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On the Hölder continuity of the generalized fractional integrals and derivatives

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In [3] and [4], the generalized fractional integrals and derivatives involving the Gauss hypergeometric function have been defined and their properties have been discussed. As their applications, solutions of boundary value problems for the Euler-Darboux equation with boundary conditions which contain our fractional integrals or derivatives have been investigated in [4], [5] and [6]. In particular, in [4] and [5] we have made use of the Hölder continuity of the generalized fractional integrals and derivatives of Hölder continuous functions. In this paper let us give several results for the continuity.

We begin to define the generalized fractional calculus.

DEFINITION. Let $\alpha>0$, β, η and a be real numbers. We define the integral

$$(1)\qquad I_{ax}^{\alpha,\beta,\eta} f = \frac{(x-a)^{-\alpha-\beta}}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}F(\alpha+\beta, -\eta;\alpha;\frac{x-t}{x-a})f(t)dt$$

for a real valued and continuous function $f(x)$ defined on (a,∞) having order $O((x-a)^k)$ near $x=a$ with $k>\max(0,\beta-\eta)-1$, where Γ is the gamma function, F means the Gauss hypergeometric function

$$F(a,b;c;z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}z^n$$

in $|z|<1$ and $(a)_n=\Gamma(a+n)/\Gamma(a)$. When $\alpha\leq 0$, by letting n be a positive integer such as $0<\alpha+n\leq 1$, we define

$$(2)\qquad I_{ax}^{\alpha,\beta,\eta} f = \frac{d^n}{dx^n} I_{ax}^{\alpha+n,\beta-n,\eta-n}f,$$

if the right hand side has a definite meaning.

The expressions (1) and (2) are understood as generalizations of the fractional integral and derivative of Riemann-Liouville, and (1) also contains the Erdélyi-Kober fractional integral, i.e.

$$(3)\qquad I_{ax}^{\alpha,-\alpha,\eta}f = R_{ax}^{\alpha}f, \quad \text{(Riemann-Liouville)},$$

$$(4)\qquad I_{ax}^{\alpha,0,\eta}f = E_{ax}^{\alpha,\eta}f, \quad \text{(Erdélyi-Kober)}.$$

Another fractional integral (and derivative) whose lower limit is a variable smaller than the constant upper limit b are defined as follows:

DEFINITION. Let $\alpha>0$, β, η and b be real numbers. We define

$$(5)\qquad J_{xb}^{\alpha,\beta,\eta}f = \frac{(b-x)^{-\alpha-\beta}}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}F(\alpha+\beta, -\eta;\alpha;\frac{t-x}{b-x})f(t)dt,$$

where $f(x)$ is a real valued and continuous function defined on $(-\infty, b)$ having order $O((b-x)^k)$ near $x=b$ with $k>\max(0,\beta-\eta)-1$. When $0<\alpha+n\leq 1$ for positive integer n, we define

$$(6)\qquad J_{xb}^{\alpha,\beta,\eta}f = (-1)^n\frac{d^n}{dx^n}J_{xb}^{\alpha+n,\beta-n,\eta-n}f,$$

if the right hand side has a definite meaning.

In what follows we restrict the above definitions to the case $a=0$ and $b=1$.

DEFINITION. Let H^k be a class of functions $\varphi(x)$ defined on $[0,1]$ which are Hölder continuous, i.e. which satisfy the inequality

$$|\varphi(x_1)-\varphi(x_2)| \leq C|x_1-x_2|^k \qquad \text{for } x_1, x_2\in[0,1],$$

where C is a constant not depending on x_1 and x_2, and k is the Hölder index with $0<k<1$.

In the following Theorems 1~5 and Lemma 2, we treat the function

$$(7)\qquad \varphi(x)\in H^k \qquad \text{for } 0<k<1,$$

and set

$$0\leq x<x+h\leq 1.$$

THEOREM 1. *If $t>0$, then $x^t\varphi(x)\in H^{\min(k,t)}$.*

PROOF. From the inequality

$$(8) \qquad |(x+h)^t\varphi(x+h)-x^t\varphi(x)| \\ \leq (x+h)^t|\varphi(x+h)-\varphi(x)|+|(x+h)^t-x^t||\varphi(x)|,$$

the result easily follows in view of the assumption (7) and the fact[1] $|(x+h)^t-x^t|\leq Ch^t$.

THEOREM 2. *If $k+t>0$ and $\varphi(0)=0$, then $x^t\varphi(x)\in H^{k+\min(0,t)}$.*

PROOF. If $t=0$, the result is obvious. If $0<t\leq 1$, the first term in the right hand side of the inequality (8), say A_1, is evidently not greater than Ch^k from the assumption (7). The second term, A_2, is also estimated by Ch^k, because, if $x\leq h$, the fact

$$(9) \qquad |\varphi(x)| \leq Cx^k$$

which is implied by $\varphi(0)=0$, and the boundedness of $|(x+h)^t-x^t|$ yield $A_2 \leq Ch^k$. And if $x>h$, by using the inequalities (9) and

$$(x+h)^t-x^t \leq x^t[(1+\frac{h}{x})^t-1] \leq x^t\cdot t\cdot\frac{h}{x}, \qquad (0<t\leq 1),$$

we obtain

$$A_2 \leq Cx^{k+t-1}h = Ch^k(\frac{h}{x})^{1-k}x^t \leq Ch^k.$$

If $t>1$, e.g. $1<t\leq 2$, by considering $x^t\varphi(x)=x\cdot x^{t-1}\varphi(x)$ and by using the above result, we have $x^{t-1}\varphi(x)\in H^k$ and then $x^t\varphi(x)\in H^k$. When $t<0$, A_1 is estimated such as

$$A_1 \leq C(x+h)^th^k = C(\frac{h}{x+h})^{-t}h^{k+t} \leq Ch^{k+t}.$$

A_2 is written as

$$(10) \qquad A_2 \leq Cx^k|(x+h)^t-x^t| = Cx^{k+t}\frac{(x+h)^{-t}-x^{-t}}{(x+h)^{-t}}$$

by virtue of (9). When $x\leq h$, using the fact $(x+h)^{-t}-x^{-t}\leq h^{-t}$, we have

$$A_2 \leq Ch^{k+t}(\frac{h}{x+h})^{-t} \leq Ch^{k+t}.$$

If $x>h$, taking into account of the inequality

$$(x+h)^{-t}-x^{-t} = x^{-t}[(1+\frac{h}{x})^{-t}-1] \leq x^{-t}(-t)\frac{h}{x}, \quad (0<-t<1),$$

we have the estimation of (10) such as

1) In what follows the single symbol C will denote a constant not depending on x and h.

$$A_2 \leq Cx^{k+t}\left(\frac{x}{x+h}\right)^{-t}(-t)\frac{h}{x} = C(-t)\left(\frac{x}{x+h}\right)^{-t}\left(\frac{h}{x}\right)^{1-k-t}h^{k+t} \leq Ch^{k+t}.$$

Thus the proof for the case $t<0$ is completed and we have obtained the result

$$|(x+h)^t\varphi(x+h)-x^t\varphi(x)| \leq Ch^{k+\min(0,t)}$$

for any t with $k+t>0$.

In order to prove the Hölder continuity of the generalized fractional integrals and derivatives of Hölder continuous functions, let us construct the following equality.

LEMMA 1. *If* $-\beta+\eta+1>0$ *and* (i) $\alpha>0$ *and* $\varphi(x)\in C^0$ *i.e.* $\varphi(x)$ *is continuous on the interval* $[0,1]$, *or* (ii) $-1<\alpha<0$ *and* $\varphi(x)\in H^k$ *with* $\alpha+k>0$, *then the following equality holds*:

$$(11)\qquad I_{0x}^{\alpha,\beta,\eta}\varphi = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}F(\alpha+\beta, -\eta;\alpha;1-\frac{t}{x})\{\varphi(t)-\varphi(x)\}dt + \frac{\Gamma(-\beta+\eta+1)}{\Gamma(-\beta+1)\Gamma(\alpha+\eta+1)}x^{-\beta}\varphi(x).$$

PROOF. If the assumption (i) holds, we have

$$(12)\qquad I_{0x}^{\alpha,\beta,\eta}\varphi = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}F(\alpha+\beta, -\eta;\alpha;1-\frac{t}{x})\varphi(t)dt$$

$$= \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}F(\alpha+\beta, -\eta;\alpha;1-\frac{t}{x})\{\varphi(t)-\varphi(x)\}dt + \varphi(x)\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}F(\alpha+\beta, -\eta;\alpha;1-\frac{t}{x})dt.$$

The second integral in the right hand side of (12) is calculated as

$$\int_0^x (x-t)^{\alpha-1}F(\alpha+\beta, -\eta;\alpha;1-\frac{t}{x})dt = x^\alpha\int_0^1 v^{\alpha-1}F(\alpha+\beta, -\eta;\alpha;v)dv = \frac{x^\alpha}{\alpha}F(\alpha+\beta, -\eta;\alpha+1;1) = \frac{\Gamma(\alpha)\Gamma(-\beta+\eta+1)}{\Gamma(-\beta+1)\Gamma(\alpha+\eta+1)}x^\alpha,$$

where the formulas (see [2])

$$(13)\qquad \frac{d}{dz}[z^{c-1}F(a,b;c;z)] = (c-1)z^{c-2}F(a,b;c-1;z) \qquad \text{and}$$

$$(14) \qquad F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \qquad \text{for } \operatorname{Re}(c-a-b)>0$$

have been used. Then the equality (11) when (i) is assumed is obtained. In the case of (ii) we have to calculate

$$(15) \qquad I_{0x}^{\alpha,\beta,\eta}\varphi = \frac{d}{dx} I_{0x}^{\alpha+1,\beta-1,\eta-1}\varphi$$

$$= \frac{d}{dx}\frac{x^{-\alpha-\beta}}{\Gamma(\alpha+1)}\int_0^x (x-t)^{\alpha}F(\alpha+\beta,-\eta+1;\alpha+1;1-\frac{t}{x})\varphi(t)dt$$

from the definition of the fractional derivative (2). Now let us consider the expression

$$(16) \qquad M(x,\varepsilon) = \frac{d}{dx}x^{-\alpha-\beta}\int_0^{x-\varepsilon}(x-t)^{\alpha}F(\alpha+\beta,-\eta+1;\alpha+1;1-\frac{t}{x})\varphi(t)dt.$$

In view of the relation (see [2])

$$\frac{d}{dz}[z^{c-1}(1-z)^{a-c+1}F(a,b;c;z)] = (c-1)z^{c-2}(1-z)^{a-c}F(a,b-1;c-1;z),$$

we have

$$(17) \qquad M(x,\varepsilon) = x^{-\alpha-\beta}\varepsilon^{\alpha}F(\alpha+\beta,-\eta+1;\alpha+1;\frac{\varepsilon}{x})\varphi(x-\varepsilon)$$

$$+ \alpha x^{-\alpha-\beta}\int_0^{x-\varepsilon}(x-t)^{\alpha-1}F(\alpha+\beta,-\eta;\alpha;1-\frac{t}{x})\varphi(t)dt$$

$$= x^{-\alpha-\beta}F(\alpha+\beta,-\eta+1;\alpha+1;\frac{\varepsilon}{x})\varepsilon^{\alpha}\{\varphi(x-\varepsilon)-\varphi(x)\}$$

$$+ \alpha x^{-\alpha-\beta}\int_0^{x-\varepsilon}(x-t)^{\alpha-1}F(\alpha+\beta,-\eta;\alpha;1-\frac{t}{x})\{\varphi(t)-\varphi(x)\}dt$$

$$+ [x^{-\alpha-\beta}F(\alpha+\beta,-\eta+1;\alpha+1;\frac{\varepsilon}{x})\varepsilon^{\alpha}$$

$$+ \alpha x^{-\alpha-\beta}\int_0^{x-\varepsilon}(x-t)^{\alpha-1}F(\alpha+\beta,-\eta;\alpha;1-\frac{t}{x})dt]\varphi(x).$$

The terms in the brackets of (17) are evaluated as follows:

$$(18) \qquad [\ \ ] = \frac{d}{dx}x^{-\alpha-\beta}\int_0^{x-\varepsilon}(x-t)^{\alpha}F(\alpha+\beta,-\eta+1;\alpha+1;1-\frac{t}{x})dt$$

$$= \frac{d}{dx}x^{-\beta+1}\int_{\frac{\varepsilon}{x}}^{1} v^{\alpha}F(\alpha+\beta,-\eta+1;\alpha+1;v)dv$$

$$= \frac{d}{dx}x^{-\beta+1}\frac{1}{\alpha+1}\{F(\alpha+\beta,-\eta+1;\alpha+2;1)$$

$$- (\frac{\varepsilon}{x})^{\alpha+1}F(\alpha+\beta,-\eta+1;\alpha+2;\frac{\varepsilon}{x})\}$$

$$= \frac{\Gamma(\alpha+1)\Gamma(-\beta+\eta+1)}{\Gamma(-\beta+1)\Gamma(\alpha+\eta+1)} x^{-\beta}$$

$$- \varepsilon^{\alpha+1}\frac{d}{dx} x^{-\alpha-\beta} F(\alpha+\beta, -\eta+1; \alpha+2; \frac{\varepsilon}{x})$$

$$= \frac{\Gamma(\alpha+1)\Gamma(-\beta+\eta+1)}{\Gamma(-\beta+1)\Gamma(\alpha+\eta+1)} x^{-\beta}$$

$$+(\alpha+\beta)\varepsilon^{\alpha+1} x^{-\alpha-\beta-1} F(\alpha+\beta+1, -\eta+1; \alpha+2; \frac{\varepsilon}{x}),$$

where the formulas (13), (14) and

$$\frac{d}{dz} z^a F(a, b; c; z) = a z^{a-1} F(a+1, b; c; z)$$

(see [2]) have been used. Since the limit

$$\lim_{\varepsilon\to 0} \varepsilon^{\alpha}\{\varphi(x-\varepsilon)-\varphi(x)\} = 0$$

is uniform from the assumption, then by passing to the limit

$$I_{0x}^{\alpha,\beta,\eta}\varphi = \frac{1}{\Gamma(\alpha+1)} \lim_{\varepsilon\to 0} M(x, \varepsilon)$$

we obtain (11) from (15)～(18) provided the assumption (ii) holds.

LEMMA 2. *The integral*

$$A(x) = \int_0^x (x-t)^{\alpha-1} F(\alpha+\beta, -\eta; \alpha; 1-\frac{t}{x})\{\varphi(t)-\varphi(x)\}dt$$

belongs to the class $H^{k+\alpha}$, *if* $0<k+\alpha<1, \alpha\neq 0$ *and* $\eta>\beta-1$.

PROOF. Since

$$F(a, b; c; z) = O((1-z)^{\min(0, c-a-b)}), \quad (z\to 1), \tag{19}$$

the kernel of the integral $A(x)$ has order $q=\min(0, -\beta+\eta)$ at $t=0$. Then to obtain the result, it is enough to prove that the integral

$$B(x) = \int_0^x (x-t)^{\alpha-1} t^q \{\varphi(t)-\varphi(x)\}dt$$

has the property

$$B(x)-B(x-h) = O(h^{k+\alpha}), \quad (h\to 0). \tag{20}$$

But (20) is shown in [4] on the same lines of [1], thus we omit the proof here.

THEOREM 3. *If* $0<k+\min(\alpha, -\beta)<1$, $\alpha\neq 0$, $\eta>\beta-1$ *and* $\varphi(0)=0$, *then* $I_{0x}^{\alpha,\beta,\eta}\varphi\in H^{k+\min(0,\alpha,-\beta)}$.

PROOF. Let us consider the expression (11). From the assumption $\varphi(0)=0$ and Theorem 2, we find

$$x^{-\beta}\varphi(x) \in H^{k+\min(0,-\beta)}. \tag{21}$$

On the other hand, Lemma 2 and Theorem 2 imply that the first term in the right hand side of (11) belongs to $H^{k+\min(\alpha,-\beta)}$, since the integral $A(x)$ in Lemma 2 is equal to zero at $x=0$. Then we have the conclusion.

THEOREM 4. *If* $0<k+\alpha<1, \alpha\neq 0$ *and* $\eta>\beta-1$, *then* $x^{\beta}I_{0x}^{\alpha,\beta,\eta}\varphi\in H^{k+\min(0,\alpha)}$

PROOF. Multiplying the equality (11) by x^{β}, we can easily see the result by noting Theorem 2 and Lemma 2.

THEOREM 5. *If* $0<k+\alpha<1, \alpha\neq 0, \beta<0$ *and* $\eta>\beta-1$, *we have* $I_{0x}^{\alpha,\beta,\eta}\varphi \in H^{\min(k,k+\alpha,-\beta)}$.

PROOF. In the proof of Theorem 3, if we replace the relation (21) by $x^{-\beta}\varphi(x)\in H^{\min(k,-\beta)}$ which follows from Theorem 1, the result is obtained.

REMARK. Completely similar results to Theorems 1~5 remain valid by replacing x^{t} by $(1-x)^{t}, \varphi(0)=0$ by $\varphi(1)=0$ and $I_{0x}^{\alpha,\beta,\eta}$ by $J_{x1}^{\alpha,\beta,\eta}$.

EXAMPLE. Let α and β be positive constants with $\alpha+\beta<1$. Constants a, b and c are such that $-\alpha<a<\alpha+2\beta-1$, $-\alpha<a+b<\beta$ and $-\alpha<a+c<\beta$. Let $\varphi_1(x)\in H^{k_1}$ and $\varphi_2(x)\in H^{k_2}$ with $\varphi_1(0)=\varphi_1(1)=0, a-\alpha-2\beta+2<k_1<1$ and $a-\beta+1<k_2<1$. We consider the functions

$$\begin{aligned} f_1(x) &= R_{0x}^{\alpha+\beta-1}\varphi_1, \\ f_2(x) &= I_{0x}^{-a+\beta-1,0,-b-\alpha-\beta+1}J_{x1}^{a+\alpha,-\alpha-\beta+1,-a-c-\alpha}\varphi_1 \qquad \text{and} \\ f_3(x) &= I_{0x}^{-a+\beta-1,0,-b-\alpha-\beta+1}\varphi_2. \end{aligned}$$

The linear combination of these functions with certain constant coefficients is just the function $R_0(x)$ defined at (3.6) in [**6**], and $R(x)=(1-x)^{c-\alpha-\beta+1}\times R_0(x)$ means the right hand side of the equation (2.5) in [**6**]. From the assumptions $\varphi_1(0)=0$ and $k_1+\alpha+\beta-1>0$ we have $f_1(x)\in H^{k_1+\alpha+\beta-1}$, in view of (3) and Theorem 3. Considering $-a+\beta-1<0$, we know that $f_3(x)\in H^{k_2-a+\beta-1}$ by Theorem 4. By noting Theorem 3 and Remark, we have

$J_{x1}^{a+\alpha,-\alpha-\beta+1,-a-c-\alpha}\varphi_1 \in H^{k_1+\alpha+\beta-1}$ from $\varphi_1(1)=0$, $a+c<\beta$ and $a+\alpha>0$. Then Theorem 4 implies $f_2(x) \in H^{k_1-a+\alpha+2\beta-2}$, where the assumption $a-\alpha-2\beta+2<k_1$ have been used. Therefore we have

$$R_0(x) \in H^{\min(k_1-a+\alpha+2\beta-2,k_2-a+\beta-1)}.$$

References

[1] G. H. HARDY and J. E. LITTLEWOOD: *Some properties of fractional integrals.* I, Math. Z. **27** (1928), 565-606.

[2] W. MAGNUS, F. OBERHETTINGER and R. P. SONI: Formulas and Theorems for the Special Functions of Mathematical Physics, Springer-Verlag (Berlin), 1966.

[3] M. SAIGO: *A remark on integral operators involving the Gauss hypergeometric functions*, Math. Rep. Kyushu Univ. **11** (1978), 135-143.

[4] M. SAIGO: *A certain boundary value problem for the Euler-Darboux equation*, Math. Japon. **24** (1979), 377-385.

[5] M. SAIGO: *A certain boundary value problem for the Euler-Darboux equation.* II, Math. Japon. **25** (1980), 211-220.

[6] M. SAIGO: *A certain boundary value problem for the Euler-Darboux equation.* III, (submitted for publication).