $\phi_1, \cdots \cdots, \phi_n$. For our purpose let us consider the linear equation

$$(2.2) \qquad (D_t^2 u_n(t), \phi_k) + (D_x u_n(t), D_x \phi_k) + \delta(D_t u_n(t), \phi_k) = (F(t), \phi_k) \quad (\delta > 0, k = 1, 2, \cdots\cdots, n).$$

This equation can be regarded as a system of ordinary differential equations and, if $F \in C(\omega; L^2)$, has a unique ω-periodic solution, which we denote by $G_n(F)(t)$. $G_n(\cdot)$ is a linear bounded operator from $C(\omega; L^2)$ to X_n,

Next, setting

$$F_n(t) = -\{\rho_0(D_t u_n) + \beta(u_n) + \rho_1(u_n, D_t u_n)\} + \delta D_t u_n + f(t),$$

we consider the equation

$$(2.3) \quad (D_t^2 u_n(t), \phi_k) + (D_x u_n(t), D_x \phi_k) + \delta(D_t u_n, \phi_k) = \lambda(F_n(t), \phi_k), \ 0 \leqq \lambda \leqq 1 \quad (k=1, 2, \cdots\cdots, n)$$

with $u_n(t) = u_n(t+\omega)$. This is equivalent to

$$(2.4) \quad u_n(t) = T_n(\lambda)(u_n)(t)$$

where

$$T_n(\lambda)(u_n)(t) = \lambda G_n(F_n)(t).$$

Our problem (2.1) is of course equivalent to (2.4) with $\lambda=1$. For $r>0$ we set

$$B_n(r) = \{u \in X_n \,|\, \|u\|_{X_n} \leqq r\}.$$

Then, it is easy to see that $T_n(\lambda)$ is a continuous and compact operator from $[0, 1] \times B_n(r)$ *to* X_n. Since (2.4) with $\lambda=0$ has a unique solution $u \equiv 0$, Leray-Shauder's degree theory implies that it suffices for the existence of a solution of (2.1) to show an a priori estimate of $\|u_n(t)\|_{X_n}$ independent of λ to the solutions u_n of (2.3). Indeed we can show the following lemma.

LEMMA 2.1.

Let $u_n(t)$ be a solution of the problem (2.3) $(0 \leqq \lambda \leqq 1)$. Then we have

$$(2.5) \quad \|u_n(t)\|_E^2 \equiv \|u_n(t)\|_{L^2}^2 + \|u_n(t)\|_{\mathring{H}^1}^2 \leqq C_0(\delta_0) < \infty$$

where $C_0(\delta_0)$ is a constant independent of n and λ (but depends on δ_0).

PROOF.

Let $u_n(t) = \sum_{j=1}^{n} \alpha_{j,n}(t) \phi_j(x)$. Multiplying (2.3) by $\alpha_{j,n}(t)$, summing up from $j=1$ to n and integrating over $[0, \omega]$ (roughly speaking, multiplying $D_t u_n$) we have

$$\begin{aligned} &\int_\omega \int_0^1 \{\rho_0(D_t u_n) D_t u_n + \rho_1(u_n D_t u_n) D_t u_n\} \, dxdt \\ &\leqq \int_\omega \int_0^1 |f(t) D_t u_n| \, dxdt \\ &\leqq \frac{1}{2k_0} \delta_0^2 + \frac{k_0}{2} \int_\omega \|D_t u_n(t)\|_{L^2}^2 dt \end{aligned}$$

and hence, by the assumption A_1,

$$(2.6)\qquad \int_\omega\int_0^1 \{\rho_0(D_t u_n)+\rho_1(u_n, D_t u_n)\} D_t u_n dxds \leq C(\delta_0).$$

Next, multiplying (2.3) by $\alpha_{j,n}(t)$, summing up from $j=1$ to n and integrating (roughly speaking, multiplying u_n) we have

$$\begin{aligned}(2.7)\qquad &\int_\omega\int_0^1 \{|D_x u_n|^2+\lambda\beta(u_n)u_n\} dxdt \\ &\leq \lambda\int_\omega \|f(t)\|_{L^2}\|u_n(t)\|_{L^2}dt+\lambda\int_\omega^1 |\rho_0(D_t u_n)\cdot u_n|\, dxds \\ &\quad +\lambda\int_\omega\int_0^1 |\rho_1(u_n, D_t u_n)\cdot u_n|\, dxds \\ &\leq \frac{1}{2}\int_\omega \|D_x u_n(t)\|_{L^2}^2 dt+C\delta_0^2+I_1+I_2.\end{aligned}$$

Now

$$\begin{aligned}(2.8)\qquad I_1 &\leq \iint_{\{(x,t)||D_t u_n(x,t)|\leq 1\}} |\rho_0(x, D_t u_n(x,t))u_n(x,t)\cdot u_n|\, dxdt \\ &\quad + \iint_{\{(x,t)||D_t u_n(x,t)|\geq 1\}} |\rho_0(x, D_t u_n(x,t))|\,|D_t u_n|\,|D_t u_n|^{-1}|u_n|\, dxdt \\ &\leq C\int_\omega\int_0^1 |u_n|\, dxdt+\int_\omega\int_0^1 \rho_0(D_t u_n)D_t u_n dxdt \ \sup_\omega\|u_n(t)\|_{L^\infty} \\ &\leq C(\int_\omega \|u(_n(t)\|_{L^2}^2 dt)^{1/2}+C(\delta_0)\ \sup_\omega\|u_n(t)\|_{\mathring{H}^1}.\end{aligned}$$

And also, by A_3,

$$\begin{aligned}(2.9)\qquad I_2 &\leq \lambda \iint_{\{(x,t)||D_t u_n(x,t)|\leq k\}} |\rho_{11}(u_n, D_t u_n)\cdot u_n|\, dxdt \\ &\quad + \iint_{\{(x,t)|\,|D_t u_n(x,t)|\geq k\}} |\rho_{11}(u_n, D_t u_n)|\,|D_t u_n|\,|D_t u_n|^{-1}|u_n|\, dxdt \\ &\quad + k_3\sum_{i=0}^m \int_\omega\int_0^1 |u_n|^{\alpha_i+1}|D_t u_n|^{\gamma_i} dxdt \\ &\leq C_K\lambda\int_\omega\int_0^1 (\beta(u_n)u_n+|u_n|^2+1)^{\theta/2} dxdt\ \sup_\omega\|u_n(t)\|_{L^\infty} \\ &\quad +K^{-1}\int_\omega\int_0^1 \rho_{11}(u_n, D_t u_n)D_t u_n dxdt\ \sup_\omega\|u_n(t)\|_{L^\infty} \\ &\quad +k_3\sum_{i=1}^m (\int_\omega\int_0^1 |D_t u_n|^{\gamma_i+1}|u_n|^{\alpha_i} dxdt)^{\gamma_i/(\gamma_i+1)}\sup_\omega\|u_n(t)\|_{L^\infty}^{1+\alpha_i/(1+\gamma_i)} \\ &\leq \frac{1}{4}\int_\omega\int_0^1 \{|D_x u_n|^2+\lambda\beta(u_n)u_n\} dxdt+C(\delta_0)(1+\sup_\omega\|u_n(t)\|_{L^\infty}^2),\end{aligned}$$

where we have used the assumptions on θ and α_i. From (2.7)-(2.9) and Lemma 1.1 we have, for ${}^{\forall}\varepsilon>0$, ${}^{\forall}\lambda\in[0,1]$,

$$(2.10)\quad \int_\omega\int_0^1\{|D_x u_n|^2+\lambda\beta(u_n)u_n\}dxdt\leqq\varepsilon\sup_\omega\|u_n(t)\|^2_{\mathring{H}1}+C(\varepsilon,\delta_0).$$

From (2.6) and (2.10) we see that there exists $t^*\in[0,\omega]$ such that

$$(2.11)\quad E_\lambda(u_n(t^*))\equiv\|u_n(t^*)\|^2_E+2\lambda\int_\omega\int_0^1\beta(u_n(t^*))u_n(t^*)dxdt$$
$$\leqq\varepsilon\sup_\omega\|u_n(t)\|^2_{\mathring{H}1}+C(\varepsilon,\delta_0)\,({}^{\forall}\varepsilon>0)$$

and hence, by (2.6) and (2.11),

$$\sup_\omega E_\lambda(u_n(t))\leqq E_\lambda(u_n(t^*))+2\int_\omega\int_0^1(|\rho_0(D_t u_n)|+|\rho_1(u_n,D_t u_n)|+|f(t)|)\times$$
$$\times|D_t u_n|dxdt$$
$$\leqq\varepsilon\sup_\omega\|u_n(t)\|^2_{\mathring{H}1}+C(\varepsilon,\delta_0)$$

which implies with $\varepsilon=\dfrac{1}{2}$

$$E_\lambda(u_n(t))\leqq C(\delta_0)<\infty\qquad\text{for } {}^{\forall}t.$$

q. e. d.

Lemma 2.1 and the equation (2.3) imply immediately

$$\|D^2u_n(t)\|_{L^2}\leqq\text{const.}<\infty$$

and hence we conclude that for any positive integer n the finite dimensional problem (2.1) has an ω-periodic solution $u_n(t)\in C^2(\omega;W_n)$.

§ 3. Furthere stimations of approximate solutions

Starting from the estimate (2.5) we proceed to the further estimations of the ω-periodic solution $u_n(t)$ of the problem (2.1) whose existence is already assured in the previous section. Hereafter we denote by $C(\eta)$ etc. various positive constants which depend on η and other known constants, but does not depend on n.

LEMMA 3.1.
We have

$$(3.1)\quad \|D_t u_n(t)\|_E\leqq C(\delta_0,\delta_1)<\infty$$

where we recall $\|D_t u_n(t)\|_E^2 \equiv \|D_t^2 u_n(t)\|_{L^2}^2 + \|D_t u_n(t)\|_{\mathring{H}^1}^2$.

PROOF.

Differentiation of the equation (2.1), which is possible by the smoothness assumptions on β, ρ_0, ρ_1 and f, gives

$$(3.2)\qquad (D_t^3 u_n, \phi_k) + (D_x D_t u_n, \phi_k) + (\frac{\partial}{\partial v}\rho_0(D_t u_n)\cdot D_t^2 u_n, \phi_k)$$
$$+ (\frac{\partial}{\partial u}\beta(u_n)\cdot D_t u_n, \phi_k) + (\frac{\partial}{\partial u}\rho_1(u_n, D_t u_n)\cdot D_t u_n, \phi_k)$$
$$+ (\frac{\partial}{\partial v}\rho_1(u_n, D_t u_n)\cdot D_t^2 u_n, \phi_k) = (D_t f, \phi_k),\ k=1, 2, \cdots\cdots, n.$$

Multiplying (3.2) by $D_t^2 u_n$ we have

$$\int_\omega\int_0^1 \{\frac{\partial}{\partial v}\rho_0(D_t u_n) + \frac{\partial}{\partial v}\rho_1(u_n, D_t u_n)\}\,|D^2 u_n|^2 dxdt$$
$$\leqq \int_\omega\int_0^1 \{\frac{\partial}{\partial u}\beta(u_n)\cdot D_t u_n| + |D_t f|\}\,|D_t^2 u_n|\,dxdt$$
$$+ \int_\omega\int_0^1 |\frac{\partial}{\partial u}\rho_1(u_n, D_t u_n)\cdot D_t u_n\cdot D_t^2 u_n|\,dxdt,$$

by Lemma 2.1, (2.6) and the assumption A_3,

$$\leqq \frac{k_0}{4}\int_\omega\int_0^1 |D^2 u_n|^2 dxdt + C(\delta_0, \delta_1)$$
$$+ C(\delta_0)\int_\omega\int_0^1 \sqrt{\rho_0(D_t u_n)\cdot D_t u_n}\cdot|D_t u_n|\,dxdt$$
$$\leqq \frac{k_0}{2}\int_\omega\int_0^1 |D_t^2 u_n|^2 dxdt + C(\delta_0)\sup_\omega \|D_t u_n\|_{L^\infty}^2 + C(\delta_0, \delta_1)$$

which implies with Lemma 1.1

$$(3.3)\qquad \int_\omega\int_0^1 \{\frac{\partial}{\partial v}\rho_0(D_t u_n) + \frac{\partial}{\partial v}\rho_1(u_n, D_t u_n)\}\,|D^2 u_n|^2 dxdt$$
$$\leqq \varepsilon \sup_\omega \|D_t u_n(t)\|_{\mathring{H}^1}^2 + C(\delta_0, \delta_1, \varepsilon)$$

for ${}^\forall\varepsilon>0$. On the other hand, multiplying (3.1) by u_n, we have

$$(3.4)\qquad \int_\omega\int_0^1 |D_x D_t u_n|^2 dxdt \leqq \int_\omega\int_0^1 |D_t^2 u_n|^2 dxdt$$
$$+ I_3 + I_4 + I_5 + \int_\omega\int_0^1 |f(x, t) D_t u_n|\,dxdt,$$

where we set

$$I_3=\int_\omega\int_0^1\left|\frac{\partial}{\partial v}\rho_0(D_tu_n)\cdot D_tu_n\cdot D^2u_n\right|dxdt,$$

$$I_4=\int_\omega\int_0^1\left|\frac{\partial}{\partial u}\rho_1(u_n, D_tu_n)\right||D_tu_n|^2dxdt$$

and

$$I_5=\int_\omega\int_0^1\left|\frac{\partial}{\partial v}\rho_1(u_n, D_tu_n)\cdot D_tu_n\cdot D_t^2u_n\right|dxdt.$$

These terms are estimated as follows.

$$I_3\leqq\left(\int_\omega\int_0^1\left|\frac{\partial}{\partial v}\rho_0(D_tu_n)\right||D_t^2u_n|^2dxdt\right)^{1/2}\left(\int_\omega\int_0^1\left|\frac{\partial}{\partial v}\rho_0(D_tu_n)\right||D_tu_n|^2dxdt\right)^{1/2},$$

by (3.3), assumption A_1 and (2.6),

$$\leqq(\varepsilon\sup_\omega\|D_tu_n(t)\|^2_{\mathring{H}^1}+C(\delta_0, \delta_1, \varepsilon))^{1/2}(C\delta_0)\sup_\omega\|D_tu_n(t)\|^2_{L^\infty}+C)^{1/2}$$

$$\leqq\varepsilon\sup_\omega\|D_tu_n(t)\|^2_{\mathring{H}^1}+C(\delta_0, \delta_1, \varepsilon)\quad({}^\forall\varepsilon>0).$$

$$I_4\leqq C(\delta_0)\int_\omega\int_0^1(\sqrt{\rho_0(D_tu_n)\cdot D_tu_n}+1)\,dxdt\sup_\omega\|D_tu_n\|^2_{L^\infty}\quad(\text{by (1.5)})$$

$$\leqq\varepsilon\sup_\omega\|D_tu_n(t)\|^2_{\mathring{H}^1}+C(\delta_0, \varepsilon)\quad({}^\forall\varepsilon>0).$$

And

$$I_5\leqq\left(\int_\omega\int_0^1\left|\frac{\partial}{\partial v}\rho_1(u_n, D_tu_n)\right||D_t^2u_n|^2dxdt\right)^{1/2}\times$$

$$\times\left(\int_\omega\int_0^1\left|\frac{\partial}{\partial v}\rho_1(u_n, D_tu_n)\right||D_tu_n|^2dxdt\right)^{1/2}$$

$$\leqq\int_\omega\int_0^1\frac{\partial}{\partial v}\rho_1(u_n, D_tu_n)|D_t^2u_n|^2dxdt$$

$$+C(\delta_0)\sup_\omega\|D_tu_n(t)\|^2_{L^\infty}\quad(\text{by (1.6) and (2.6)}$$

$$\leqq\varepsilon\sup_\omega\|D_tu_n(t)\|^2_{\mathring{H}^1}+C\delta_0, \delta_1, \varepsilon).$$

Thus we have, for ${}^\forall\varepsilon>0$,

$$(3.5)\qquad \int_\omega\int_0^1|D_xu_n(t)|^2dxdt\leqq\varepsilon\sup_\omega\|D_tu_n(t)\|^2_{\mathring{H}^1}+C(\delta_0, \delta_1, \varepsilon)$$

From (3.3) and (3.5),

$$\int_{\omega}\|D_t u_n)\|_E^2 dt \leqq \varepsilon \sup_{\omega} \|D_t u_n(t)\|_{\mathring{H}1}^2 + C(\delta_0, \delta_1, \varepsilon)$$

and hence, there exists $t^* \in [0, \omega]$ such that

$$(3.6) \qquad \|D_t u_n(t^*)\|_E^2 \leqq \varepsilon \sup_{\omega} \|D_t u_n(t)\|_{\mathring{H}1}^2 + C(\delta_0, \delta_1, \varepsilon)$$

for ${}^{\forall}\varepsilon > 0$. By the equation (3.2) and (3.6) we have

$$\sup_{\omega} \|D_t u_n(t)\|_E^2 \leqq \|D_t u_n(t^*)\|_E^2 + 2\int_{\omega}\int_0^1 \{\frac{\partial}{\partial v}\rho_0(D_t u_n) + \frac{\partial}{\partial v}\rho_1(u_n, D_t u_n)\} |D^2 u_n|^2 dx dt$$

$$\leqq \varepsilon \sup_{\omega} \|D_t u_n(t)\|_{\mathring{H}1}^2 + C(\delta_0, \delta_1, \varepsilon), \quad (\varepsilon > 0)$$

which yields immediately, taking $\varepsilon = \frac{1}{2}$,

$$(3.7) \qquad \|D_t u_n(t)\|_E^2 \leqq C(\delta_0, \delta_1) < \infty \qquad \text{for } {}^{\forall}t.$$

q. e. d.

LEMMA 3. 2.

$$(3.8) \qquad \|D_t^2 u_n(t)\|_E \leqq C(\delta_0, \delta_1, \delta_2) < \infty \qquad \textit{for } {}^{\forall}t.$$

PROOF.

Differentiating the equation (3.2) we have

$$(3.9) \qquad \begin{aligned} &(D_t^4 u_n, \phi_k) + (D_x D_t^2 u_n, D_x \phi_k) + (\frac{\partial}{\partial v}\rho_0(D_t u_n)\cdot D_t^3 u_n + \\ &+ \frac{\partial}{\partial v}\rho_1(u_n, D_t u_n)\cdot D_t^3 u_n, \phi_k) + ((\frac{\partial^2}{\partial v^2}\rho_0 + \frac{\partial^2}{\partial v^2}\rho_1)(D_t^2 u_n)^2, \phi_k) \\ &+ (g_0, \phi_k) = (D_t^2 f, \phi_k), \quad k = 1, 2, \cdots\cdots, n, \end{aligned}$$

where

$$\begin{aligned} g_0(t) = &\frac{\partial}{\partial u}\beta(u_n)\cdot D_t^2 u_n + \frac{\partial}{\partial u}\rho_1(u_n, D_t u_n)\cdot D_t^2 u_n \\ &+ \frac{\partial^2}{\partial u^2}\beta(u_n)\cdot(D_t u_n)^2 + \frac{\partial^2}{\partial u^2}\rho_1(u_n, D_t u_n)\cdot(D_t u_n)^2\cdot \\ &+ 2\frac{\partial^2}{\partial u \partial v}\rho_1(u_n, D_t u_n)\cdot D_t u_n \cdot D_t^2 u_n \end{aligned}$$

Multiplying (3.9) by $D_t^3 u_n$,

$$\int_{\omega}\int_0^1 (\frac{\partial}{\partial v}\rho_0(D_t u_n) + \frac{\partial}{\partial v}\rho_1(u_n, D_t u_n)) |D_t^3 u_n|^2 dx dt$$

$$(3.10)\qquad \leqq \int_\omega \int_0^1 \{|\frac{\partial^2}{\partial v^2}\rho_0(D_t u_n)| + |\frac{\partial^2}{\partial v^2}\rho_1(u_n, D_t u_n)|\}|D_t^2 u_n|^2 |D_t^3 u_n| dxdt$$
$$+\int_\omega \int_1^0 (|g_0(x, t)| + |D_t^2 f|)|D_t^3 u_n| dxdt.$$

By Lemmas 2.1 and 3.1 we can get easily

$$|\frac{\partial^2}{\partial v^2}\rho_0(D_t u_n)| + |\frac{\partial^2}{\partial v^2}\rho_1(u_n, D_t u_n)| + \int_\omega \|g_0(t)\|_{L^2}^2 dt \leqq C(\delta_0, \delta_1) < \infty$$

and hence

$$\int_\omega \int_0^1 (\frac{\partial}{\partial v}\rho_0(D_t u_n) + \frac{\partial}{\partial v}\rho_1(u_n, D_t u_n))|D_t^3 u_n|^2 dxdt$$
$$\leqq C(\delta_0, \delta_1)\int_\omega \int_0^1 |D_t^2 u_n|^2 |D_t^3 u_n| dxdt + C(\delta_0, \delta_1, \delta_2)$$
$$\leqq C(\delta_0. \delta_1)(\int_\omega \|D_t^2 u_n\|_{L^2}^2 dt)^{1/2} \cdot (\int_\omega \|D_t^3 u_n\|_{L^2}^2 dt)^{1/2} \times$$
$$\times \sup_\omega \|D_t^2 u_n(t)\|_{L^\infty} + C(\delta_0, \delta_1, \delta_2)$$
$$\leqq C(\delta_0, \delta_1, \delta_2)(1 + \sup_\omega \|D_t^2 u_n(t)\|_{L^\infty}^2)$$
$$(3.11)\qquad \leqq \varepsilon \sup_\omega \|D_t^2 u_n(t)\|_{\mathring{H}^1}^2 + C(\delta_0, \delta_1, \delta_2, \varepsilon), \quad \varepsilon > 0.$$

On the other hand, multiplying $D_t^2 u_n$,

$$\int_\omega \|D_x D_t^2 u_n(t)\|^2 dt \leqq \int_\omega \|D_t^3 u_n\|^2 dt + \int_\omega \int_0^1 \{|\frac{\partial}{\partial v}\rho_0 + \frac{\partial}{\partial v}\rho_1||D_t^3 u_n||D_t^2 u_n|$$
$$+ (|\frac{\partial^2}{\partial v^2}\rho_0| + |\frac{\partial^2}{\partial v^2}\rho_1|)|D_t^2 u_n|^3\} dxdt$$
$$+\int_\omega \int_0^1 (|g_0| + |D_t^2 f|)|D_t^2 u_n| dxdt$$

using Lemmas 2.1, 3.1 and (3.11),

$$(3.12)\qquad \leqq C(\delta_0, \delta_1, \delta_2)(1 + \sup_\omega \|D_t^2 u_n(t)\|_{L^\infty}^2)$$
$$\leqq \varepsilon \sup_\omega \|D_t^2 u_n(t)\|_{\mathring{H}^1}^2 + C(\delta_0, \delta_1, \delta_2, \varepsilon), \quad \varepsilon > 0.$$

From (3.11) and (3.12) we can conclude, as in the proof of Lemma 3.1,

$$\|D_t^2 u_n(t)\|_E^2 \leqq C(\delta_0, \delta_1, \delta_2) < \infty$$

which proves Lemma 3.2. q.e.d.

Finally we derive the following

LEMMA 3.3.

(3.13) $\|D_t^3 u_n(t)\|_E \leqq C(\delta_0, \delta_1, \delta_2, \delta_3) < \infty.$

PROOF.

Differentiating the equation (3.9) one more time we have

$$(3.14)\quad (D_t^5 u_n, \phi_k) + (D_x D_t^3 u_n, D_x \phi_k) + (\{\frac{\partial}{\partial v}\rho_0 + \frac{\partial}{\partial v}\rho_1\}\cdot D_t^4 u_n, \phi_k)$$
$$+ (g_1, \phi_k) = (D_t^3 f, \phi_k),\ k=1, 2, \cdots\cdots, n,$$

where

$$g_1(x,t) = (\frac{\partial^3}{\partial u^3}\beta + \frac{\partial^3}{\partial u^3}\rho_1)\cdot(D_t u_n)^3 + 3(\frac{\partial^2}{\partial u^2}\beta + \frac{\partial^2}{\partial u^2}\rho_1)\cdot D_t u_n \cdot D_t^2 u_n$$
$$+ (\frac{\partial}{\partial u}\beta + \frac{\partial}{\partial u}\rho_1)\cdot D_t^3 u_n + 3(\frac{\partial^2}{\partial v^2}\rho_0 + \frac{\partial^2}{\partial v^2}\rho_1)\cdot D_t^2 u_n \cdot D_t^3 u_n$$
$$+ (\frac{\partial^3}{\partial v^3}\rho_0 + \frac{\partial^3}{\partial v^3}\rho_1)(D_t^2 u_n)^3 + 3\frac{\partial^2}{\partial u \partial v}\rho_1\{(D_t u_n \cdot D_t^3 u_n) + (D_t^2 u_n)^2\} +$$
$$+ 3\frac{\partial^3}{\partial u^2 \partial v}\rho_1\cdot(D_t u_n)^2\cdot D_t^2 u_n + 3\frac{\partial^3}{\partial u \partial v^2}\rho_1\cdot D_t u_n\cdot(D_t^2 u_n)^2.$$

By use of Lemmas 2.1, 3.1 and 3.2, we see easily

$$(3.15)\quad \int_\omega \|g_1(t)\|_{L^2}^2 ds \leqq C(\delta_0, \delta_1, \delta_2) < \infty$$

and hence, multiplying (3.14) by $D_t^4 u_n$,

$$(3.16)\quad \int_\omega \int_0^1 \{\frac{\partial}{\partial v}\rho_0(D_t u_n) + \frac{\partial}{\partial v}\rho_1(u_n, D_t u_n)\}\,|D_t^4 u_n|^2 dx dt$$
$$\leqq C(\delta_0, \delta_1, \delta_2, \delta_3).$$

Also, multiplying (3.14) by $D_t^3 u_n$, we have easily

$$(3.17)\quad \int_\omega \|D_x D_t^3 u_n(t)\|_{L^2}^2 dt \leqq C(\delta_0, \delta_1, \delta_2, \delta_3) < +\infty.$$

The estimates (3.16) and (3.17) imply immediately

$$\|D_t^3 u_n(t)\|_E^2 \leqq C(\delta_0, \delta_1, \delta_2, \delta_3) < \infty$$

q, e. d.

Now we have finished the estimations of the ω-periodic approximate solutions $u_n(t)$, $n=1, 2, 3, \cdots\cdots$.

§ 4. Passage to the limit

On the basis of the estimates in the previous section we shall show that a subsequence of $\{u_n(t)\}$ converges in various norms, and that the limit function $u(t)$ is a required ω-periodic classical solution of the problem (0.1)-(0.2).

Since $\mathring{H}_1$ is compactly imbedded in $C([0,1])$ Ascoli-Alzera's lemma implies

(4.1) $\quad D_t^i u_n(x,t) \to D_t^i u(x,t)$ uniformly on $[0,1]\times \mathrm{R}$ for $i=0,1,2,$

and

(4.2) $\quad D_t^3 u_n \to D_t^3 u$ in L^2 uniformly on $R,$

along a subsequence of $\{u_n(t)\}$.
Moreover, by a standard compactness argument, we see

(4.3) $\quad D_t^i u_n \to D_t^i u$ weakly* in $L^\infty(\omega; H_1)$ and weakly in $L^2(\omega; L^2)$ for $i=0,1,2,3,4.$

The limit function $u(x,t)$ satisfies, of course,

$$D_t^i u \in L^\infty(\omega; L^2),\ \ D_t^i u \in L^\infty(\omega; H_1),\ i=0,1,2,3$$

and

(4.14) $\quad \sum_{i=0}^{3} \|D_t^i u(t)\|_E \leqq C(\delta_0, \delta_1, \delta_2, \delta_3) < \infty.$

Thus, taking the limit in (2.1), (3.2) and (3.9) along the subsequence we have

$$\frac{d^2}{dx^2} D_t^i u = D_t^i F(x,t) \qquad \text{for a.e. } t \in R,\ i=0,1,2,$$

where

$$F(x,t) = D_t^2 u + \rho_0(x, D_t u) + \beta(x,u) + \rho_1(x,u,D_t u) + f(x,t).$$

(The equation (4.5) should be understood in the generalized sense)

Since

$$F \in C(\omega; H_1),\ D_t F \in L^\infty(\omega; H_1) \cap C(\omega; L^2),$$
$$D_t^2 F \in L^\infty(\omega; L^2) \text{ and } D_t^i \in \mathring{H}_1 \text{ (for } {}^\forall t \in R)$$

we have from (4.5) (note that $D_t^i u \in \mathring{H}_1$ for ${}^\forall t \in R, i=0,1,2$)

(4.6) $u \in C(\omega; H_3), D_t u \in L^\infty(\omega; \mathring{H}_1 \cap C(\omega; H_2)$
and $D_t^2 u \in L^\infty(\omega; H_2)$.

(4.6) implies $F \in L^\infty(\omega; H_2)$, and by (4.5) we see

$u \in L^\infty(\omega; H_4)$.

Thus we have proved the following theorem.

THEOREM. 4.1
Under the assumptions (A.1)-(A.4), *the problem* (0.1)-(0.2) *admits, at least, one solution* u *which satisfies*

(4.7) $D_t^i u \in L^\infty(\omega; H_{4-i} \cap \mathring{H}_1)\ (i=0, 1, 2, 3), D_t^4 u \in L^\infty(\omega; L^2)$

and

$$\|D_t^i u(t)\|_{H_{4-i}} \leqq C\Big(\sum_{i=0}^{3} \delta_i, \sum_{i=0}^{3} \nu_i\Big) < \infty \text{ a.e. } t,$$

for $i=0, 1, 2, 3, 4$, *where* $H_0 \equiv L^2$.

By a standard regularity argument and by the Sobolev's lemma we obtain immediately from (4.7)

(4.8) $u \in C^3(\omega; L^2) \cap C^2(\omega; \mathring{H}_1) \cap C^1(\omega; H_2 \cap \mathring{H}_1) \cap C(\omega; H_3 \cap \mathring{H}_1)$
$\subset C^2([0, 1] \times R)$,

that is, the solution u of Theorem 4.1 is a classical ω-periodic solution of (0.1)-(0.2).

REMARK 1.
By Sobolev's lemma and (4.7) we can obtain in fact

$u \in C^{2+\alpha}([0.1] \times R)$ for $0 < {}^\forall \alpha < 1$.

REMARK 2.
As is easily ssen from the proof we may assert

$C\Big(\sum_{i=0}^{3} \delta_i, \sum_{i=0}^{2} \nu_i\Big) \to 0$ as $\sum_{i=0}^{3} \delta_i + \sum_{i=0}^{2} \nu_i \to 0$.

Refereoces

[1] L. AMERIO and G. PROUSE, Almost Periodic Functions and Functional Equations, Van Nostrand, Princeton, 1971.

[2] M. BIROLI, *Bounded or almost periodic solution of the nonlinear vibrating mumbrance equation*, Ricerche Mat. **22**, Fas. 2(1973), 190-202.

[3] J.C. CLEMENTS, *Existence theorem for some nonlineare quations of evolution*, Canad. J. Math., 22-4(1970), 726-745.

[4] F.A. FICKEN and B.A. FLEISHMAN, *Initial value and time-periodic solutions for a nonlinear wave equation*, Comm. Pure Appl. Math. **10**(1957), 331-356.

[5] T. KAKITA, *On the existence of time-periodic solutions of some nonlinear evolution equations*, Applicable Analysis, **4**(1974), 63-76.

[6] N. KRYLOVA, *Periodic solutions of hyperbolic partial differential equation with quadratic dissipative term*, Czech. Math. J. **20**(95) (1970), 375-405.

[7] J.L. LIONS, Quelques Méthodes de Resolution des Problemes aux Limites Non Linéaires, Dunod, Paris, 1969.

[8] M. NAKAO, *Bounded, periodic or almost periodic solutions of nonlinear hyperbolic partial differential equations*, J. Differential Equations, **23**(1977), 368-386.

[9] M. NAKAO, *Decay of solutions of some nonlinear wave equations in one space dimension*, Funkcialaj Ekvacioj, Vol. **20**(1977), 223-236.

[10] M. NAKAO, *Bounded, periodic and almost periodic classical solutions of some nonlinear wave equations with a dissipative term*, J. Math. Soc. Japan, Vol. **30** (1978) 375-394.

[11] G. PRODI, *Soluzioni periodiche dell-oquazione delle onde termine dissipativo non lineare*, Rend. Sem. Math. Univ. Padova, **36**(1966), 37-49.

[12] P.H. RABINOWITZ, *Periodic solutions of nonlinear hyperbolic partial differential equations*, Comm. Pure Appl. Math., **20**(1967), 145-205.

[13] W.V. WAHL, *Periodic solutions of nonlinear wave equations with a dissipative term*, Math. Ann., **190**(1971), 313-322.

[14] M. YAMAGUCHI, *Existence and stability of bounded and almost periodic solutions of perturbed quasilinear partial differential equations of standard type*, Proc. Fac. Science, Tokai Univ. **14**(1978), 45-57.