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A property that a disjunction $A \vee B$ is provable if and only if A is provable or B is provable is called the disjunction property. In this note we show that a kind of disjunction property holds in propositional knowledge systems introduced by McCarthy [2]. We make use of the notation and results in Sato [3].

1. Formal systems of knowledge

1. 1. Well formed formulas

Let $Pr=\{p_1, p_2, \cdots\}$ be a denumerable set of propositional variables. Let $Sp=\{S_0, S_1, S_2, \cdots\}$ be a denumerable set of symbols for persons, where S_0 is a constant for a particular person (any "FOOL") and will be denoted by O. Let $T=\{\bar{1}, \bar{2}, \cdots\}$ be the set of numerals denoting the corresponding positive integers. For simplicity we will identify $\bar{n}$ with the integer n. Intuitively elements in T denote time. The set of well formed formulas is the least set Wff defined by:

(W1) $p\in Pr$ implies $p\in Wff$;
(W2) $\alpha, \beta\in Wff$ implies $(\sim\alpha)$, $(\alpha\supset\beta)$, $(\alpha\wedge\beta)$, $(\alpha\vee\beta)\in Wff$;
(W3) $S\in Sp$, $t\in T$, $\alpha\in Wff$ implies $(St\alpha)\in Wff$.

ABBREVIATION: Parentheses are usually omitted. $[St]\alpha=St\alpha$ (read "S knows α at time t").

1. 2. Formal systems

Here we define three modal systems $KT3$, $KT4$ and $KT5$ of knowledge due to McCarthy [2]. We first define the logical system $KT3$.

The inference rules for $KT3$ are:

(R1) $\dfrac{\alpha \quad \alpha \supset \beta}{\beta}$ (modus ponens)

(R2) $\dfrac{\alpha}{[St]\alpha}$ (for all S in Sp and t in T)

The rule (R1) says that you may infer β if you could prove α and $\alpha \supset \beta$, and similarly for (R2). The following are the axiom schemata for $KT3$ and their intuitive meanings.

(A1) Substitution instances of tautologies,
(A2) $[St]\alpha \supset \alpha$ (What is known is true.),
(A3) $[Ot]\alpha \supset [Ot][St]\alpha$ (What FOOL knows at time t, FOOL knows at time t that everyone knows it at time t.),
(A4) $[St]\alpha \wedge [St](\alpha \supset \beta) \supset [St]\beta$ (Everyone can do modus ponens.),
(A5) $[St]\alpha \supset [Su]\alpha$, where $t \leq u$ (What is known remains to be known.).

In (A1)-(A5), α, β denote arbitrary elements in Wff, S denotes arbitrary element in Sp, and t, u denote elements in T.

$KT4$ is obtained from $KT3$ by adding the following axiom schema:

(A6) $[St]\alpha \supset [St][St]\alpha$ (When a person knows something, he knows that he knows it.).

$KT5$ is defined by adjoining the following axiom schema to $KT4$.

(A7) $\sim [St]\alpha \supset [St] \sim [St]\alpha$ (When a person does not know something, he knows that he does not know it.).

REMARK 1. We have chosen the names $KT3$, $KT4$ and $KT5$ because they correspond to the modal systems T, $S4$ and $S5$. (More precisely they correspond to the bi-modal systems $S4$-T, $S4$-$S4$ and $S5$-$S5$.) And sometimes we will refer to our systems as KTi ($i=3, 4, 5$).

McCarthy introduced the axiom schemata (A6) and (A7) to characterize ***introspective*** nature of one's knowledge and call (A6) positive introspective axiom and (A7) negative one.

2. Disjunction property

2. 1. Definition of Kripke-type models

A *model* is a triple $\langle W; r, v \rangle$, where

(S1) W is any nonempty set (of *possible worlds*), (S2) $r : Sp \times T \to 2^{W \times W}$,

(S3) $v : Pr \to 2^{W}$.

Given a model $M = \langle W; r, v \rangle$, a relation $\models \subseteq W \times Wff$ is defined as follows:

(E1) If $\alpha \in Pr$ then $w \models \alpha$ iff $w \in v(\alpha)$.

(E2) If $\alpha = \sim\beta$ then $w \models \alpha$ iff $w \not\models \beta$.

(E3) If $\alpha = \beta \supset \gamma$ then $w \models \alpha$ iff $w \not\models \beta$ or $w \models \gamma$.

(E4) If $\alpha = \beta \wedge \gamma$ then $w \models \alpha$ iff $w \models \beta$ and $w \models \gamma$.

(E5) If $\alpha = \beta \vee \gamma$ then $w \models \alpha$ iff $w \models \beta$ or $w \models \gamma$.

(E6) If $\alpha = [St]\beta$ then $w \models \alpha$ iff for all $w' \in W$ such that $(w, w') \in r(S, t)$, $w' \models \beta$.

We will write "$w \models \alpha$(in M)" if we wish to make M explicit.

A formula α is called ***valid*** in a model M iff for all $w \in W$, $w \models \alpha$. A model which satisfies the following condition is called *KT3-model*.

(M1) $r(O,t) \supseteq r(S,t)$ for any $S \in Sp$ and $t \in T$,

(M2) $r(S,u) \supseteq r(S,t)$ for any $S \in Sp$ and $u, t \in T$ such that $u \leq t$,

(M3) $r(S,t)$ is reflexive for any $S \in Sp$ and $t \in T$,

(M4) $r(O,t)$ is transitive for any $t \in T$.

A model is a *KT4-model* if it satisfies (M1)-(M4) and

(M5) $r(S,t)$ is transitive for any $S \in Sp$ and $t \in T$.

A model is a *KT5-model* if it satisfies (M1)-(M5) and

(M6) $r(S,t)$ is symmetric for any $S \in Sp$ and $t \in T$.

The following theorem due to Sato [**3**] is fundamental.

THEOREM 1. *For any $\alpha \in Wff$, α is a theorem of KTi if and only if α is valid in all KTi-models* ($i = 3, 4, 5$).

PROOF. See Sato [**3**].

2. 2. Disjunction property

We will show that a kind of disjunction property holds in $KT3$ or $KT4$ using Theorem 1. The proof technique used here is essentially due to Kripke [**2**].

THEOREM 2. $[S^1t_1]\alpha_1 \vee \cdots \vee [S^nt_n]\alpha_n$ *is provable in KTi* ($n \geq 1$), *if and only if for some j* ($1 \leq j \leq n$) $[S^jt_j]\alpha_j$ *is provable in KTi, where $i = 3$ or 4.*

PROOF. Since if-part is trivial, we show only-if-part. It is sufficent to show that for some j, α_j is derivable because of the rule (R2). Suppose none of α_j is derivable: then let $M_j = (W_j; r_j, v_j)$ ($1 \leq j \leq n$) be a countermodel to α_j (Theorem 1). Namely there exists $w_j \in W_j$ such that $w_j \not\models \alpha_j$ (in M_j).

We can assume without loss of generality that W_j is mutually disjoint with each other. Then we can construct a model $M=(W;r,v)$ as follows:

(i) $W=\bigcup_{j=1}^{n} W_j \cup \{w\}$, where $w \notin \bigcup_{j=1}^{n} W_j$.

(ii) For $u, v \in W$, $S \in Sp$, $t \in T$, $(u, v) \in r(S, t)$ if and only if either there exists j such that

(1) $u, v \in W_j$ and $(u, v) \in r(S, t)$, or

(2) $u=w$.

(iii) $v: Pr \to 2^W$ is defined as $v(p)=\bigcup_{j=1}^{n} v_j(p)$ for $p \in Pr$.

Using the mutual disjointness of W_j, we can easily show that $r(S, t)$ is reflexive or transitive if $r_j(S, t)(1 \leq j \leq n)$ is, and $r(O, t) \supseteq r(S, t)$, $r(S, u) \supseteq r(S, t)(u \leq t)$ for all $S \in Sp$ and $u, t \in T$.

Now we verify by induction that $u \in W_j$, $u \models \alpha$(in M_j) iff $u \models \alpha$(in M) for every $\alpha \in Wff$. For if α is in Pr then this follows from the definition (iii). If $\alpha=\beta \supset \gamma$, $\alpha=\beta \wedge \gamma$, $\alpha=\beta \vee \gamma$, or $\alpha=\sim\beta$, the inductive step is easy. If $\alpha=[St]\beta$, and the statement has been verified for β, $u \models \alpha$ (in M_j) iff for all $u' \in M_j$ such that $(u, u') \in r_j(S, t)$, $u' \models \beta$(in M_j). But $u' \models \beta$(in M_j) iff $u' \models \beta$ (in M) by induction hypothesis. Using the mutual disjointness of $W_k(1 \leq k \leq n)$ and the condition (ii), if $u \in W_j$, we have $(u, u') \in r_j(S, t)$ iff $(u, u') \in r(S, t)$. Therefore $u \models \alpha$(in M_j) iff $u \models \alpha$(in M).

Hence in particular, since $w_j \not\models \alpha_j$ (in M_j)$(1 \leq j \leq n)$, we have for all j, $w_j \not\models [S^j t_j]\alpha_j$(in M). Since $(w, w_j) \in r(S^j, t_j)$ $(1 \leq j \leq n)$, there holds $w \not\models [S^j t_j]\alpha_j$ (in M). Hence $w \not\models [S^1 t_1]\alpha_1 \vee \cdots \vee [S^n t_n]\alpha_n$, and M is a countermodel to $[S^1 t_1]\alpha_1 \vee \cdots \vee [S^n t_n]\alpha_n$. This is a contradiction.

REMARK 2. Theorem 2 fails for $KT5$. For example consider the following instance of the axiom (A6), $\sim[St]p \supset [St]\sim[St]p$, where p is in Pr. Equivalently $[St]p \vee [St]\sim[St]p$ is provable in $KT5$, but neither p nor $\sim[St]p$ is derivable.

REMARK 3. The systems reported herein have their origin in researches on Artificial Intelligence. And the disjuction property plays an important role in applying our systems to formal analysis of some well-known puzzles. The reader is referred to Sato [3] for details.

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