# On a property of the Appell hypergeometric function F_1

Saigo, Megumi
Department of Mathematics, College of General Education, Kyushu University

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# On a property of the Appell hypergeometric function $F_1$

Megumi SAIGO

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In the discussions of the boundary value problems for the Euler-Darboux equation with the boundary conditions involving the generalized fractional integrals or derivatives [3], [4], it has been required to know the behavior of the hypergeometric series near boundary points of the domains of convergence. That is, in [3] the formula

$$(1)\qquad F(a,b;a+b;1-\rho x) = -\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\{2\gamma+\psi(a)+\psi(b)+\log x+\log\rho\}+o(1),\quad (\rho\to+0)$$

for $x>0$ is used (see (3.6) and (3.7) in [3]), which is implied by the formula (see [2])

$$(2)\qquad F(a,b;a+b;z)=\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(n!)^2} \times[2\psi(n+1)-\psi(a+n)-\psi(b+n)-\log(1-z)](1-z)^n,$$
$$(|\arg(1-z)|<\pi,\ |1-z|<1),$$

where $F$ means the Gauss hypergeometric function

$$(3)\qquad F(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}z^n$$

in $|z|<1$ together with its analytic continuation, $(a)_n=\Gamma(a+n)/\Gamma(a)$, $\Gamma$ is the gamma function, $\psi(z)=\Gamma'(z)/\Gamma(z)$ is the psi function and $\gamma=-\psi(1)$ signifies the Euler constant.

In the case of [4], the Appell hypergeometric function

$$(4)\qquad F_1(a;b,b';c;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m(b')_n}{(c)_{m+n}m!n!}x^m y^n$$

having the domain of convergence such that $\max(|x|,|y|)<1$ is used and a similar formula for $F_1$ to (1) is needed. (see (4.7) and (4.8) in [4]) So we have to establish the formula (IV. 9) in [4], which is the following:

$$(5)\qquad F_1(a;b,b';a+b+b';1-\rho x,1-\rho y)$$
$$=\frac{\Gamma(a+b+b')}{\Gamma(a)\Gamma(b+b')}\left\{\frac{b'}{b+b'}\left(1-\frac{y}{x}\right){}_3F_2\left(1,1,b'+1;2,b+b'+1;1-\frac{y}{x}\right)\right.$$
$$\left.-[2\gamma+\psi(a)+\psi(b+b')+\log x+\log\rho]\right\}+\mathrm{o}(1),\quad(\rho\to+0),$$

for $x>0, y>0$ and $|1-\frac{y}{x}|<1$. To demonstrate this formula is the purpose of this work. ${}_3F_2$ in the formula (5) stands for the generalized hypergeometric function

$$(6)\qquad {}_3F_2(a,b,c;d,e;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n(c)_n}{(d)_n(e)_n n!}z^n$$

in $|z|<1$ together with its analytic continuation.

For the function $F_1$ the expansion

$$(7)\qquad F_1(a;b,b';c;x,y)=\sum_{m=0}^{\infty}\frac{(a)_m(b')_m}{(c)_m m!}F(a+m,b+b'+m;c+m;x)(y-x)^m$$

is proved in [1].

More general hypergeometric function of two variables than the Appell one is the Kampé de Fériet function (see [1])

$$(8)\qquad F\left(\begin{array}{c|c|c} \mu & a_1,\cdots,a_\mu & \\ \nu & b_1,b_1';\cdots;b_\nu,b_\nu' & x,y \\ \rho & c_1,\cdots,c_\rho & \\ \sigma & d_1,d_1';\cdots;d_\sigma,d_\sigma' & \end{array}\right)$$
$$=\sum_{m,n=0}^{\infty}\frac{\prod_{i=1}^{\mu}(a_i)_{m+n}\prod_{i=1}^{\nu}(b_i)_m(b_i')_n}{\prod_{i=1}^{\rho}(c_i)_{m+n}\prod_{i=1}^{\sigma}(d_i)_m(d_i')_n}\frac{x^m y^n}{m!n!},\qquad(\mu+\nu\leq\rho+\sigma+1)$$

which is defined for all complex $x$ and $y$ if $\mu+\nu<\rho+\sigma+1$, for $\max(|x|,|y|)<1$ if $\mu+\nu=\rho+\sigma+1$ and $\mu\leq\rho$, and for $|x|^{1/(\mu-\rho)}+|y|^{1/(\mu-\rho)}<1$ if $\mu+\nu=\rho+\sigma+1$ and $\mu>\rho$. The function (5) is known as a special case of (8) by setting $\mu=\nu=\rho=1$ and $\sigma=0$.

In order to prove the formula (5), we shall prepare one more formula (see [2]):

$$(9)\qquad F(a,b;a+b-m;z)$$
$$=\frac{\Gamma(m)\Gamma(a+b-m)}{\Gamma(a)\Gamma(b)}(1-z)^{-m}\sum_{n=0}^{m-1}\frac{(a-m)_n(b-m)_n}{n!(1-m)_n}(1-z)^n$$

$$+\frac{(-1)^m\Gamma(a+b-m)}{\Gamma(a-m)\Gamma(b-m)}\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{n!\,(n+m)!}(1-z)^n$$
$$\times[\psi(n+1)+\psi(m+n+1)-\psi(a+n)-\psi(b+n)-\log(1-z)],$$
$$(|\arg(1-z)|<\pi,\ |1-z|<1,\ m=\text{positive integer}).$$

PROOF OF (5). If we notice the formulas (7), (2) and (9), we have for small $\rho>0$

$$(10)\qquad F_1(a;b,b';a+b+b';1-\rho x,1-\rho y)$$
$$=\sum_{m=0}^{\infty}\frac{(a)_m(b')_m}{(a+b+b')_m m!}F(a+m,b+b'+m;a+b+b'+m;1-\rho x)[\rho(x-y)]^m$$
$$=\sum_{m=1}^{\infty}\frac{(a)_m(b')_m}{(a+b+b')_m m!}\ \frac{\Gamma(m)\Gamma(a+b+b'+m)}{\Gamma(a+m)\Gamma(b+b'+m)}\sum_{n=0}^{m-1}\frac{(a)_n(b+b')_n}{n!\,(1-m)_n}(\rho x)^n\,(1-\frac{y}{x})^m$$
$$+\sum_{m=0}^{\infty}\frac{(a)_m(b')_m}{(a+b+b')_m m!}\ \frac{\Gamma(a+b+b'+m)}{\Gamma(a)\Gamma(b+b')}(-1)^m\sum_{n=0}^{\infty}\frac{(a+m)_n(b+b'+m)_n}{n!\,(n+m)!}(\rho x)^{m+n}$$
$$\times[\psi(n+1)+\psi(m+n+1)-\psi(a+m+n)-\psi(b+b'+m+n)-\log\rho x](1-\frac{y}{x})^m,$$

Dividing the sums in (10), we have

$$(11)\qquad F_1(a;b,b';a+b+b';1-\rho x,1-\rho y)$$
$$=\frac{\Gamma(a+b+b')}{\Gamma(a)\Gamma(b+b')}\{S_1+S_2+S_3+S_4-[2\gamma+\psi(a)+\psi(b+b')+\log\rho x]\},$$

where

$$S_1=\sum_{m=1}^{\infty}\frac{(b')_m(m-1)!}{(b+b')_m m!}(1-\frac{y}{x})^m,$$
$$S_2=\sum_{m=2}^{\infty}\ \sum_{n=1}^{m-1}\frac{(b')_m(a)_n(b+b')_n(m-1)!}{(b+b')_m(1-m)_n n!\,m!}(1-\frac{y}{x})^m(\rho x)^n,$$
$$S_3=\sum_{n=1}^{\infty}\frac{(a)_n(b+b')_n}{(n!)^2}(\rho x)^n[2\psi(n+1)-\psi(a+n)-\psi(b+b'+n)-\log\rho x],$$
$$S_4=\sum_{m=1}^{\infty}\ \sum_{n=0}^{\infty}\frac{(a)_m(b')_m(a+m)_n(b+b'+m)_n}{m!\,n!\,(m+n)!}(\rho x)^{m+n}(\frac{y}{x}-1)^m$$
$$\times[\psi(n+1)+\psi(m+n+1)-\psi(a+m+n)-\psi(b+b'+m+n)-\log\rho x].$$

Let us investigate these sums. $S_1$ can be represented by using the function ${}_3F_2$ such as

$$(12)\qquad S_1=\frac{b'}{b+b'}(1-\frac{y}{x})\,{}_3F_2(1,1,b'+1;2,b+b'+1;1-\frac{y}{x}).$$

Since $(1-m)_n=(-1)^n(m-1)!/(m-n-1)!$ for $m\geq n+1$, by setting $k=n-1$, and $l=m-n-1$, $S_2$ can be written in terms of the Kampé de Fériet function (8):

$$S_2 = -\frac{ab'(b'+1)}{2(b+b'+1)}\rho x\left(1-\frac{y}{x}\right)^2 F\left(\begin{array}{c|l|l} 1 & b'+2 & \\ 3 & a+1, 1; b+b'+1, 1; 1, 1 & \rho(y-x),\ 1-\frac{y}{x} \\ 2 & b+b'+2, 3 & \\ 1 & 2, 1 & \end{array}\right).$$

Thus we have

$$(13) \qquad S_2 = \mathrm{O}(\rho), \quad (\rho\to+0).$$

Dividing the summand of $S_3$ into such terms that one of them contains $\log\rho x$ and the other does not, we obtain

$$\begin{aligned}(14) \qquad S_3 = & -\rho x(\log\rho x)a(b+b')\,{}_3F_2(a+1, b+b'+1, 1; 2, 2; \rho x) \\ & +\rho x a(b+b')\sum_{n=0}^{\infty}\frac{(a+1)_n(b+b'+1)_n}{(2)_n(2)_n}(\rho x)^n \\ & \times[2\psi(n+2)-\psi(a+1+n)-\psi(b+b'+1+n)].\end{aligned}$$

Since $\psi(z+1)=\psi(z)+1/z$ and $\psi(z)$ diverges to infinity as $z\to\infty$, it is easily seen that the second term on the right hand side of (14) has a positive radius of convergence. Then we conclude that

$$(15) \qquad S_3 = \mathrm{o}(1), \quad (\rho\to+0).$$

Similarly, by noting that $(a+m)_n=(a)_{m+n}/(a)_m$, $S_4$ can be written in the form

$$\begin{aligned} S_4 = & -\rho(\log\rho x)ab'(y-x)F\left(\begin{array}{c|l|l} 2 & a+1, b+b'+1 & \\ 2 & b'+1, 1; 1, 1 & \rho(y-x), \rho x \\ 1 & 2 & \\ 2 & b+b'+1, 1; 2, 1 & \end{array}\right) \\ & +\rho ab'(y-x)\sum_{m,n=0}^{\infty}\frac{(a+1)_{m+n}(b+b'+1)_{m+n}(b'+1)_m}{(2)_{m+n}(b+b'+2)_m(2)_m n!}[\rho(y-x)]^m(\rho x)^n \\ & \times[\psi(n+1)+\psi(m+n+2)-\psi(a+1+m+n)-\psi(b+b'+1+m+n)],\end{aligned}$$

and we have

$$(16) \qquad S_4 = \mathrm{o}(1), \quad (\rho\to+0).$$

Therefore (11)~(13), (15) and (16) yield the formula (5) and our proof is completed.

## References

[1] P. APPELL et J. KAMPÉ de FÉRIET: Fonctions Hypergéométriques et Hypersphériques, Polynomes d'Hermite, Gauthier-Villars (Paris), 1926.

[2] W. MAGNUS, F. OBERHETTINGER and R. P. SONI: Formulas and Theorems for the Special Functions of Mathematical Physics, Springer-Verlag (Berlin), 1966.

[3] M. SAIGO: *A certain boundary value problem for the Euler-Darboux equation.* II, Math. Japon. **25**(1980), 211-220.

[4] M. SAIGO: *A certain boundary value problem for the Euler-Darboux equation.* III, (submitted for publication).