Remarks on the Morrey-Sobolev type imbedding theorems in the strong L^(P, λ) spaces

Ono, Akira
Department of Mathematics, College of General Education, Kyushu University

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Remarks on the Morrey-Sobolev type imbedding theorems in the strong $\mathscr{L}^{(p,\lambda)}$ spaces

Akira ONO

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Introduction

The Morrey-Sobolev type imbedding theorems, that is the theorems introducing the regularity of function u defined on the n-Euclidean space when the gredient u_x belongs to some normed space of functions, have been studied by various authors and have proved to be very important tools in the study of partial differential equations.

We have proved theorems of this kind in the previous papers [5], [6], [7] and [8] concerning the spaces $\mathscr{L}_r^{(p,\lambda)}$ (the $\mathscr{L}^{(p,\lambda)}$ spaces of strong type r, where $1<p<\infty$, $-p<\lambda<n$ and $1\leqq r<\infty$) which was introduced by G. Stampacchia in [11]. These results are closely analogous to Stampacchia's theorem for the space $\mathscr{L}^{(p,\lambda)}=\mathscr{L}_\infty^{(p,\lambda)}$. However, in the case of $u_x\in \mathscr{L}_r^{(p,\lambda)}$, where $1<p<\lambda<n$ and $\frac{n}{\lambda}\tilde{p}\leqq r$ $(\frac{1}{\tilde{p}}=\frac{1}{p}-\frac{1}{\lambda})$, the analogous estimate of u is still not obtained.

In this paper, we shall prove at first that an analogous result holds for $r=\frac{n}{\lambda}\tilde{p}$. Furthermore, although in the case of $1<p<\infty$, $0<\lambda<n$, $1\leqq r<\infty$ and $\frac{n}{r}=\frac{\lambda}{p}$ the estimates of u was given, these are different forms from the theorems in [8]. Therefore, we shall give the estimates of same form which improve the results in [7].

This article is organized as follows:

In § 1, the definition of the space M^p (the weak L^p space) and main results are stated.

The proofs of the results are given in § 2. For the proof, we make use of theorems due to John-Nirenberg [3], Stampacchia [11] and the author [7] (with Furusho), [8].

§ 3 is devoted to state the version of the theorems in the unified forms with the results of [8].

1. Preliminaries

We always consider subfamilies of integrable functions defined on a fixed bounded cube Q_0 in the n-Euclidean space E^n and call subcube Q of Q_0 if it is a subcube Q of Q_0 with parallel edges to those of Q_0. In addition, we denote the measure of Q by $|Q|$ and the mean value of function $u(x)=u(x_1, \cdots, x_n)$ over Q by u_Q: $u_Q=|Q|^{-1}\int_Q u(x)\,dx$.

Now, as for the definitions of the spaces $\mathscr{L}_r^{(p,\lambda)}$, $\mathscr{H}_r^{\alpha}=\mathscr{L}_r^{(1,-\alpha)}$ $(0<\alpha<1)$ and Lip (a, p) $(0<a<\infty)$, we refer [7], [9] and [11]. And, although the following space is defined in [11], we state the definition of it because of its important roles in this paper.

DEFINITION. A function u is said to belong to the space M^p (the weak L^p space) if there exists a constant $K=K(u)$ satisfying the following inequality:

$$\text{meas.}\{x\in Q_0:\ |u(x)|>\sigma\}\leqq\left(\frac{K}{\sigma}\right)^p \tag{1.1}$$

We denote min. K for which (1.1) holds by $\|u\|_{M^p(Q_0)}$, endowed with this norm the space M^p is a Banach space.

REMARK 1.1. We note that the following inequality holds:

$$\|u\|_{L^q(Q_0)}\leqq C(p,\ q)\|u\|_{M^p(Q_0)} \tag{1.2}$$

where q is an arbitrary constant satisfying $1<q<p$. Moreover, we make the following:

REMARK 1.2. By a suitable extension to E^n of an arbitrary function u belonging to $\mathscr{L}_r^{(p,\lambda)}(Q_0)$ u belongs to the space $\mathscr{L}_r^{(p,\lambda)}$ on the concentric and parallel cube Q_0' with twice edge lengths of that of Q_0 and has a support contained in a fixed cube Q_0'' such as $Q_0 \subsetneqq Q_0'' \subsetneqq Q_0'$.

Now, our main results read as follows:

THEOREM 1. *Let p, λ, r are constants satisfying $1<p<\lambda<n$ and $r=\frac{n}{\lambda}\tilde{p}$ $\left(\frac{1}{\tilde{p}}=\frac{1}{p}-\frac{1}{\lambda}\right)$. Suppose that u is a function such that the gredient u_x belongs to the space $\mathscr{L}_r^{(p,\lambda)}(Q_0)$. Then u belongs to the space $\mathscr{L}_r^{(\tilde{p},\lambda)}$ and*

(1.3) $$[u]_{\mathscr{L}_r^{(\tilde{p},\lambda)}} \leqq C\,\|u_x\|_{\mathscr{L}_r^{(p,\lambda)}}$$

where C is a constant independent of u. [1]

THEOREM 2. *Let u be a function such that the gredient* u_x *belongs to the space* $\mathscr{L}_{p_\lambda}^{(p,\lambda)}(Q_0)$ $(p_\lambda = \frac{n}{\lambda}p)$. *Then, the following estimates hold for u:*

(1) $1<p<\lambda<n$, *then u belongs to* $\mathscr{L}_{r_1}^{(\tilde{p},\lambda)}$ *and*

(1.4) $$[u]_{\mathscr{L}_{r_1}^{(\tilde{p},\lambda)}} \leqq C\|u_x\|_{\mathscr{L}_{p_\lambda}^{(p,\lambda)}}$$

where r_1 *is an arbitrary constant greater than* p_λ.

(2) $1<p=\lambda$, *then u belongs to* $\mathscr{L}_{r_1}^{(1,0)}$ *and*

(1.5) $$[u]_{\mathscr{L}_{r_1}^{(1,0)}} \leqq C\|u_x\|_{\mathscr{L}_{p_\lambda}^{(p,\lambda)}}$$

where r_1 *is a constant as in* (1).

(3) $0<\lambda<p$, *then u belongs to* $\mathscr{H}_{r_1}^{1-\frac{\lambda}{p}} = \mathscr{L}_{r_1}^{(1,\frac{\lambda}{p}-1)}$ *and*

(1.6) $$[u]_{\mathscr{H}_{r_1}^{1-\frac{\lambda}{p}}} \leqq C\|u_x\|_{\mathscr{L}_{p_\lambda}^{(p,\lambda)}}$$

where r_1 *is a constant as in* (1).

2. Proof of the Theorems

We need some lemmas for the proof of Theorem 1.

LEMMA 1. *The space* $\mathscr{L}_r^{(p,\lambda)}$ *is isomorphic to the space* $\mathscr{L}_r^{(1,\frac{\lambda}{p})}$ *and*

(2.1) $$C(n,\ p,\ \lambda,\ r)\ [u]_{\mathscr{L}_r^{(p,\lambda)}} \leqq [u]_{\mathscr{L}_r^{(1,\frac{\lambda}{p})}} \leqq [u]_{\mathscr{L}_r^{(p,\lambda)}}$$

where $1<p<\infty$, $-p<\lambda<n$ *and* $\frac{\lambda}{p} \leqq \frac{n}{r}$.

REMARK 2.1. This lemma is proved in [2] and [4] independently ($-p<\lambda<0$), [3] ($\lambda=0$) and [7] ($0<\lambda<n$) respectively.

LEMMA 2. [3] *Let p be a constant greater than unity and u be a function belonging to the space* $\mathscr{L}_p^{(1,\frac{n}{p})} = N^{(p,0)}$ *(for the definition see* [11]).

1) Throughout this paper C denotes always the constant independent of u and sometimes we indicate the arguments on which C depends.

Then u belongs to the space M^p and

(2. 2) $$\|u-u_{Q_0}\|_{M^p(Q_0)}\leqq C\|u\|_{\mathscr{L}_p^{(1,\frac{n}{p})}(Q_0)}.$$

Now, we are going to give the

PROOF OF THE THEOREM 1.

Let $\{Q_j\}$ be an arbitrary system of finite disjoint subdubes of Q_0. As the gredient u_x belongs to the L^p space, u belongs to the L^{p*} space by Sobolev's lemma $\left(\frac{1}{p^*}=\frac{1}{p}-\frac{1}{n}\right)$. And, we have (see [10] Chapt. 6, §6)

$$\|u(x)-u_{Qj}\|_{L^{p*}(Q_j)}=C(n)\|\varDelta*\frac{1}{|x|^{n-2}}*u(x)-(\varDelta*\frac{1}{|x|^{n-2}}*u(x))_{Qj}\|_{L^{p*}(Q_j)}$$

$$=C(n)\|\sum_k\frac{x_k}{|x|^n}*u_{x_k}(x)-(\sum_k\frac{x_k}{|x|^n}*u_{x_k}(x))_{Q_j}\|_{L^{p*}(Q_j)}$$

While, it is well-known that the Sobolev's constant depends on $|Q_j|$ in general. However, in this case the constant may be supposed to depend on $|Q_0|$ in place of $|Q_j|$ by precise investigation of [1], 5. 7–5. 10. Hence, we have

$$\leqq C(n,\ p,\ |Q_0|)\|u_x(x)-(u_x)_{Q_j}\|_{L^p(Q_j)}$$

On the other hand, if we put $\frac{\mu-n}{p^*}=\frac{\lambda-n}{p}$ that is $\frac{\mu}{p^*}=\frac{\lambda}{p}-1$, then

$$\frac{n}{r}-\frac{\mu}{p^*}=\frac{\lambda}{\tilde{p}}-\frac{\lambda}{p}+1=0$$

Hence,

$$\Big[\sum_j\Big\{(|Q_j|^{\frac{\mu-n}{np^*}}\|u-u_{Q_j}\|_{L^{p*}(Q_j)})\Big\}^r\Big]^{\frac{1}{r}}$$

$$\leqq C(n,\ p,|Q_0|)\Big[\sum_j\Big\{|Q_j|^{\frac{\lambda-n}{np}}\|u_x-(u_x)_{Q_j}\|_{L^p(Q_j)}\Big\}^r\Big]^{\frac{1}{r}}$$

$$\leqq C(n,\ p,|Q_0|)[u_x]_{\mathscr{L}_r^{(p,\lambda)}(Q_0)}$$

As the system $\{Q_j\}$ is arbitrary, this means that u belongs to the space $\mathscr{L}_r^{(p*,\mu)}$. On the other hand, we have verified that the equality $\frac{\mu}{p^*}=\frac{n}{r}=\frac{\lambda}{\tilde{p}}$ holds. Therefore, u belongs to the space $\mathscr{L}_r^{(\tilde{p},\lambda)}$ by Lemma 1.

This completes the proof of Theorem 1.

COROLLARY. *Under the condition of Theorem* 1, *u belongs to the space* $M^{\frac{n}{\lambda}\tilde{p}}(Q_0)$.

PROOF. By Lemma 1 the space $\mathscr{L}_r^{(\tilde{p},\lambda)}$ is isomorphic to the space $\mathscr{L}_r^{(1,\frac{\lambda}{\tilde{p}})}=\mathscr{L}_r^{(1,\frac{n}{r})}$ and the conclusion is immediate by Lemma 2.

Before proceeding to give the proof of Theorem 2, we state the following:

LEMMA 3. [11] *Let p, λ, r and r_1 be constants satisfying* $1<p<\infty$, $-p<\lambda<n$ *and* $1\leqq r<r_1<\infty$. *Then, we have*

(2.3) $$[u]_{\mathscr{L}_{r_1}^{(p,\lambda)}}\leqq[u]_{\mathscr{L}_r^{(p,\lambda)}}$$

LEMMA 4. [8] *Let u be a function such that the gredient u_x belongs to the space $\mathscr{L}_r^{(p,\lambda)}(Q_0)$, where $1<p<\infty$, $0<\lambda<n$ and $p_\lambda<r<\infty$. Then the following estimates hold for u.*

(1) *$p<\lambda$ and $p_\lambda<r<\frac{n}{\lambda}\tilde{p}$, then u belongs to $\mathscr{L}_r^{(\tilde{p},\lambda)}$ and*

(2.4) $$[u]_{\mathscr{L}_r^{(\tilde{p},\lambda)}}\leqq C\|u_x\|_{\mathscr{L}_r^{(p,\lambda)}}$$

(2) *$p=\lambda$, then u belongs to $\mathscr{L}_r^{(1,0)}$ and*

(2.5) $$[u]_{\mathscr{L}_r^{(1,0)}}\leqq C\|u_x\|_{\mathscr{L}_r^{(p,\lambda)}}$$

(3) *$p>\lambda$, then u belongs to $\mathscr{H}_r^{1-\frac{\lambda}{p}}=\mathscr{L}_r^{(1,\frac{\lambda}{p}-1)}$ and*

(2.6) $$[u]_{\mathscr{H}_r^{1-\frac{\lambda}{p}}}\leqq C\|u_x\|_{\mathscr{L}_r^{(p,\lambda)}}$$

PROOF OF THE THEOREM 2.

We can deduce the conclusion directly by taking $r=p_\lambda$ and r_1 arbitrarily close to p_λ in Lemma 3 and applying Lemma 4.

REMARK 2.2. In [7], we have stated a theorem in the different form from Theorem 2, that is "Under the same condition as in Theorem 2, u belongs to $\mathscr{L}_{r_1}^{(r_1,n-r_1)}(Q_0)$ and similar estimates hold for u, where r_1 is an arbitrary constant such as $1<r_1<p_\lambda$.". Therefore, we can easily verify that Theorem 2 is an improvement of the former one, considering for example the cases: (i) $n=p_\lambda(p=\lambda)$ and (ii) $n<p_\lambda$ $(p>\lambda)$.

3. Comments on the imbedding theorems

In this paper, we have proved two imbedding theorems for special r, that is $r=\frac{n}{\lambda}\tilde{p}$ and $r=p_\lambda$. However, in the previous paper [8], we obtained results in more general situations. Combining these results, we would like to mention them in the unified form.

THEOREM A. *Let u be a function such that the gredient u_x belongs to the space* $\mathscr{L}_r^{(p,\lambda)}(Q_0)$. where $1<p<\infty$, $0<\lambda<n$ and $\frac{\lambda}{p}\leqq\frac{n}{r}<\frac{\lambda}{p}+1$. *Then, the following estimates hold for u.*

(1) *$p<\lambda$, then u belongs to the space $\mathscr{L}_{r_1}^{(\tilde{p},\lambda)}$ and*

$$[u]_{\mathscr{L}_{r_1}^{(\tilde{p},\lambda)}}\leqq C\|u_x\|_{\mathscr{L}_r^{(p,\lambda)}} \tag{3.1}$$

where r_1 is an arbitrary constant greater than p_λ.

(2) *$p=\lambda$, then u belongs to the space $\mathscr{L}_{r_1}^{(1,0)}$ and*

$$[u]_{\mathscr{L}_{r_1}^{(1,0)}}\leqq C\|u_x\|_{\mathscr{L}_r^{(p,\lambda)}} \tag{3.2}$$

where r_1 is a constant as in (1).

(3) *$p>\lambda$, then u belongs to the space $\mathscr{H}_{r_1}^{1-\frac{\lambda}{p}}=\mathscr{L}_{r_1}^{(1,\frac{\lambda}{p}-1)}$* and

$$[u]_{\mathscr{H}_{r_1}^{1-\frac{\lambda}{p}}}\leqq C\|u_x\|_{\mathscr{L}_r^{(p,\lambda)}} \tag{3.3}$$

where r_1 is a constant as in (1).

THEOREM B. *If we assume* "$\frac{n}{r}<\frac{\lambda}{p}$" *in place of the condition* "$\frac{\lambda}{p}\leqq\frac{n}{r}<\frac{\lambda}{p}+1$" *in Theorem A, similar estimates hold for u.*

(1) *$p<\lambda$, $p_\lambda<r\leqq\frac{n}{\lambda}\tilde{p}$ and r_1 is replaced by r.*

(2) *$p=\lambda$, $p_\lambda<r<\infty$ and r_1 is as in* (1).

(3) *$p>\lambda$, $p_\lambda<r<\infty$ and r_1 is as in* (1).

REMARK 3.1. As was mentioned in the introduction, these results are closely analogous to Stampacchia's theorem in [11] for the space $\mathscr{L}^{(p,\lambda)}$ ($=\mathscr{L}_\infty^{(p,\lambda)}$). Therefore, it seems to us that improvement of Theorems A and B is very difficult.

REMARK 3.2. In the case of $p<\lambda$ and $\frac{n}{\lambda}\tilde{p}<r$, we do not obtain an analogous result to the above theorems. The difficulty to solve the problem is caused by the fact that to attain the supremum the constants depend on $|Q_j|$ and may tend to infinity as $|Q_j|$ tend to 0.

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