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Iyanaga, Manabu
Department of Mathematics, College of General Education, Kyushu University

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A differential geometry on the Hilbert manifold

Manabu IYANAGA

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§ 1. Introduction

The aims of this paper are to introduce the curvature of H-type on the Hilbert manifold M and to derive the structure equation and the Bianchi's identity by means of the exterior covariant differentiation for the $H(M)$-valued differential forms. A law of the exterior covariant differentiation is axiomatized by a map on the $C^{(n)}(M, R)$-module $\Omega_p^{(n)}(M, H(M))$ which satisfies certain requirements. The connection form of H-type is introduced by means of this exterior covariant differentiation and it is an assignment of a matrix valued one-form to every moving frame $\mathfrak{b}(x)$ on M.

In § 2 we perform the investigation of H-valued differential forms on M and obtain the same results as the case of real valued differential forms on the differentiable manifold. The definitions of the exterior multiplication and the exterior differentiation are generalized.

In § 3 we construct the moving frames on M and the fibre space $H_x(M)$ at every point x of M which has the same structure as the Hilbert space H. The $H(M)$-valued differential forms of class $C^{(n)}$ on M are defined with the help of the H-valued differential forms of class $C^{(n)}$ on M and it is showed that they are expressed in terms of the moving frame $\mathfrak{b}(x)$. The set $\Omega_p^{(n)}(M, H(M))$ of all $H(M)$-valued differential forms of class $C^{(n)}$ on M is a $C^{(n)}(M, R)$-module. The exterior differential of $\Omega_p \in \Omega_p^{(n)}(M, H(M))$ with respect to $\mathfrak{b}(x)$ is defined with the help of the exterior differential of H-valued differential forms on M and we obtain the familiar properties of the exterior differentiation.

In brief, the exterior covariant differentiation is a linear continuous map $D_H: \Omega_p^{(n)}(M, H(M)) \to \Omega_{p+1}^{(n-1)}(M, H(M))$ such that $D_H(\omega_q \wedge \Omega_p) = d\omega_q + (-1)^q \omega_q \wedge D_H \Omega_p$ for $\omega_q \in \Omega_p^{(n)}(M, R)$, $\Omega_p \in \Omega_p^{(n)}(M, H(M))$. The connection form ω_H of H-type with respect to $\mathfrak{b}(x)$ is defined by $D_H \mathfrak{b}(x) = \mathfrak{b}(x) \omega_H(x)$ and the curvature form R_H is defined by $D^2{}_H \mathfrak{b}(x) = \mathfrak{b}(x) R_H(x)$. The structure equation $R_H = d\omega_H + \omega_H \wedge \omega_H$ is obtained and the Bianchi's identity is derived under a condition.

Throughout this paper it is assumed that the Hilbert spaces are separable and by the base of the Hilbert space we mean the base $Ae = (Ae_1, \cdots\cdots, Ae_n, \cdots\cdots)$ made by an orthonormal base e and the bounded linear operator A which has the bounded inverse operator A^{-1}.

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§ 2. *H*-valued differential forms on the Hilbert manifold

1. $C^{(n)}$-diffeomorphisms

DEFINITION.

Let E, F be two Banach spaces and let V, W be open sets of E, F respectively. We say that a map f: $V \to W$ is *$C^{(n)}$-diffeomorphism* if the following conditions are satisfied:

(1) f is a bijection of V onto W,

(2) the maps f, f^{-1} are of class $C^{(n)}$.

Let $\mathfrak{e} = (\mathfrak{e}_1, \cdots\cdots, \mathfrak{e}_n, \cdots\cdots)$, $\mathfrak{f} = (\mathfrak{f}_1, \cdots\cdots, \mathfrak{f}_n, \cdots\cdots)$ be bases of two Hilbert spaces E, F. Giving a map $f: E \to F$, we expand the map f in terms of the base $\mathfrak{f}$:

$$f(x) = \sum_{i=1}^{\infty} f^i(x)\mathfrak{f}_i, \text{ for } x \in E.$$

PROPOSITION 1.

Suppose that a function f is of class $C^{(n)}$ in an open set V of E. Then f has the continuous partial derivatives $\partial_{j_1} \cdots\cdots \partial_{j_m} f(x)$ $(m \leqq n; j_1, \cdots\cdots,$

$j_m=1, 2, \cdots\cdots)$ *in* V *and we have*

$$f^{(m)}(x)(y_1, \cdots\cdots, y_m)=\sum_{j_1,\cdots\cdots,j_m=1}^{\infty}\partial j_1\cdots\cdots\partial j_m f(x) y_1{}^{j_1}\cdots\cdots y_m{}^{j_m},$$

$$\textit{for } x\in V \textit{ and for } y_1, \cdots\cdots, y_m\in E,$$

where $y_1=\sum_{j=1}^{\infty} y_1^j \mathfrak{e}_j, \cdots\cdots, y_m=\sum_{j=1}^{\infty} y_m^j \mathfrak{e}_j.$

PROPOSITION 2.

Suppose that a map $f: V\to F$ *is of class* $C^{(n)}$ *in* V. *Then every function* $f^i: V\to R$ *is of class* $C^{(n)}$ *in* V *and we have*

$$f^{(m)}(x)=\sum_{i=1}^{\infty} f^{i\,(m)}(x)\mathfrak{f}_i, \quad \textit{for } x\in V.$$

These propositions can be proved by induction on m. As an immediate consequence from Proposition 1, 2,

PROPOSITION 3.

Suppose that a map $f: V\to F$ *is of class* $C^{(n)}$ *in* V. *Then we have*

$$f^{(m)}(x)(y_1, \cdots\cdots, y_m)=\sum_{i=1}^{\infty}\left\{\sum_{j_1,\cdots\cdots,j_m=1}^{\infty}\partial j_1\cdots\cdots\partial j_m f^i(x) y_1{}^{j_1}\cdots\cdots y_m{}^{j_m}\right\}\mathfrak{f}_i.$$

2. Vector fields

By M we denote a Hilbert manifold of class $C^{(n)}$ whose base space is the Hilbert space E and we denote by $T_x(M)$ the tangent space of M at $x\in M$. By I_x we denote the index set of all charts at x.

Given an index $\alpha\in I_x$ and a tangent vector $\mathfrak{X}\in T_x(M)$, there exists an unique vector $X\in E$ such that $X_\alpha=(X, \alpha)\in\mathfrak{X}$. Thus we identify a representative X_α of $\mathfrak{X}$ with this unique vector $X\in E$ and we can consider a map $\phi_\alpha(x)$ of $T_x(M)$ into E such that $\phi_\alpha(x)(\mathfrak{X})=X_\alpha$. We define an inner product of $\mathfrak{X}, \mathfrak{Y}\in T_x(M)$ by $<\mathfrak{X}, \mathfrak{Y}>=<\phi_{\alpha_0}(x)(\mathfrak{X}), \phi_{\alpha_0}(x)(\mathfrak{Y})>$ where α_0 is an arbitrary fixed index belonging to I_x. Consequently we get a Hilbert space $T_x(M)$. Then a map $\phi_{\alpha_0}(x): T_x(M)\to E$ given by $X_{\alpha_0}=\phi_{\alpha_0}(x)(\mathfrak{X})$ is an isomorphism where X_{α_0} is the representative of $\mathfrak{X}$ identified with the vector of E.

We shall denote the tangent vector at $x\in M$ by X or X_x instead of $\mathfrak{X}$.

We shall go here to express the tangent vector by the derivation. Let

$(U_\alpha, \varphi_\alpha)$, (U_β, φ_β) be two charts at $x \in M$ and X_α, X_β be two representatives of $X \in T_x(M)$. Let U be an open set containing x. By $X(f)$ we shall mean the real value given by $X(f) = f'_\alpha(x_\alpha) X_\alpha$ where $f \in C^{(n)}(U, R)$, $f_\alpha = f \circ \varphi_\alpha^{-1}$ and $x_\alpha = \varphi_\alpha(x)$. This definition is independent of the choice of representatives of X:

$$(2.1)\quad X(f) = f'_\alpha(x_\alpha) X_\alpha = f'_\beta(x_\beta) X_\beta.$$

We have clearly

$$(2.2)\quad \begin{aligned} &(X+Y)(f) = X(f) + Y(f), \text{ for } X, Y \in T_x(M), \\ &(\lambda X)(f) = \lambda X(f), \text{ for } \lambda \in R. \end{aligned}$$

By applying Proposition 1 of §2.1 to the formulas (2.1) and (2.2), we get

$$X(f) = \sum_{i=1}^{\infty} X_\alpha^i \frac{\partial f_\alpha}{\partial x_\alpha^i}(x_\alpha) = \sum_{i=1}^{\infty} X_\beta^i \frac{\partial f_\beta}{\partial x_\beta^i}(x_\beta),$$

$$\sum_{i=1}^{\infty} (X_\alpha^i + Y_\alpha^i) \frac{\partial f_\alpha}{\partial x_\alpha^i}(x_\alpha) = \sum_{i=1}^{\infty} X_\alpha^i \frac{\partial f_\alpha}{\partial x_\alpha^i}(x_\alpha) + \sum_{i=1}^{\infty} Y_\alpha^i \frac{\partial f_\alpha}{\partial x_\alpha^i}(x_\alpha),$$

$$\sum_{i=1}^{\infty} (\lambda X_\alpha^i) \frac{\partial f_\alpha}{\partial x_\alpha^i}(x_\alpha) = \lambda \sum_{i=1}^{\infty} X_\alpha^i \frac{\partial f_\alpha}{\partial x_\alpha^i}(x_\alpha),$$

where X_α^i, X_β^i, x_α^i, x_β^i are respectively components of X_α, X_β, x_α, x_β with respect to the base $\mathfrak{e}$. These facts show that a tangent vector X may be expressed by $X = \sum_{i=1}^{\infty} X^i \frac{\partial}{\partial x^i}$ and that $X+Y$, λX may be executed as follows:

$$\sum_{i=1}^{\infty} X^i \frac{\partial}{\partial x^i} + \sum_{i=1}^{\infty} Y^i \frac{\partial}{\partial x^i} = \sum_{i=1}^{\infty} (X^i + Y^i) \frac{\partial}{\partial x^i},$$

$$\lambda \sum_{i=1}^{\infty} X^i \frac{\partial}{\partial x^i} = \sum_{i=1}^{\infty} (\lambda X^i) \frac{\partial}{\partial x^i}.$$

Consequently we may identify a tangent vector $X = \sum_{i=1}^{\infty} X^i \mathfrak{e}_i$ with a corresponding derivation $\sum_{i=1}^{\infty} X^i \frac{\partial}{\partial x^i}$.

Finally we shall define the vector field of class $C^{(n)}$. Let us put $T(M) = \bigcup_{x \in M} T_x(M)$ and let U be an open set of M containing an arbitrary point $x_0 \in M$.

DEFINITION.

By a vector field on M we mean a map $X: M \to T(M)$ such that $X(x)$ belongs to $T_x(M)$. By a vector field of class $C^{(n)}$ on M we mean a vector field on M such that a map $\phi_{\alpha_0}(X): U \to E$ given by $\phi_{\alpha_0}(X)(x) = \phi_{\alpha_0}(x)(X(x))$ belongs to $C^{(n)}(U, E)$.

By $\mathfrak{X}^{(n)}(M)$ we denote a set of all vector fields of class $C^{(n)}$ on M. $\mathfrak{X}^{(n)}(M)$ is a $C^{(n)}(M, R)$-module.

Next, we define the bracket $[X, Y]$ *for* $X, Y \in \mathfrak{X}^{(n)}(M)$ as a map from $C^{(n)}(M, R)$ into $C^{(n-2)}(M, R)$ as follows:

$$[X, Y] = \bar{X}(\bar{Y}(f)) - \bar{Y}(\bar{X}(f)),$$

for $\bar{X} = \sum_{i=1}^{\infty} \bar{X}_i(x) \frac{\partial}{\partial x^i}$ *and* $\bar{Y} = \sum_{i=1}^{\infty} \bar{Y}_i(x) \frac{\partial}{\partial x^i}$,

where $\bar{X}(x) = \phi_{\alpha_0}(x)(X(x)) = \sum_{i=1}^{\infty} \bar{X}_i(x)\mathfrak{e}_i$, $\bar{Y}(x) = \phi_{\alpha_0}(x)(Y(x)) = \sum_{i=1}^{\infty} \bar{Y}_i(x)\mathfrak{e}_i$.

The bracket $[X, Y]$ is a vector field of $C^{(n-2)}$ on M and we have

$$[Y, X] = -[X, Y],$$

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0,$$

$$\text{for } X, Y, Z \in \mathfrak{X}^{(n)}(M).$$

3. *H*-valued differential forms

Let H be a Hilbert space. Let U be an open set containing an arbitrary point $x_0 \in M$. We denote $\bigcup_{x \in M} \mathscr{A}_p(T_x(M), H)$ by $\mathscr{A}_p(T(M), H)$ where $\mathscr{A}_p(T_x(M), H)$ is the set of all p-linear alternating continuous maps of $T_x(M)$ into H.

DEFINITION.

By a H-valued differential p-form on M we mean a map $\Omega_p: M \to \mathscr{A}_p(T(M), H)$ such that $\Omega_p(x)$ belongs to $\mathscr{A}_p(T_x(M), H)$ for each point $x \in M$. By a H-valued differential p-form Ω_p of class $C^{(n)}$ on M we mean a H-valued differential p-form such that a map $\Omega_p(X_1, \cdots\cdots, X_p): U \to H$ given by $\Omega_p(X_1, \cdots\cdots, X_p)(x) = \Omega_p(x)(X_1(x), \cdots\cdots, X_p(x))$ belongs to $C^{(n)}(U, H)$ where $X_1, \cdots\cdots, X_p \in \mathfrak{X}^{(n)}(M)$.

We write $\Omega_p(x: X_1, \cdots\cdots, X_p)$ for $\Omega_p(x)(X_1(x), \cdots\cdots, X_p(x))$. By $\Omega_p^{(n)}(M, H)$ we denote a set of all H-valued differential p-forms of class $C^{(n)}$ on M. This $\Omega_p^{(n)}(M, H)$ is a real vector space under the natural addition and scalar multiplication and becomes a $C^{(n)}(M, R)$-module by the introduction of a multiplication $(f\Omega_p)(x) = f(x)\Omega_p(x)$.

PROPOSITION.

Let $\mathfrak{Y}=(\mathfrak{Y}_1, \cdots\cdots, \mathfrak{Y}_n, \cdots\cdots)$ be a base of H. A H-valued differential p-form $\Omega_p \in \Omega_p^{(n)}(M, H)$ is uniquely expressed in terms of $\mathfrak{Y}$ as follows:

$$\Omega_p(x) = \sum_{i=1}^{\infty} \omega_p^i(x)\mathfrak{Y}_i,$$

where $\omega_p^i \in \Omega_p^{(n)}(M, R)$ $(i=1, 2, \cdots\cdots)$.

PROOF. We see readily by Proposition 2 of § 2.1 that $\omega_p^i (i=1, 2, \cdots\cdots)$ are of class $C^{(n)}$.

4. Exterior multiplication

Let H_1, H_2 be Hilbert spaces and $H_1 \otimes H_2$ be the direct product of H_1 and H_2. Let $\mathfrak{Y}^{(1)}=(\mathfrak{Y}_1^{(1)}, \cdots\cdots, \mathfrak{Y}_n^{(1)}, \cdots\cdots)$, $\mathfrak{Y}^{(2)}=(\mathfrak{Y}_1^{(2)}, \cdots\cdots, \mathfrak{Y}_n^{(2)}, \cdots\cdots)$ be the bases of H_1, H_2 respectively.

DEFINITION.

By the exterior product of $\Phi_p \in \Omega_p^{(n)}(M, H_1)$ and $\Psi_q \in \Omega_q^{(n)}(M, H_2)$, denoted by $\Phi_p \wedge \Psi_q$, we mean a H-valued differential $(p+q)$–form such that

$$\begin{aligned}&(\Phi_p \wedge \Psi_q)(x : X_1, \cdots\cdots, X_{p+q}) \\ &\qquad = \sum_{\sigma} \varepsilon(\sigma) \Phi_p(x : X_{\sigma(1)}, \cdots\cdots, X_{\sigma(p)}) \otimes \Psi_q(x : X_{\sigma(p+1)}, \cdots\cdots, X_{\sigma(p+q)}),\end{aligned}$$

for $x \in M$ and $X_1, \cdots\cdots, X_{p+q} \in \mathfrak{X}^{(n)}(M)$,
where the sum is done over all permutations σ of $\{1, 2, \cdots\cdots, p+q\}$ satisfying $\sigma(1) < \cdots\cdots < \sigma(p)$, $\sigma(p+1) < \cdots\cdots < \sigma(p+q)$
and

$$\varepsilon(\sigma) = \begin{cases} +1, & \text{for even permutation } \sigma, \\ -1, & \text{for odd permutation } \sigma. \end{cases}$$

The exterior product $\Phi_p \wedge \Psi_q$ is of class $C^{(n)}$.

PROPOSITION.

For $\Phi_p \in \Omega_p^{(n)}(M, H_1)$, $\Psi_q \in \Omega_q^{(n)}(M, H_2)$ expressed in the forms $\Phi_p(x) = \sum_{i=1}^{\infty} \varphi_p^i(x)\mathfrak{Y}_i^{(1)}$ and $\Psi_q(x) = \sum_{i=1}^{\infty} \psi_q^i(x)\mathfrak{Y}_i^{(2)}$, we have

$$(\Phi_p \wedge \Psi_q)(x) = \sum_{i,j=1}^{\infty} (\psi_p^i \wedge \varphi_q^j)(x)\mathfrak{Y}_i^{(1)} \otimes \mathfrak{Y}_j^{(2)}.$$

PROOF. This proposition can be proved by using the fact that the direct product is continuous with respect to each factor.

5. Exterior differentiation

DEFINITION.

The exterior differential $d\Omega_p$ of $\Omega_p \in \Omega_p^{(n)}(M. H)$ is defined by the following formula:

$$d\Omega_p(x: X_1, \cdots\cdots, X_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} \Omega_p'(x: X_1, \cdots\cdots, \hat{X}_i, \cdots\cdots, X_{p+1})(\bar{X}_i)$$
$$+ \sum_{1 \leq i < j \leq p+1} (-1)^{i+j} \Omega_p(x: [X_i, X_j], X_1, \cdots\cdots, \hat{X}_i, \cdots\cdots, \hat{X}_j, \cdots\cdots, X_{p+1}),$$

for $X_1, \cdots\cdots, X_{p+1} \in \mathfrak{X}^{(n)}(M)$,
where the symbol $\wedge$ means that the term is omitted and $\bar{X}_i$ means $\bar{X}_i(x) = \Phi_{\alpha_0}(x)(X_i(x))$. Henceforth we shall omit the symbol $-$.

PROPOSITION 1.

The exterior differential $d\Omega_p$ of $\Omega_p \in \Omega_p^{(n)}(M, H)$ is a H-valued differential $(p+1)$-form of class $C^{(n-1)}$ on M.

PROPOSITION 2.

We have, for $\Phi_p \in \Omega_p^{(n)}(M, H_1)$ and $\Psi_q \in \Omega_q^{(n)}(M, H_2)$,

$$\begin{aligned} &d(\Phi_p \wedge \Psi_q) = (d\Phi_p) \wedge \Psi_q + (-1)^p \Phi_p \wedge (d\Psi_q), \\ (5.1)\quad &dd\Omega_p = 0, \ (n \geqq 2). \end{aligned}$$

We omit the proofs of the above two propositions, since those are merely straight-forward calculations.

PROPOSITION 3.

Suppose that Ω_p is a H-valued differential p-form of class $C^{(n)}$ on M expressed in the form $\Omega_p(x) = \sum_{i=1}^{\infty} \omega_p^i(x) \mathfrak{Y}_i$. Then the exterior differential $d\Omega_p$ of Ω_p is given by $d\Omega_p(x) = \sum_{i=1}^{\infty} d\omega_p^i(x) \mathfrak{Y}_i$.

PROOF. This is seen by using the definition of the exterior differentiation and Proposition 2 of §2.1.

§ 3. Connection form and curvature form

1. H(M)-valued vector fields

By an isomorphism on the Hilbert space H we mean a one-to-one linear continuous map of H onto itself. By $GL(H)$ we denote a set of all isomorphisms on H. This $GL(H)$ is a group and becomes a Banach space by introducing the norm in a well-known manner.

Let $(U_\alpha, \varphi_\alpha)$, (U_β, φ_β) be charts at an arbitrary point $x_0 \in M$. By $\mathfrak{U}_\alpha$ we mean a set of all maps $A_\alpha : U_\alpha \to GL(H)$ such that two maps A_α, A_α^{-1} are of class $C^{(n)}$ in U_α where $A_\alpha(x)A_\alpha^{-1}(x)=1$ for $x \in U_\alpha$. By $\mathfrak{F}_{x_0}$ we denote a family of all pairs (U_α, A_α) such that $A_\alpha \in \mathfrak{U}_\alpha$, $\alpha \in I_{x_0}$. For $A_\alpha \in \mathfrak{U}_\alpha$ and $A_\beta \in \mathfrak{U}_\beta$, we define their product $A_\alpha A_\beta$ by $(A_\alpha A_\beta)(x) = A_\alpha(x) A_\beta(x)$ for $x \in U_\alpha \cap U_\beta$. We denote by $A_{\alpha\beta}$ the product $A_\alpha A_\beta^{-1}$. A map $A_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(H)$ is of class $C^{(n)}$ and we see

$$
(1.1) \quad \begin{aligned} &A_{\alpha\beta}(x) = A_{\beta\alpha}^{-1}(x), \ \text{ for } x \in U_\alpha \cap U_\beta, \\ &A_{\alpha\gamma}(x) = A_{\alpha\beta}(x) A_{\beta\gamma}(x), \ \text{ for } x \in U_\alpha \cap U_\beta \cap U_\gamma. \end{aligned}
$$

Let $\mathfrak{Y} = (\mathfrak{Y}_1, \cdots\cdots, \mathfrak{Y}_n, \cdots\cdots)$ be a base of H and let $(U_{\alpha_0}, A_{\alpha_0})$, (U_α, A_α) be elements of $\mathfrak{F}_x$ where α_0 is an arbitrary fixed index belonging to I_x. We consider a base $\mathfrak{Y}A_{\alpha_0\alpha}(x) = (A_{\alpha_0\alpha}\mathfrak{Y}_1, \cdots\cdots, A_{\alpha_0\alpha}\mathfrak{Y}_n, \cdots\cdots)$ of H. By writing $\mathfrak{b}_\alpha(x) = \mathfrak{Y}A_{\alpha_0\alpha}(x)$ and $\mathfrak{b}_n^\alpha(x) = A_{\alpha_0\alpha}\mathfrak{Y}_n$, we get a base $\mathfrak{b}_\alpha(x) = (\mathfrak{b}_1^\alpha(x), \cdots\cdots, \mathfrak{b}_n^\alpha(x), \cdots\cdots)$ of H. We define the following operation: $\mathfrak{b}_\alpha(x) A_{\alpha\beta}(x) = (A_{\alpha_0\alpha}(x)\{A_{\alpha\beta}(x)\mathfrak{Y}_1\}, \cdots\cdots, A_{\alpha_0\alpha}(x)\{A_{\alpha\beta}(x)\mathfrak{Y}_n\}, \cdots\cdots)$. Thus we get

$$
(1.2) \quad \mathfrak{b}_\beta(x) = \mathfrak{b}_\alpha(x) A_{\alpha\beta}(x).
$$

DEFINITION 1.

A base $\mathfrak{b}_\alpha(x)$ of H is said to be a *moving frame of H-type* on the Hilbert manifold M.

The relation (1.2) gives the transformation law between two moving frames $\mathfrak{b}_\alpha(x)$ and $\mathfrak{b}_\beta(x)$.

We denote by Z_α an element $(Z, \alpha) \in H \times I_x$ and define the operation AZ_α by $(AZ, \alpha) \in H \times I_x$ for $A \in GL(H)$. We say that Z_α, $Z_\beta \in H \times I_x$ are equivalent if there exist two pairs (U_α, A_α), $(U_\beta, A_\beta) \in \mathfrak{F}_x$ such that

$$
(1.3) \quad Z_\beta = A_{\beta\alpha} Z_\alpha.
$$

The equivalence thus defined satisfies the axiom of equivalence relation by

(1.1). By a fibre element $\mathfrak{Z}$ at $x\in M$ we mean an equivalence class $\{Z_\alpha\}$ of elements of $H\times I_x$. We denote by $H_x(M)$ a set of all fibre elements $\mathfrak{Z}$ at x. We introduce the addition and the multiplication into $H_x(M)$ as follows: $\mathfrak{Z}_1+\mathfrak{Z}_2=\{(Z_1+Z_2)_\alpha\}, \lambda\mathfrak{Z}_1=\{(\lambda Z_1)_\alpha\}$ for $\mathfrak{Z}_1=\{(Z_1)_\alpha\}, \mathfrak{Z}_2=\{(Z_2)_\alpha\}\in H_x(M)$ and $\lambda\in R$. Thus $H_x(M)$ is a real vector space.

DEFINITION 2.

The vector space $H_x(M)$ is said to be *a fibre space of H-type at $x\in M$.*

Given $\alpha\in I_x$ and $\mathfrak{Z}\in H_x(M)$, there exists an unique vector $Z\in H$ such that $Z_\alpha=(Z,\alpha)\in\mathfrak{Z}$. Thus we identify a representative Z_α of $\mathfrak{Z}$ with this unique vector $Z\in H$ and we consider a map $\Phi_\alpha(x):H_x(M)\to H$ such that $\Phi_\alpha(x)(\mathfrak{Z})=Z_\alpha$. We define an inner product of $\mathfrak{Z}_1, \mathfrak{Z}_2\in H_x(M)$ by $<\mathfrak{Z}_1\ \mathfrak{Z}_2>=<\Phi_{\alpha_0}(x)(\mathfrak{Z}_1), \Phi_{\alpha_0}(x)(\mathfrak{Z}_2)>$. Consequently we get a fibre space $H_x(M)$ which has the same structure as the Hilbert space H:

PROPOSITION 1.

A map $\Phi_{\alpha_0}(x)$ is an isomorphism of $H_x(M)$ onto H.

We denote by Z or Z_x the fibre element $\mathfrak{Z}\in H_x(M)$.

We proceed to express a fibre element $Z\in H_x(M)$ in terms of a moving frame on M. This corresponds to express a tangent vector $X\in T_x(M)$ as the derivation. We consider a vector of H denoted by $\mathfrak{b}_\alpha(x)Z_\alpha$: $\mathfrak{b}_\alpha(x)Z_\alpha=\sum_{i=1}^\infty Z_\alpha^i\mathfrak{b}_i^\alpha(x)$ where $Z_\alpha=\sum_{i=1}^\infty Z_\alpha^i\mathfrak{Y}_i$. We see $\mathfrak{b}_\alpha(x)Z_\alpha=\mathfrak{b}_\beta Z_\beta$ from (1.2) and (1.3). This shows that a vector $\mathfrak{b}_\alpha(x)Z_\alpha\in H$ is irrelevant to the representatives of $Z\in H_x(M)$. Therefore we may identify a fibre element Z with the vector $\mathfrak{b}_\alpha(x)Z_\alpha$.

PROPOSITION 2.

A fibre element $Z\in H_x(M)$ is expressed in the form: $Z=\sum_{i=1}^\infty Z_\alpha^i\mathfrak{b}_i^\alpha(x)$ where $Z_\alpha=\sum_{i=1}^\infty Z_\alpha^i\mathfrak{Y}_i$ for a representative Z_α of Z.

Henceforth we write for $Z\in H_x(M)$ in the form: $Z=\sum_{i=1}^\infty Z^i\mathfrak{b}_i(x)$.

By the fibre bundle of H-type we mean a set $H(M)=\bigcup_{x\in M}H_x(M)$.

DEFINITION 3.

A map $Z:U\to H(M)$ such that $Z(x)\in H_x(M)$ for $x\in U$, is said to be a *$H(M)$-valued vector field on M.* A $H(M)$-valued vector field Z on M is said to be *of class $C^{(n)}$ on M* if a map $\Phi_{\alpha_0}\circ Z:U\to H$ given by $(\Phi_{\alpha_0}\circ Z)(x)=\Phi_{\alpha_0}(x)(Z(x))$ for $x\in U$, is of class $C^{(n)}$ in U.

By $\mathfrak{Z}^{(n)}(M)$ we mean a set of all $H(M)$-valued vector fields of class $C^{(n)}$ on M. For $Z\in\mathfrak{Z}^{(n)}(M)$ we write $Z(x)=\sum_{i=1}^{\infty}Z^i(x)\mathfrak{b}_i(x)$.

2. $H(M)$-valued differential forms

By the differential p-form bundle of H-type on M we mean a set $\mathscr{A}_p(T(M),H(M))=\bigcup_{x\in M}\mathscr{A}_p(T_x(M),H_x(M))$.

DEFINITION 1.

A map Ω_p: $U\to\mathscr{A}_p(T(M),H(M))$ such that $\Omega_p(x)\in\mathscr{A}_p(T_x(M),H_x(M))$ for $x\in U$, is said to be a *$H(M)$-valued differential p-form on M*. A $H(M)$-valued differential p-form Ω_p on M is said to be *of class $C^{(n)}$ on M* if a map $(\Phi_{\alpha_0}\circ\Omega_p)(X_1,\cdots\cdots,X_p):U\to H$ given by $(\Phi_{\alpha_0}\circ\Omega_p)(X_1,\cdots\cdots,X_p)(x)=\Phi_{\alpha_0}(x)(\Omega_p(x:X_1,\cdots\cdots,X_p))$ is of *class $C^{(n)}$ in U* where $X_1,\cdots\cdots,X_p\in\mathfrak{X}^{(n)}(M)$.

We denote by $\Omega_p^{(n)}(M,H(M))$ a set of all $H(M)$-valued differential p-forms of class $C^{(n)}$ on M. A set $\Omega_p^{(n)}(M,H(M))$ becomes a $C^{(n)}(M,R)$-module.

PROPOSITION 1.

A $H(M)$-valued differential p-form $\Omega_p\in\Omega_p^{(n)}(M,H(M))$ is uniquely expressed in terms of $\mathfrak{b}(x)$ in the form $\Omega_p(x)=\sum_{i=1}^{\infty}\omega_p^i(x)\mathfrak{b}_i(x)$ where $\omega_p^i\in\Omega_p^{(n)}(M,R)$.

PROOF. This is derived from Proposition of §2.3 and Proposition 2 of §3.1.

Let H_1,H_2 be Hilbert spaces and $H_x^1(M),H_x^2(M)$ be fibre spaces of H_1, H_2-type at $x\in M$. Let us consider $H=H_1\otimes H_2, H_x(M)=H_x^1(M)\otimes H_x^2(M)$. An isomorphism $\Phi_{\alpha_0}(x):H_x(M)\to H$ is introduced as a linear map such that $\Phi_{\alpha_0}(x)(Z)=\Phi_{\alpha_0}^1(x)(Z_1)\otimes\Phi_{\alpha_0}^2(x)(Z_2)$ for $Z=Z_1\otimes Z_2$ where $\Phi_{\alpha_0}^1(x),\Phi_{\alpha_0}^2(x)$ are the isomorphisms of $H_x^1(M),H_x^2(M)$ onto H_1,H_2. Let $H_1(M),H_2(M),H(M)$ be fibre bundles of H_1,H_2, H-type on M respectively. The exterior product $\Omega_p^1\wedge\Omega_q^2$ of $\Omega_p^1\in\Omega_p^{(n)}(M,H_1(M))$ and $\Omega_q^2\in\Omega_q^{(n)}(M,H_2(M))$ is defined in the same manner as in the definition of §2.4.

PROPOSITION 2.

For $\Omega_p^1\in\Omega_p^{(n)}(M,H_1(M)),\Omega_q^2\in\Omega_q^{(n)}(M,H_2(M))$ expressed in the forms

$\Omega_p^1(x)=\sum_{i=1}^{\infty}\overset{1}{\omega}{}_p^i(x)\mathfrak{b}_i^1(x)$ *and* $\Omega_q^2(x)=\sum_{i=1}^{\infty}\overset{2}{\omega}{}_q^i\mathfrak{b}_i^2(x)$, *we have* $\Omega_p^1\wedge\Omega_q^2\in\Omega_{p+q}^{(n)}(M, H(M))$ *and* $(\Omega_p^1\wedge\Omega_q^2)(x)=\sum_{i,j=1}^{\infty}(\overset{1}{\omega}{}_p^i\wedge\overset{2}{\omega}{}_q^j)(x)\mathfrak{b}_i^1(x)\otimes\mathfrak{b}_j^2(x)$ *where* $\mathfrak{b}_1(x)=(\mathfrak{b}_1^1(x), \cdots\cdots, \mathfrak{b}_n^1(x), \cdots\cdots)$, $\mathfrak{b}_2(x)=(\mathfrak{b}_1^2(x), \cdots\cdots, \mathfrak{b}_n^2(x), \cdots\cdots)$ *are moving frames on* M *which are the bases of* $H_x^1(M), H_x^2(M)$.

We shall discuss the exterior differential of $H(M)$-valued differential forms with respect to a moving frame on M. For $\Omega_p\in\Omega_p{}^{(n)}(M, H(M))$ written in the form $\Omega_p(x)=\sum_{i=1}^{\infty}\omega_p^i(x)\mathfrak{b}_i(x)$, we consider the differential form $\tilde{\Omega}_p(x)=\sum_{i=1}^{\infty}\omega_p^i(x)\mathfrak{Y}_i$. We see $\tilde{\Omega}_p\in\Omega_p{}^{(n)}(M, H)$, since $\tilde{\Omega}_p(x)=A^{-1}(x)(\Phi_{\alpha_0}\circ\Omega_p)(x)$ where $A(x)\in GL(H)$ such that $\mathfrak{b}_i(x)=A(x)\mathfrak{Y}_i(i=1, 2, \cdots\cdots)$. We get $d\tilde{\Omega}_p(x)=\sum_{i=1}^{\infty}d\omega_p^i(x)\mathfrak{Y}_i$ by Proposition 3 of §2.5 and we see $d\Omega_p\in\Omega_{p+1}^{(n-1)}(M, H)$. Hence we have $(Ad\tilde{\Omega}_p)(x)=\sum_{i=1}^{\infty}d\omega_p^i(x)\mathfrak{b}_i(x)$ and $Ad\tilde{\Omega}_p\in\Omega_{p+1}^{(n-1)}(M, H(M))$ where we mean $(Ad\tilde{\Omega}_p)(x)(X_1, \cdots\cdots, X_{p+1})=A(x)d\tilde{\Omega}_p(x{:}X_1, \cdots\cdots, X_{p+1})$ for $X_1, \cdots\cdots, X_{p+1}\in\mathfrak{X}^{(n)}(M)$.

DEFINITION 2.

For $\Omega_p\in\Omega_p{}^{(n)}(M, H(M))$ written in the form $\Omega_p(x)=\sum_{i=1}^{\infty}\omega_p^i(x)\mathfrak{b}_i(x)$, the exterior differential $d\Omega_p$ *of* Ω_p with respect to $\mathfrak{b}(x)$ is defined by $d\Omega_p(x)=\sum_{i=1}^{\infty}d\omega_p^i(x)\mathfrak{b}_i(x)$.

PROPOSITION 3.

$$d(\Omega_p^1\wedge\Omega_q^2)=(d\Omega_p^1)\wedge\Omega_q^2+(-1)^p\Omega_p^1\wedge d\Omega_q^2, \quad d^2\Omega_p=0.$$

PROOF. The exterior differential of $H(M)$-valued differential forms with respect to a moving frame on M is determined by the exterior differential of real valued differential forms. Therefore this proposition is derived from Proposition 2 of §2.5.

3. Exterior covariant differentiation

DEFINITION 1.

The exterior covariant differentiation of H-type is a linear map D_H: $\Omega_p{}^{(n)}(M, H(M))\to\Omega_{p+1}^{(n-1)}(M, H(M))$ (over R) satisfying the following requirements: (A_1) a map Ψ_H given by $\Psi_H(\Omega_p(x{:}X_1, \cdots\cdots, X_p))=(D_H\Omega_p)(x{:}X_1, \cdots\cdots, X_{p+1})$ is continuous, (A_2) $D_H(\omega_q\wedge\Omega_p)=d\omega_q\wedge\Omega_p+(-1)^q\omega_q\wedge D_H\Omega_p$ for $\omega_q\in\Omega_q{}^{(n)}(M, R)$, (A_3) $D_HC=0$ for constant C.

If there is no posibility of confusion, we omit the symbol of the type.

DEFINITION 2.

The exterior covariant differentiation $D_{H_1\otimes H_2}$ *of* $H_1\otimes H_2$*-type* is defined by $D_{H_1\otimes H_2}(\Omega_p^1\wedge\Omega_q^2)=(D_{H_1}\Omega_p^1)\wedge\Omega_q^2+(-1)^p\Omega_p^1\wedge(D_{H_2}\Omega_q^2)$ *for* $\Omega_p^1\in\Omega_p{}^{(n)}(M, H_1(M))$, $\Omega_q^2\in\Omega_q{}^{(n)}(M, H_2(M))$.

PROPOSITION.

For $\Omega_p\in\Omega_p{}^{(n)}(M, H(M))$ *expressed in the form* $\Omega_p(x)=\sum_{i=1}^{\infty}\omega_p^i(x)\mathfrak{b}_i(x)$, *we have* $D\Omega_p(x)=\sum_{i=1}^{\infty}d\omega_p^i(x)\mathfrak{b}_i(x)+(-1)^p\sum_{i=1}^{\infty}\omega_p^i(x)\wedge D\mathfrak{b}_i(x)$.

PROOF. From $\mathfrak{b}_i(x)=1\wedge\mathfrak{b}_i(x)$, we see $\Omega_p(x)=\sum_{i=1}^{\infty}\omega_p^i(x)\wedge\mathfrak{b}_i(x)$ where 1 denotes a base of R. In virtue of (A_1) and (A_2), we have

$$(3.1)\qquad D\Omega_p(x)=\sum_{i=1}^{\infty}d\omega_p^i(x)\mathfrak{b}_i(x)+(-1)^p\sum_{i=1}^{\infty}\omega_p^i(x)\wedge D\mathfrak{b}_i(x).$$

4. Connection forms

We shall introduce the connection form by means of the exterior covariant differentiation.

We have in virtue of (A_1)

$$(4.1)\qquad D\mathfrak{b}_i(x)=\sum_{j=1}^{\infty}\omega_i{}^j(x)\mathfrak{b}_j(x),$$

where $\omega_i{}^j\in\Omega_1{}^{(n-1)}(M, R)$ and the matrix $\omega(x:X)=(\omega_i{}^j(x:X))$ is a bounded linear operator on $H_x(M)$ where $X\in\mathfrak{X}^{(n)}(M)$. We write (4.1) in the form $D\mathfrak{b}(x)=\mathfrak{b}(x)\omega(x)$. For $\Omega_0\in\Omega_0{}^{(n)}(M, H(M))$ expressed in the form $\Omega_0(x)=\sum_{i=1}^{\infty}\omega^i(x)\mathfrak{b}_i(x)$, we get from (3.1) and (4.1)

$$(4.2)\qquad D\Omega_0(x)=\sum_{i=1}^{\infty}\{d\omega^i(x)+\sum_{j=1}^{\infty}\omega_j{}^i(x)\omega^j(x)\}\mathfrak{b}_i(x).$$

Inserting $d\omega^i(x)=\sum_{k=1}^{\infty}\partial_k\omega^i(x)dx^k$ and $\omega_j{}^i(x)=\sum_{k=1}^{\infty}\Gamma_j{}^i{}_k(x)dx^k$ into (4.2), we obtain

$$D\Omega_0(x:X)=\sum_{i,j=1}^{\infty}\nabla_{Xj}{}^i(x)\omega^j(x)\mathfrak{b}_i(x),\quad\text{for } X\in\mathfrak{X}^{(n)}(M),$$

where

$$(4.3)\qquad \nabla_{Xj}{}^i(x)=\sum_{k=1}^{\infty}dx^k(X)(\delta_j^i\partial_k+\Gamma_j{}^i{}_k(x)).$$

DEFINITION 1.

We say that $\omega=(\omega_i{}^j)$, $\Gamma=(\Gamma_i{}^j{}_k)$, $\nabla_X=(\nabla_{X i}{}^j)$ are respectively *the connection form, the connection, the covariant derivation in the direction of X of H-type with respect to $\mathfrak{b}(x)$.*

PROPOSITION 1.

For $\Omega_0^1, \Omega_0^2 \in \Omega_0{}^{(n)}(M, H(M))$ expressed in the forms $\Omega_0^1(x)=\sum_{i=1}^{\infty} \overset{1}{\omega}{}^i(x)\mathfrak{b}_i(x)$ and $\Omega_0^2(x)=\sum_{i=1}^{\infty} \overset{2}{\omega}{}^i(x)\mathfrak{b}_i(x)$, we have

$$\nabla_{X+Y}\overset{1}{\omega}=\nabla_X\overset{1}{\omega}+\nabla_Y\overset{1}{\omega},$$

$$\nabla_X(\overset{1}{\omega}+\overset{2}{\omega})=\nabla_X\overset{1}{\omega}+\nabla_X\overset{2}{\omega},$$

$$\nabla_{\lambda X}\overset{1}{\omega}=\lambda\nabla_X\overset{1}{\omega}, \quad for\ \lambda\in R,$$

$$\nabla_X(f\overset{1}{\omega})=f\nabla_X\overset{1}{\omega}+X(f)\,\overset{1}{\omega}, \quad for\ f\in C^{(n)}(M, R),$$

where $\overset{1}{\omega}=(\overset{1}{\omega}{}^1, \overset{1}{\omega}{}^2, \cdots\cdots, \overset{1}{\omega}{}^n, \cdots\cdots)$ *and* $\overset{2}{\omega}=(\overset{2}{\omega}{}^1, \overset{2}{\omega}{}^2, \cdots\cdots, \overset{2}{\omega}{}^n, \cdots\cdots)$.

PROOF. The above formulas are derived easily from (4.3).

We consider the dual space $H_x^*(M)$ of $H_x(M)$ which is said to be the fibre space of H^*-type at $x\in M$ and get the dual fibre bundle $H^*(M)=\bigcup_{x\in M} H_x^*(M)$ of $H(M)$. For $X\in H_x(M)$ and $X^*\in H_x^*(M)$, we denote by $<X^*|X>$ the value of the bounded linear functional X^* at X. We denote by $\Omega_{(p,q)}(M, H(M))$ a set of all exterior products $\Omega_p^*\wedge\Omega_q$ where $\Omega_p^*\in\Omega_p{}^{(n)}(M, H^*(M))$, $\Omega_q\in\Omega_q{}^{(n)}(M, H(M))$.

DEFINITION 2.

The contraction C is a map $C:\Omega_{(p,q)}(M, H(M))\rightarrow C^{(n)}(M, R)$ given by $C(\Omega_p^*\wedge\Omega_q)(x:X_1, \cdots\cdots, X_{p+q})=<\Omega_p^*(x;X_1, \cdots\cdots, X_p)|\Omega_q(x:X_{p+1}, \cdots\cdots, X_{p+q})>$ for $X_1, \cdots\cdots, X_{p+q}\in\mathfrak{X}^{(n)}(M)$.

We set up the following requirement:

(R_1) the exterior covariant differentiation commute with the contraction.

PROPOSITION 2.

Let $\mathfrak{b}^*(x)=(\mathfrak{b}^1(x), \cdots\cdots, \mathfrak{b}^n(x), \cdots\cdots)$ be the dual moving frame of $\mathfrak{b}(x)$ $=(\mathfrak{b}_1(x), \cdots\cdots, \mathfrak{b}_n(x), \cdots\cdots):<\mathfrak{b}^i(x)|\mathfrak{b}_j(x)>=\delta^i_j$. Assume that $D_H\mathfrak{b}(x)=\mathfrak{b}(x)\omega_H(x)$ and $D_{H*}\mathfrak{b}^*(x)=\mathfrak{b}^*(x)\omega_{H*}(x)$. Then we have $\omega_H(x)+\omega_{H*}(x)=0$.

PROOF. By (A_3) and (R_1), we get $CD_{H*\otimes H}(\mathfrak{b}^i(x)\wedge\mathfrak{b}_j(x))=D_{H*\otimes H}C(\mathfrak{b}^i(x)\wedge\mathfrak{b}_j(x))=D_{H*\otimes H}\delta^i_i=0$. By the definition of $D_{H*\otimes H}$, we see

$$CD_{H*\otimes H}(\mathfrak{b}^i(x)\wedge\mathfrak{b}_j(x))=C(\{\sum_{k=1}^{\infty}\omega_{H*k}{}^i(x)\mathfrak{b}^k(x)\}\wedge\mathfrak{b}_j(x)$$
$$+\mathfrak{b}^i(x)\wedge\{\sum_{k=1}^{\infty}\omega_{Hj}{}^k(x)\mathfrak{b}_k(x)\})$$
$$=\sum_{k=1}^{\infty}\omega_{H*k}{}^i(x)\langle\mathfrak{b}^k(x)\,|\,\mathfrak{b}_j(x)\rangle+\sum_{k=1}^{\infty}\omega_{Hj}{}^k(x)\langle\mathfrak{b}^i(x)\,|\,\mathfrak{b}_k(x)\rangle$$
$$=\omega_{H*j}{}^i(x)+\omega_{Hj}{}^i(x).$$

Consequently we have $\omega_{H*j}{}^i(x)+\omega_{Hj}{}^i(x)=0$.

DEFINITION 3.

Let ω_H be a connection form of H-type. The connection form ω_{H*} *of* H*-type such that $\omega_H+\omega_{H*}=0$, is said to be ***the dual connection form of*** ω_H.

EXAMPLE: For $\Omega_p\in\Omega_p{}^{(n)}(M,(H\otimes H^*)(M))$ expressed in the form $\Omega_p(x)=\sum_{i,j=1}^{\infty}\omega_{p},{}^i_j(x)\mathfrak{b}_i(x)\otimes\mathfrak{b}^j(x)$, we get

$$D_{H\otimes H*}\Omega_p(x)=\sum_{i,j=1}^{\infty}\{d\omega_{p},{}^i_j(x)+\sum_{k=1}^{\infty}\omega_{Hk}{}^i\wedge\omega_{p},{}^k_j(x)$$
$$-\sum_{k=1}^{\infty}\omega_{Hj}{}^k(x)\wedge\omega_{p},{}^i_k(x)\}\mathfrak{b}_i(x)\otimes\mathfrak{b}^j(x). \tag{4.4}$$

By a matrix differential p-form A we mean a matrix such that its all elements are real valued differential p-forms. By the exterior differential dA we mean the exterior differential of all elements of A. The exterior product $A\wedge B$ of two matrix differential forms A, B is defined in the same manner in the matrix multiplication where the product of matrix elements must be replaced by the exterior product.

PROPOSITION 3.

Suppose that $\mathfrak{b}_\alpha(x),\mathfrak{b}_\beta(x)$ *are moving frames of the same type joined by* $\mathfrak{b}_\beta(x)=\mathfrak{b}_\alpha(x)A_{\alpha\beta}(x)$ *and that* $\omega_\alpha,\omega_\beta$ *are connection forms with respect to* $\mathfrak{b}_\alpha(x),\mathfrak{b}_\beta(x)$. *Then the transformation low of connection forms is given by* $\omega_\beta(x)=A_{\beta\alpha}(x)dA_{\alpha\beta}(x)+A_{\beta\alpha}(x)\omega_\alpha(x)A_{\alpha\beta}(x)$.

PROOF. We have $D\mathfrak{b}_\beta(x)=\mathfrak{b}_\alpha(x)dA_{\alpha\beta}(x)+D\mathfrak{b}_\alpha(x)A_{\alpha\beta}(x)$. By using $D\mathfrak{b}_\alpha(x)=\mathfrak{b}_\alpha(x)\omega_\alpha(x)$ and $D\mathfrak{b}_\beta(x)=\mathfrak{b}_\beta(x)\omega_\beta(x)$, we see $\mathfrak{b}_\beta(x)\omega_\beta(x)=\mathfrak{b}_\alpha(x)dA_{\alpha\beta}(x)+\mathfrak{b}_\alpha(x)\omega_\alpha(x)A_{\alpha\beta}(x)$ and hence $\mathfrak{b}_\alpha(x)A_{\alpha\beta}(x)\omega_\beta(x)=\mathfrak{b}_\alpha(x)dA_{\alpha\beta}(x)+\mathfrak{b}_\alpha(x)\omega_\alpha(x)A_{\alpha\beta}(x)$. Thus we get $\omega_\beta(x)=A_{\beta\alpha}(x)dA_{\alpha\beta}(x)+A_{\beta\alpha}(x)\omega_\alpha(x)A_{\alpha\beta}(x)$.

EXAMPLE: In case $A_{\beta\alpha}(x)=(\partial x^i_\beta/\partial x^j_\alpha)_{i,j=1,2,},\cdots\cdots$, we obtain the transformation law of connections:

$$\overset{\beta}{\Gamma}{}_j{}^i{}_k(x)=\sum_{l,m,n=1}^{\infty}\frac{\partial x_\alpha^m}{\partial x_\beta^j}\frac{\partial x_\alpha^n}{\partial x_\beta^k}\frac{\partial x_\beta^i}{\partial x_\alpha^l}\overset{\alpha}{\Gamma}{}_m{}^l{}_n(x)+\sum_{l=1}^{\infty}\frac{\partial^2 x_\alpha^l}{\partial x_\beta^j\partial x_\beta^k}\frac{\partial x_\beta^i}{\partial x_\alpha^l},$$

where $\overset{\alpha}{\Gamma}=(\overset{\alpha}{\Gamma}{}_j{}^i{}_k)$, $\overset{\beta}{\Gamma}=(\overset{\beta}{\Gamma}{}_j{}^i{}_k)$ are connections with respect to $\mathfrak{b}_\alpha(x)=(\partial/\partial x_\alpha^1, \partial/\partial x_\alpha^2, \cdots\cdots)$, $\mathfrak{b}_\beta(x)=(\partial/\partial x_\beta^1, \partial/\partial x_\beta^2, \cdots\cdots)$. This is the generalization of the transformation law of affine connections.

5. Curvature forms and structure equation

DEFINITION.

By the curvature form with respect to $\mathfrak{b}(x)$ we mean the matrix differential 2-form R such that $D(D\mathfrak{b}(x))=\mathfrak{b}(x)R(x)$. In case $D\mathfrak{b}(x)=\mathfrak{b}(x)\omega(x)$, the curvature form R is said to be *the curvature form of the connection form* ω.

THEOREM 1 (Structure equation).

Let ω be a connection form and R its curvature form. We have $R=d\omega+\omega\wedge\omega$.

PROOF. By (A_1) and (A_2), we get

$$\begin{aligned} D(D\mathfrak{b}(x)) &= D(\mathfrak{b}(x)\omega(x)) \\ &= \mathfrak{b}(x)d\omega(x)+D\mathfrak{b}(x)\wedge\omega(x) \\ &= \mathfrak{b}(x)d\omega(x)+\mathfrak{b}(x)\omega(x)\wedge\omega(x) \\ &= \mathfrak{b}(x)(d\omega(x)+\omega(x)\wedge\omega(x)). \end{aligned}$$

Therefore we have $R(x)=d\omega(x)+\omega(x)\wedge\omega(x)$.

LEMMA.

Let $\omega_j^i(i,j=1,2,\cdots\cdots)$ be real valued differential 1-forms of class $C^{(n)}$ on M such that $\sum_{i=1}^{\infty}|\omega_j^i(x:X)|^2<\infty$, $\sum_{j=1}^{\infty}|\omega_j^i(x:X)|^2<\infty$ for $x\in U$ and $X\in\mathfrak{X}^{(n)}(M)$ where U is an open set of M. Then we have $d(\sum_{k=1}^{\infty}\omega_k^i(x)\wedge\omega_j^k(x))=\sum_{k=1}^{\infty}d(\omega_k^i(x)\wedge\omega_j^k(x))$.

The proof is a straightforward calculation.

Let R_j^i be the (i,j)-element of a curvature form R with respect to $\mathfrak{b}(x)$. We set a condition for R such that $\sum_{i,j=1}^{\infty}|R_i^j(x:X_1, X_2)|^2<\infty$ for $x\in U$ and $X_1, X_2\in\mathfrak{X}^{(n)}(M)$. The differential form Ω given by $\Omega(x)=\sum_{i,j=1}^{\infty}R_j^i(x)\mathfrak{b}_i(x)$

$\otimes \mathfrak{b}^j(x)$ is a $(H\otimes H^*)(M)$-valued differential 2-form of class $C^{(n-2)}$.

THEOREM 2 (Bianchi's identity).

We have $D\Omega=0$ under the above condition.

PROOF. By (4.4) we see readily

$$D\Omega(x)=\sum_{i,j=1}^{\infty}\{dR_j^i(x)+\sum_{k=1}^{\infty}\omega_k{}^i(x)\wedge R_j^k(x)-\sum_{k=1}^{\infty}\omega_j{}^k(x)\wedge R_k^i(x)\}\mathfrak{b}_i(x)\otimes\mathfrak{b}^j(x).$$

By structure equation and Lemma, we get

$$\begin{aligned}
dR_j^i(x)&+\sum_{k=1}^{\infty}\omega_k{}^i(x)\wedge R_k^j(x)-\sum_{k=1}^{\infty}\omega_j{}^k(x)\wedge R_k^i(x)\\
&=d\{d\omega_j{}^i(x)+\sum_{k=1}^{\infty}\omega_k{}^i(x)\wedge\omega_j{}^k(x)\}\\
&\quad+\sum_{k=1}^{\infty}\omega_k{}^i(x)\wedge\{d\omega_j{}^k(x)+\sum_{l=1}^{\infty}\omega_l{}^k(x)\wedge\omega_j{}^l(x)\}\\
&\quad-\sum_{k=1}^{\infty}\omega_j{}^k(x)\wedge\{d\omega_k{}^i(x)+\sum_{l=1}^{\infty}\omega_l{}^i(x)\wedge\omega_j{}^l(x)\}\\
&=\sum_{k=1}^{\infty}d\omega_k{}^i(x)\wedge\omega_j{}^k(x)-\sum_{k=1}^{\infty}\omega_k{}^i(x)\wedge d\omega_j{}^k(x)\\
&\quad+\sum_{k=1}^{\infty}\omega_k{}^i(x)\wedge d\omega_j{}^k(x)-\sum_{k=1}^{\infty}\omega_j{}^k(x)\wedge d\omega_k{}^i(x)\\
&\quad+\sum_{k,l=1}^{\infty}\omega_k{}^i(x)\wedge\omega_l{}^k(x)\wedge\omega_j{}^l(x)\\
&\quad-\sum_{k,l=1}^{\infty}\omega_j{}^k(x)\wedge\omega_l{}^i(x)\wedge\omega_k{}^l(x).
\end{aligned}$$

Now the 1st and the 4th terms, the 2nd and the 3rd terms, the 5th and the 6th terms in the righthand side of the above formula cancel respectively. Thus we have

$$dR_j^i(x)+\sum_{k=1}^{\infty}\omega_k{}^i(x)\wedge R_j^k(x)-\sum_{k=1}^{\infty}\omega_j{}^k(x)\wedge R_k^i(x)=0.$$

This completes our proof of Theorem 2.

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