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A quasilinear degenerate parabolic equation

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Introduction

In this paper, we consider degenerate parabolic equations of the form:

$$Lu \equiv u_t - ((u_x)^{2m+1})_x + \mu u - \alpha u^{p+1} - \beta u^q (u_x)^n = 0 \qquad (1)$$

in which subscripts denote partial differentiation, m, n, p, q are positive numbers and $\mu. \alpha, \beta$ are constants.

Equation (1) is nolinear and of degenerate second order parabolic type. At point where $u_x \neq 0$, (1) is parabolic, but at point where $u_x = 0$ it is not.

Our first aim here is to find a global classical solution of the Initial-Boundary Value Problem (I. B. V. P.) and the Initial Value Problem (I. V. P.) for (1). There are many papers which study weak solutions of degenerate parabolic equations ([2], [3], [9], [10], etc). We can show the global existence of classical solution by the methods which have been already applied in [4], [5].

It is well known that if u is a solution of the Cauchy problem: $u_t = u_{xx} + \mu u$ in $S \equiv (-\infty, +\infty) \times (0, T)$, $u(x, 0) = u_0(x)$ $(-\infty < x < +\infty)$ $(\mu \geqq 0)$, where $u_0(x)$ is a given bounded nonnegative continuous function on $(-\infty, +\infty)$, then $u(x, t) > 0$ in S, and in particular if $u_0(x)$ has compact support as a function of x, then $u(x, t)$ does not have this property for any $t > 0$. For (1), this situation is different. Our second aim is is to study this one for (1). Let u be a bounded classical solution of (I. V. P.). We can conclude that if $u_0(x) \geqq 0$ on $(-\infty, +\infty)$ and $u_0(x)$ has compact support as a function of x then $u(x, t)$ continues to have this property for any $t \in (0, T]$. There are many papers which study an interface for degenarate parabolic equations ([1], [6], [7], [8], [12, 13], etc.).

In § 1, we give the existence theorems of the solution for the (I. B. V. P.) and (I. V. P.). In § 2, we study the propagation of support of solution of (I. V. P.).

1. Existence theorems

In this section, we assume that m, n, p, q are positive integers with $n\geqq m+1$ and μ is a positive constant.

We first assume that usual function spaces $C^k[a,b]$, $L^p(a,b)$, $\mathring{H}^k(a,b)$, $H^k(a,b)$, $\mathscr{B}^k$, $H^k(-\infty,+\infty)=H^k$, $C^j(0,T;\mathring{H}^k(\mathrm{a,b}))$ are well known. We put $(u,v)_j=\int_a^b u^{(j)}v^{(j)}dx=\int_a^b D_x^j u D_x^j v dx$ and $((u,v))_j=(u,v)_0+(u,v)_j$ $(j=0, 1,2,\dots)$. We call $(u,v)_j$ (j)-inner product and put $\|u\|_j^2=((u,u))_j$, $|u|_j^2=(u,u)_j$.

Now let (a,b) be a finite interval.

We consider the problem:

$$(\mathrm{I})\quad \begin{cases} Lu=0 & x\in(a,b),\ 0<t<\infty, \\ u(x,0)=u_0(x) & x\in(a,b), \\ u(a,t)=u(b,t)=0 & 0<t<\infty. \end{cases}$$

THEOREM 1. *Suppose that $u(x,t)$ is a solution of the problem* (I), *belonging to $\bigcap_{j=0}^{2}C^j(0,T;\mathring{H}^{k+1-j}(a,b))$ where $0<T<\infty$ and $k\geqq 3$. Then there exist monotone increasing functions $\Phi(y),\psi(y)$ $(y\geqq 0)$ with $\Phi(0)=\psi(0)=0$ such that*

(i) $(\|u(t)\|_k^2)_t\leqq 2\|u(t)\|_k^2\{-\mu+\Phi(\|u(t)\|_k)\}$,

(ii) $(\|u_t(t)\|_{k-1}^2)_t\leqq 2\|u_t(t)\|_{k-1}^2\{-\mu+\psi(\|u(t)\|_k)\}$.

PROOF. The details follow from the arguments in §2 of [5]. We shall sketch only its outline.

By taking (0)-inner product and (k)-inner product to both sides of (1) with $\mathrm{u}(t)$, we have

$$(\frac{1}{2}|u(t)|_0^2)_t+\mu|u|_0^2\leqq\alpha\int_a^b u^{p+2}dx+\beta\int_a^b((u_x)^n)u^{q+1}dx,$$

$$(\frac{1}{2}|u(t)|_k^2)_t+\mu|u|_k^2+\int_a^b D_x^{k-1}((2m+1)(u_x)^{2m}u_{xx})D_x^{k+1}\mathrm{u}dx$$
$$=\beta((u_x^n)u^q,u)_k+\alpha(u^{p+1},u)_k.$$

Now, applying Sobolev's lemma, it follows that

$$(2m+1)\int_a^b D_x^{k-1}((u_x)^{2m}u_{xx})D_x^{k+1}u\,dx\geqq\frac{1}{3}\int_a^b(D_x^{k+1}u)^2(u_x)^{2m}dx-C\|u(t)\|_k^{2m+2},$$

$$\beta((u_x)^n u^q,\ u)_k\leqq\frac{1}{3}\int_a^b(u_x)^{2m}(D_x^{k+1}u)^2dx+C\|u\|_k^{n+q+1}+C\|u(t)\|_k^{2(n-m+q)},$$

and $\alpha(u^{p+1}, u)_k \leqq C\|u(t)\|_k^{p+2}$,

where C are constants depending on Sobolev constant, k, n, m and q. Here we note that constants C, C_{ij}, appearing in the below, are similar ones. Consequently we obtain

$$\frac{1}{2}(\|u(t)\|_k^2)_t+\mu\|u(t)\|_k^2$$
$$\leqq C_{11}\|u(t)\|_k^{2m+2}+C_{12}\|u(t)\|_k^{2(n-m+q)}+C_{13}\|u(t)\|_k^{n+q+1}+C_{14}\|u(t)\|_k^{p+2}.$$

Next by differentiating (1) with respect to t, we have

$$u_{tt}=((2m+1)(u_x)^{2m}u_{xt})_x-\mu u_t+\alpha(p+1)u^p u_t$$
$$+\beta(n(u_x)^{n-1}u_{xt}u^q+qu^{q-1}(u_x)^n u_t). \tag{2}$$

By taking (0)-inner product and $(k-1)$-inner product to both side of (2) with $u_t(x, t)$, we get

$$\frac{1}{2}(|u_t(t)|_0^2)_t+\mu|u_t(t)|_0^2\leqq C\|u(t)\|_k^p|u_t(t)|_0^2$$
$$+C\|u(t)\|_k^{n+q-1}|u_t(t)|_0^2+C\|u(t)\|_k^{n+q-1}|u_t(t)|_0|u_t(t)|_1$$

and $\frac{1}{2}(|u_t(t)|_{k-1}^2)_t+\mu|u_t(t)|_{k-1}^2+(2m+1)((u_x)^{2m}u_{xt}, u_{xt})_{k-1}$

$$=\alpha(p+1)(u^p u_t, u_t)_{k-1}+\beta\{n(u^q(u_x)^{n-1}u_{xt}, u_t)_{k-1}+q(u^{q-1}(u_x)^n u_t, u_t)_{k-1}\}.$$

Here we know that

$$(2m+1)((u_x)^{2m}u_{xt}, u_{xt})_{k-1}\geqq\int_a^b (u_x)^{2m}(D_x^k u_t)^2 dx-C\|u(t)\|_k^{2m}\|u_t(t)\|_{k-1}^2,$$

$$\beta n(u^q(u_x)^{n-1}u_{xt}, u_t)_{k-1}\leqq\int_a^b (u_x)^{2m}(D_x^k u_t)^2 dx+$$
$$+C\|u(t)\|_k^{2(n+q-m-1)}\|u_t(t)\|_{k-1}^2+C\|u(t)\|_k^{n-1+q}\|u_t(t)\|_{k-1}^2$$

and $\beta q(u^{q-1}(u_x)^n u_t, u_t)_{k-1}\leqq C\|u(t)\|_k^{n+q-1}\|u_t(t)\|_{k-1}^2$.

Therefore we obtain

$$\frac{1}{2}(\|u_t(t)\|_{k-1}^2)_t+\mu\|u_t(t)\|_{k-1}^2$$
$$\leqq C_{21}\|u(t)\|_k^{2m}\|u_t(t)\|_{k-1}^2+C_{22}\|u(t)\|_k^{n-1+q}\|u_t(t)\|_{k-1}^2$$
$$+C_{23}\|u(t)\|_k^{2(n-m+q-1)}\|u_t(t)\|_{k-1}^2+C_{24}\|u(t)\|_k^p\|u_t(t)\|_{k-1}^2.$$

Thus, setting $\Phi(y)=C_{11}y^{2m}+C_{12}y^{2(n-m+q-1)}+C_{13}y^{n+q-1}+C_{14}y^p$ and $\psi(y)=C_{21}y^{2m}+C_{22}y^{2(n-m+q-1)}+C_{23}y^{n+q-1}+C_{24}y^p$, we have (i) and (ii).

Let us set R_k^1, R_k^2 as unique positive root of the equations $\Phi(y)=\mu$ and $\psi(y)=\mu$ respectively. And we put $R_k=\min(R_k^1, R_k^2)$.

THEOREM 2. *Suppose that u_0 satisfies $u_0\in\mathring{H}^{k+1}(a, b)$ $(k\geqq 4)$, $\|u_0\|_k<R_k$.*

Then the problem (I) *admits a uniqu esolution* $u(x,t)$ *belonging to* $\bigcap_{j=0}^{2} C^j(0,\infty;H^{k-1-j}(a,b))$. *Moreover* $u(x,t)$ *satisfies* $\|u(t)\|_{k-1}<R_k$ *for* $t\in(0,\infty)$.

Secondly we shall consider the problem:

$$\text{(II)}\quad \begin{cases} Lu=0 & ((x,t)\in(-\infty,+\infty)\times(0,+\infty)), \\ u(x,0)=u_0(x) & (x\in(-\infty,+\infty)). \end{cases}$$

THEOREM 3. *Suppose that* u_0 *satisfies* $u_0\in H^{k+1}$, $\|u_0\|_k<R_k(k\geqq 5)$. *Then the problem* (II) *has a unique solution* $u(x,t)\in C^0(0,\infty;H^{k-2})\cap C^1(0,\infty;H^{k-3})$.

The proof of Theorems 2 and 3 can be carried out in a similar manner to that used for the analogous assertions in [5].

2. Propagation of support

In this section we assume the existence of a global classical solution of (I. V. P.):

$$\text{(III)}\quad \begin{cases} Lu\equiv u_t-((u_x)^{2m+1})_x+\mu u-\alpha u^{p+1}-\beta u^{q+1}(u_x)^n=0 \\ \qquad\qquad\qquad\qquad\qquad \text{in } S\equiv(-\infty<x<+\infty,0<t<\infty), \\ u(x,0)=u_0(x)\ \ (-\infty<x<\infty), \end{cases}$$

where m, p,q are positive numbers and μ is a nonnegative contant.

We note that in §1 we have proved the global existence of a classical solution of the problem (III) for the case of m,p,q being positive integers with $n\geqq m+1$ and μ a positive constant.

Now, by the standard maximum principle, we can easily verify that if $u(x,t)$ is a bounded classical solution of (III), $|u|+|u_x|\leqq M_0$ in $\bar{S}$ and $0\leqq u_0(x)\leqq M_0(x\in(-\infty,\infty))$, then $M_0\geqq u(x,t)\geqq 0$ in $\bar{S}$.

Here we study some properties of the bounded solution of the problem (III). In the below, we shall use the auxiliary functions $z(t,x)$ which have been employed in ([6], [7], [8], [12]).

THEOREM 4. *Let* $u(x,t)$ *be a global classical bounded solution of the problem* (III). *Suppose that* $|u|+|u_x|\leqq M_0$ *in* S *and there exist constant* a_1, b_1 *such that* $u_0(x)=0(x\notin(a_1,b_1))$ *and* $0<u_0(x)\leqq M_0(x\in(a_1,b_1))$. *Then there exist constants* A_1,B_1 *such that* $u(x,t)\equiv 0$ $(x\notin(A_1,B_1),\ t\geqq 0)$ *in the following cases*:

i) $\mu>0,\alpha\leqq 0$; ii) $\mu>0,\alpha M_0^{p-1}-\mu<0,\alpha>0$; iii) $\mu=0,\alpha<0,m>(p/2)$.

PROOF. It is sufficient to verify that there exists a constant $B_1(>b_1)$ such that $u(x,t)=0$ $(B_1\leqq x<\infty, 0\leqq t<\infty)$.
We put

$$z(x,t)=\begin{cases} M_0\left(\dfrac{B-x}{B-b_1}\right)^{\omega} & b_1\leqq x\leqq B \\ 0 & B\leqq x \end{cases}$$

where $\omega(>2)$ is suitable constant.

Since z satisfies the relation $Lz>0$ in $(b_1, B_1)\times(0,\infty)$ for sufficiently large number $B_1(>b_1)$ and $z(x,0)\geqq 0=u(x,0)$ $(x\geqq b_1)$, $z(b_1,t)=M_0\geqq u(b_1,t)$ $(0\leqq t<\infty)$, applying maximum principle, we have $z(x,t)\geqq u(x,t)$ in $(b_1\leqq x<\infty,\ 0<t<\infty)$.

Hence $u(x,t)\equiv 0$ in $(B_1\leqq x<\infty,\ 0\leqq t<\infty)$.

Simiraly we can verify that there exists a number $A_1(<a_1)$ such that $u(x,t)=0$ in $(-\infty<x\leqq A_1,\ 0\leqq t<\infty)$.

THEOREM 5. *Suppose that $u(x,t)$ is a global classical solution of the problem* (III) *and there exists a point $(x_0,t_0)\in S$ such that $u(x_0,t_0)>0$. Then $u(x_0,t)>0$ for all $t(\geqq t_0)$.*

PROOF. Without loss of generality, we may take $(x_0,t_0)=(0,0)$.

Now we put

$$z(x,t)=\begin{cases} (\rho^2-x^2)^{\omega}e^{-\sigma t} & |x|\leqq\rho, 0<t<\infty, \\ 0 & |x|\geqq\rho, 0<t<\infty \end{cases}$$

in which $\omega(>2)$ and the constants $\sigma>0$, $\rho>0$ are chosen suitably.

We can easily verify that for sufficiently large $\sigma>0$ $Lz<0$ in $\{|x|<\rho, 0<t<\infty\}$.

Now, since $u(0,0)>0$, we can select sufficiently small $\rho>0$ such that $z(x,0)=(\rho^2-x^2)\leqq u(x,0)$ in $|x|<\rho$. By $u(\pm\rho,t)\geqq 0=z(\pm\rho,t)$ for $t\geqq 0$, using maximum principle, we get $z(x,t)\leqq u(x,t)$ in $\{|x|<\rho, 0\leqq t<\infty\}$.
Hence $0<u(0,t)$ for any $t(\geqq 0)$.

Here we set $\lambda_1(t)=\inf\{x\in(-\infty,\infty); u(x,t)>0\}$ and $\lambda_2(t)=\sup\{x\in(-\infty,\infty); u(x,t)>0\}$.

THEOREM 6. *Under the assumptions of Theorem 4, there exist a function $\lambda_1(t)$ which is monotonically decreasing and a function $\lambda_2(t)$ which is monotonically increasing such that for $t\in[0,\infty)$ $u(x,t)>0$ if and only if $x\in(\lambda_1(t), \lambda_2(t))$ and $A_1\leqq\lambda_1(t)$, $\lambda_2(t)\leqq B_1$ $(t\in[0,\infty))$.*

PROOF. At first, from Theorem 4, we may easily assert that the above functions $\lambda_1(t)$, $\lambda_2(t)$ are well defined on$[0,\infty)$ and statisfy $A_1 \leqq \lambda_1(t)$, $\lambda_2(t) \leqq B_1$on$[0,\infty)$.

Here, we may assume that $u(x_0, t_0)=0$ at $(x_0, t_0) \in S$ where $A_1 \leqq x_0 \leqq B_1$.

It is sufficient to verify that if $A_1 < x_0 \leqq a_1$, then $x_0 \leqq \lambda_1(t_0)$. For sufficiently small $\delta>0$, we have that $L\delta>0$ in $(A_1, x_0)\times(0, t_0)$ and $0=u(x_0, t)=u(A_1, t) <\delta(t\in[0, t_0))$. Hence we have $\delta \geqq u(x, t)$ in $[A_1, x_0]\times[0, t_0]$. By the arbitrariness of δ, we obtain $u(x, t_0)=0(A_1 \leqq x \leqq x_0)$. Therefore $A_1 < x_0 \leqq \lambda_1(t_0)$.

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