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A remark on integral operators involving the Gauss hypergeometric functions

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The fractional calculus has been investigated by many mathematicians [**11**]. In their works the Riemann-Liouville operator (R-L) was the most central, while Erdélyi and Kober defined their operator (E-K) in connection with the Hankel transform [**6**]. Thereafter various generalizations have been made [**11**]. Here we shall define a certain integral operator involving the Gauss hypergeometric function. (cf. Definition 1) Such an integral was first treated by Love [**7**] as an integral equation. However, if we regard the integral as an operator with a slight change, it will contain as special cases both R-L and E-K owing to reduction formulas for the Gauss function by restricting the parameters. The more interesting fact is that for our operator two kinds of product rules may be made up by virtue of Erdélyi's formulas [**3**], which were first proved by using the method of fractional integration by parts in the R-L sense. From the rules, of course, the ones for R-L and E-K are deduced. Moreover, our operator is representable by products of R-L's, from which it is possible to obtain the integrability and estimations of Hardy-Littlewood type. We shall also state, in parallel, formulas for an integral operator on the interval (x, ∞), which is an extension of operators of Weyl and another Erdélyi-Kober. (cf. Definition 2) Then a formula of integration by parts for our operators is obtained. In Section 3 commutative relations will be given for the sake of the Mellin transform.

Since 1969, A. M. Nahušev [**9**] and the other authors in USSR (see [**12**]) have studied various problems for degenerate hyperbolic equations and equations of mixed type with boundary conditions containing integrals or derivatives of fractional order in the R-L sense. Results for such problems involving our operators and more properties of the operators will be published elsewhere.

1. Definitions

DEFINITION 1. Let $\alpha>0, \beta$ and η be real numbers.[1] The integral operator $I_x^{\alpha,\beta,\eta}$, which acts on certain functions $f(x)$ on the interval $(0,\infty)$, is defined by

$$(1.1)\quad I_x^{\alpha,\beta,\eta}f=\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}F(\alpha+\beta,-\eta;\alpha;1-\frac{t}{x})f(t)dt,$$

where Γ is the gamma function, F denotes the Gauss hypergeometric series

$$(1.2)\quad {}_2F_1(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}z^n,\quad |z|<1$$

and its analytic continuation into $|\arg(1-z)|<\pi$, and $(a)_n=\Gamma(a+n)/\Gamma(a)$.

DEFINITION 2. Under the same assumptions in Definition 1, the integral operator $J_x^{\alpha,\beta,\eta}$ is defined by

$$(1.3)\quad J_x^{\alpha,\beta,\eta}f=\frac{1}{\Gamma(\alpha)}\int_x^{\infty}(t-x)^{\alpha-1}t^{-\alpha-\beta}F(\alpha+\beta,-\eta;\alpha;1-\frac{x}{t})f(t)dt.$$

NOTE 1. When $\alpha+\beta=0$ or $\beta=0$, I and J are reduced to the following integral operators:

$$(1.4)\quad I_x^{\alpha,-\alpha,\eta}f=\frac{1}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}f(t)dt\equiv R_x^{\alpha}f,\quad \text{(Riemann-Liouville)}$$

$$(1.5)\quad I_x^{\alpha,0,\eta}f=\frac{x^{-\alpha-\eta}}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}t^{\eta}f(t)dt\equiv E_x^{\alpha,\eta}f,\quad \text{(Erdélyi-Kober)}$$

$$(1.6)\quad J_x^{\alpha,-\alpha,\eta}f=\frac{1}{\Gamma(\alpha)}\int_x^{\infty}(t-x)^{\alpha-1}f(t)dt\equiv W_x^{\alpha}f,\quad \text{(Weyl)}$$

$$(1.7)\quad J_x^{\alpha,0,\eta}f=\frac{x^{\eta}}{\Gamma(\alpha)}\int_x^{\infty}(t-x)^{\alpha-1}t^{-\alpha-\eta}f(t)dt\equiv K_x^{\alpha,\eta}f,\quad \text{(Erdélyi-Kober)}$$

by virtue of the formulas $F(0,b;c;z)=1$ and

$$(1.8)\quad F(a,b;a;z)=(1-z)^{-b}.$$

NOTE 2. The following equalities will be useful in the future:[2]

$$(1.9)\quad E^{\alpha,\eta}f=x^{-\alpha-\eta}R^{\alpha}x^{\eta}f,$$

$$(1.10)\quad K^{\alpha,\eta}f=x^{\eta}W^{\alpha}x^{-\alpha-\eta}f.$$

Making use of the relation $F(a,b;c;z)=F(b,a;c;z)$, we have

THEOREM 1.

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1) The following results are valid for complex numbers Re $\alpha>0$, β and η, but for brevity we shall confine ourselves to the real case.

2) The suffix x in the above operators will be omitted in the following discussions unless it is needed.

(1.11) $$I^{\alpha,\beta,\eta}x^{\beta-\eta}f=I^{\alpha,\eta,\beta}f,$$

(1.12) $$J^{\alpha,\beta,\eta}x^{\alpha+\beta+\eta}f=J^{\alpha,-\alpha-\eta,-\alpha-\beta}f.$$

The formula

(1.13) $$F(a,b;c;z)=(1-z)^{c-a-b}F(c-a,c-b;c;z)$$

implies

THEOREM 2.

(1.14) $$I^{\alpha,\beta,\eta}f=x^{-\alpha-\beta-\eta}I^{\alpha,-\alpha-\eta,-\alpha-\beta}f,$$

(1.15) $$J^{\alpha,\beta,\eta}f=x^{\eta-\beta}J^{\alpha,\eta,\beta}f.$$

COROLLARY.

(1.16) $$I^{\alpha,\beta,0}f=x^{-\alpha-\beta}R^{\alpha}f,\ \ I^{\alpha,\beta,-\alpha}f=x^{-\beta}E^{\alpha,-\alpha-\beta}f,$$

(1.17) $$J^{\alpha,\beta,0}f=x^{-\beta}K^{\alpha,\beta}f,\ \ J^{\alpha,\beta,-\alpha}f=x^{-\alpha-\beta}W^{\alpha}f.$$

2. Some properties of I and J

Let $1\leq p<\infty$. L_p denotes a class of real functions which are measurable and p-th power integrable on the interval $(0,\infty)$ with the norm $\|\cdot\|_p$. L_∞ denotes a class of real, measurable and essentially bounded functions on $(0,\infty)$ with the norm $\|\cdot\|_\infty$.

If we combine results of Hardy and Littlewood [**5**], Kober [**6**] and Flett [**4**] by keeping in mind Note 2, we obtain

LEMMA 1. *Let $1\leq p\leq q\leq\infty$, $a<1-\frac{1}{p}$ and $b>a-\frac{1}{p}$. If functions $f(x)$ and $g(x)$ satisfy $x^a f\in L_p$ and $x^b g\in L_p$, and $\alpha>\frac{1}{p}-\frac{1}{q}$, where α may be equal to $\frac{1}{p}-\frac{1}{q}$ except the cases $1=p<q<\infty$ and $1<p<q=\infty$, then $x^{\frac{1}{p}-\frac{1}{q}-\alpha+a}\times R^{\alpha}f$ and $x^{\frac{1}{p}-\frac{1}{q}-\alpha+b}W^{\alpha}g$ belong to L_q and there hold the estimations*[3)]

(2.1) $$\|x^{\frac{1}{p}-\frac{1}{q}-\alpha+a}R^{\alpha}f\|_q\leq C\|x^a f\|_p,$$

(2.2) $$\|x^{\frac{1}{p}-\frac{1}{q}-\alpha+b}W^{\alpha}g\|_q\leq C\|x^b g\|_p.$$

DEFINITION 3. Let $1\leq p\leq q\leq\infty$. The condition $A_1(\alpha,\beta,\eta;a;p,q)$ means that the members satisfy

(1) $\quad a<\min(0,-\beta+\eta)-\frac{1}{p}+1,$

3) In what follows the single symbol C will denote a constant depending on the parameters appeared in the inequality.

(2) (i) $\alpha\geq-\beta+\frac{1}{p}-\frac{1}{r}\geq\frac{1}{p}-\frac{1}{q}$ for $p\leq r\leq q$, or

(ii) $\alpha\geq-\eta+\frac{1}{p}-\frac{1}{r}\geq\frac{1}{p}-\frac{1}{q}$ for $p\leq r\leq q$.

(If $1=p<q<\infty$ or $1<p<q=\infty$, one of the equal signs in (i) and (ii) should be excluded.) If, instead of (1), we assume

(1)′ $\quad b > -\min(\beta, \eta)-\frac{1}{p}$,

then that is called the condition $A_2(\alpha, \beta, \eta; b; p, q)$.

THEOREM 3. *Let* $1\leq p\leq q\leq\infty$. *Assume the conditions* $A_1(\alpha, \beta, \eta; a; p, q)$ *and* $A_2(\alpha, \beta, \eta; b; p, q)$, *then* $x^{\frac{1}{p}-\frac{1}{q}+\beta+a}I^{\alpha,\beta,\eta}f$ *and* $x^{\frac{1}{p}-\frac{1}{q}+\beta+b}J^{\alpha,\beta,\eta}g$ *belong to* L_q *for any functions* $f(x)$ *and* $g(x)$ *with* $x^a f\in L_p$, $x^b g\in L_p$, *and there hold the estimations*

$$\|x^{\frac{1}{p}-\frac{1}{q}+\beta+a}I^{\alpha,\beta,\eta}f\|_q \leq C\|x^a f\|_p, \tag{2.3}$$

$$\|x^{\frac{1}{p}-\frac{1}{q}+\beta+b}J^{\alpha,\beta,\eta}g\|_q \leq C\|x^b g\|_p. \tag{2.4}$$

Furthermore the following decompositions are valid:

Case (i);

$$I^{\alpha,\beta,\eta}f=x^{-\alpha-\beta-\eta}R^{\alpha+\beta}x^{\eta}R^{-\beta}f=R^{-\beta}x^{-\alpha-\eta}R^{\alpha+\beta}x^{\eta-\beta}f, \tag{2.5}$$

$$J^{\alpha,\beta,\eta}g=x^{\eta-\beta}W^{\alpha+\beta}x^{-\alpha-\eta}W^{-\beta}g=W^{-\beta}x^{\eta}W^{\alpha+\beta}x^{-\alpha-\beta-\eta}g. \tag{2.6}$$

Case (ii);

$$I^{\alpha,\beta,\eta}f=x^{-\alpha-\beta-\eta}R^{\alpha+\eta}x^{\beta}R^{-\eta}x^{\eta-\beta}f=R^{-\eta}x^{-\alpha-\beta}R^{\alpha+\eta}f, \tag{2.7}$$

$$J^{\alpha,\beta,\eta}g=W^{\alpha+\eta}x^{-\alpha-\beta}W^{-\eta}g=x^{\eta-\beta}W^{-\eta}x^{\beta}W^{\alpha+\eta}x^{-\alpha-\beta-\eta}g. \tag{2.8}$$

PROOF. For the simplicity we shall consider the case $1<p<q<\infty$ and (ii). From the assumptions $a<1-\frac{1}{p}$, $\alpha+\eta\geq\frac{1}{p}-\frac{1}{r}$ for $p<r<q$, $x^a f\in L_p$ and by noting Lemma 1 we obtain $x^{\frac{1}{p}-\frac{1}{r}-\alpha-\eta+a}R^{\alpha+\eta}f\in L_r$ and

$$\|x^{\frac{1}{p}-\frac{1}{r}-\alpha-\eta+a}R^{\alpha+\eta}f\|_r \leq C\|x^a f\|_p. \tag{2.9}$$

On the other hand $a<-\beta+\eta-\frac{1}{p}+1$ implies $\frac{1}{p}-\frac{1}{r}+\beta-\eta+a<1-\frac{1}{r}$, then, for any function $F(x)$ with $x^{\frac{1}{p}-\frac{1}{r}+\beta-\eta+a}F\in L_r$, we have $x^{\frac{1}{p}-\frac{1}{q}+\beta+a}R^{-\eta}F\in L_q$ and

$$\|x^{\frac{1}{p}-\frac{1}{q}+\beta+a}R^{-\eta}F\|_q \leq C\|x^{\frac{1}{p}-\frac{1}{r}+\beta-\eta+a}F\|_r, \tag{2.10}$$

because $-\eta\geq\frac{1}{r}-\frac{1}{q}$. Substituting $F=x^{-\alpha-\beta}R^{\alpha+\eta}f$ into (2.10) and using (2.9), we have

$$\|x^{\frac{1}{p}-\frac{1}{q}+\beta+a}R^{-\eta}x^{-\alpha-\beta}R^{\alpha+\eta}f\|_q \leq C\|x^a f\|_p. \tag{2.11}$$

Now consider the integral

(2.12) $R^{-\eta}x^{-\alpha-\beta}R^{\alpha+\eta}f$

$$=\frac{1}{\Gamma(-\eta)\Gamma(\alpha+\eta)}\int_0^x (x-t)^{-\eta-1}t^{-\alpha-\beta}\int_0^t (t-u)^{\alpha+\eta-1}f(u)\,du\,dt.$$

Since the interchangeability of the order of integrations is guaranteed by the above statements and by Fubini's theorem, then (2.12) is equal to

(2.13) $$\frac{1}{\Gamma(-\eta)\Gamma(\alpha+\eta)}\int_0^x f(u)\int_u^x (x-t)^{-\eta-1}(t-u)^{\alpha+\eta-1}t^{-\alpha-\beta}\,dt\,du$$

$$=\frac{x^{-\alpha-\beta}}{\Gamma(-\eta)\Gamma(\alpha+\eta)}\int_0^x (x-u)^{\alpha-1}f(u)\int_0^1 v^{-\eta-1}(1-v)^{\alpha+\eta-1}[1-(1-\frac{u}{x})v]^{-\alpha-\beta}dvdu$$

$$=\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x (x-u)^{\alpha-1}F(\alpha+\beta,-\eta;\alpha;1-\frac{u}{x})f(u)du=I^{\alpha,\beta,\eta}f,$$

where we have used the formula [8]

(2.14) $$F(a,b;c;z)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt,$$

$$\mathrm{Re}\ c>\mathrm{Re}\ b>0,\quad |\arg(1-z)|<\pi.$$

Hence from (2.11), (2.12) and (2.13) we obtain (2.3). For the other pairs of p and q with $1\leq p\leq q\leq\infty$, the validity of the theorem may be assured by a similar manner. The rest of the decompositions for I can be concluded by the use of Theorems 1 and 2. Proofs for J are parallel.

NOTE 3. Taking into account of Note 2, we may substitute E (or K) instead of some one or two R (or W) in (2.5) and (2.7) (or (2.6) and (2.8)), and obtain the other decompositions, e.g.

$$I^{\alpha,\beta,\eta}=x^{-\gamma}E^{\alpha+\beta,\eta-\gamma}x^{\gamma}R^{-\beta}=x^{-\beta-\gamma-\delta}E^{\alpha+\beta,\eta-\beta-\gamma-\delta}x^{\gamma}E^{-\beta,-\delta}x^{\delta}\quad\text{etc.}$$

For want of space the whole formulas are not mentioned here.

Between I and J, there holds the following generalized fractional integration by parts.

THEOREM 4. *Let* $1\leq p\leq\infty$, $1\leq q\leq\infty$ *and* $\frac{1}{p}+\frac{1}{q}\geq 1$. *Suppose that constants* a, b, α, β *and* η *satisfy the conditions*:

$a<\min(0,-\beta+\eta)-\frac{1}{p}+1$, $a+b=1-\frac{1}{p}-\frac{1}{q}+\beta$ *and*

(i) $\alpha\geq-\beta+\frac{1}{p}-\frac{1}{r}\geq\frac{1}{p}+\frac{1}{q}-1$ *for* $p\leq r\leq\frac{q}{q-1}$ *or*

(ii) $\alpha\geq-\eta+\frac{1}{p}-\frac{1}{r}\geq\frac{1}{p}+\frac{1}{q}-1$ *for* $p\leq r\leq\frac{q}{q-1}$.

(If $1=p<q<\infty$ *or* $1=q<p<\infty$, *one of the equal signs in* (i) *and* (ii) *should be omitted.) If* $x^a f\in L_p$ *and* $x^b g\in L_q$, *then there holds the equality*

$$(2.15)\qquad \int_0^\infty g(x)I_x^{\alpha,\beta,\eta}f\,dx=\int_0^\infty f(x)J_x^{\alpha,\beta,\eta}g\,dx.$$

PROOF. The left hand side of (2.15) is equal to

$$(2.16)\qquad \frac{1}{\Gamma(\alpha)}\int_0^\infty g(x)x^{-\alpha-\beta}\int_0^x(x-t)^{\alpha-1}F(\alpha+\beta,-\eta;\alpha;1-\frac{t}{x})f(t)\,dt\,dx.$$

The assumptions and Theorem 3 imply $x^{-b}I_x^{\alpha,\beta,\eta}|f|\in L_{q/(q-1)}$. Thus $|g(x)|\times I_x^{\alpha,\beta,\eta}|f|\in L_1$. Therefore we may interchange the order of integrations in (2.16).

$$(2.17)\qquad \int_0^\infty g(x)I_x^{\alpha,\beta,\eta}f\,dx$$
$$=\frac{1}{\Gamma(\alpha)}\int_0^\infty f(t)\int_t^\infty(x-t)^{\alpha-1}x^{-\alpha-\beta}F(\alpha+\beta,-\eta;\alpha;1-\frac{t}{x})g(x)\,dx\,dt$$
$$=\int_0^\infty f(t)J_t^{\alpha,\beta,\eta}g\,dt.$$

THEOREM 5. *Let $\alpha>\gamma>0$. Under the same assumptions in Theorem 3, there hold the following decompositions*:

$$(2.18)\qquad I^{\alpha,\beta,\eta}f=I^{\gamma,\delta,\eta}I^{\alpha-\gamma,\beta-\delta,\gamma+\eta}f=I^{\alpha-\gamma,\beta-\delta,\eta+\gamma+\delta}I^{\gamma,\delta,\eta-\beta+\delta}f,$$
$$(2.19)\qquad J^{\alpha,\beta,\eta}g=J^{\alpha-\gamma,\beta-\delta,\gamma+\eta}J^{\gamma,\delta,\eta}g=J^{\gamma,\delta,\eta-\beta+\delta}J^{\alpha-\gamma,\beta-\delta,\eta+\gamma+\delta}g.$$

PROOF. The results follow from Erdélyi's formulas [3]

$$(2.20)\qquad F(a,b;c;z)=\frac{\Gamma(c)}{\Gamma(\lambda)\Gamma(c-\lambda)}\int_0^1 t^{\lambda-1}(1-t)^{c-\lambda-1}(1-tz)^{-a'}$$
$$F(a-a',b;\lambda;tz)F(a',b-\lambda;c-\lambda;\frac{z(1-t)}{1-tz})dt$$
$$(2.21)\qquad =\frac{\Gamma(c)}{\Gamma(\lambda)\Gamma(c-\lambda)}\int_0^1 t^{\lambda-1}(1-t)^{c-\lambda-1}(1-tz)^{r-a-b}$$
$$F(r-a,r-b;\lambda;tz)\,F(a+b-r,r-\lambda;c-\lambda;\frac{z(1-t)}{1-tz})dt,$$
$$\mathrm{Re}\ c>\mathrm{Re}\ \lambda>0,\quad |\arg(1-z)|<\pi.$$

Here the change of the order of integrations is clear by the similar arguments in the previous discussions, so we shall omit precise proofs.

NOTE 4. (2.18) and (2.19) are useful as product rules in writting in the forms

$$(2.22)\qquad I^{\alpha,\beta,\eta}I^{\gamma,\delta,\alpha+\eta}=I^{\alpha+\gamma,\beta+\delta,\eta},\quad I^{\alpha,\beta,\eta}I^{\gamma,\delta,\eta-\beta-\gamma-\delta}=I^{\alpha+\gamma,\beta+\delta,\eta-\gamma-\delta},$$
$$(2.23)\qquad J^{\gamma,\delta,\alpha+\eta}J^{\alpha,\beta,\eta}=J^{\alpha+\gamma,\beta+\delta,\eta},\quad J^{\gamma,\delta,\eta-\beta-\gamma-\delta}J^{\alpha,\beta,\eta}=J^{\alpha+\gamma,\beta+\delta,\eta-\gamma-\delta}.$$

3. Further formulas for *I* and *J*

Let us consider the Mellin transform

$$\mathscr{M}\{\varphi(x);z\}=\int_0^\infty x^{z-1}\varphi(x)\ dx, \tag{3.1}$$

where z is a complex variable.

LEMMA 2. *For* $\operatorname{Re} c>0$, $\operatorname{Re} f>0$, $\operatorname{Re} z>\max\{\operatorname{Re}(-p),\ \operatorname{Re}(a+b-c-p),\ \operatorname{Re}(-d+f-p+q),\ \operatorname{Re}(e-p+q)\}$, *there holds*

$$\begin{aligned}\mathscr{M}\{w^{c+f-1}(1-w)^{d-f+p-q}&\int_0^1 v^{c-1}(1-v)^{f-1}(1-vw)^{q-d}\\ &F(a,b;c;vw)\,F(d,e;f;\frac{w(1-v)}{1-vw})\ dv;z\}\\ =\Gamma(c)\Gamma(f)&\frac{\Gamma(z+p)\Gamma(z-a-b+c+p)\Gamma(z+d-f+p-q)\Gamma(z-e+p-q)}{\Gamma(z-a+c+p)\Gamma(z-b+c+p)\Gamma(z+p-q)\Gamma(z+d-e+p-q)}.\end{aligned} \tag{3.2}$$

PROOF. Substitute the formulas

$$\mathscr{M}\{(1-x)^{c-1}F(a,b;c;1-x)H(1-x);z\}=\frac{\Gamma(c)\Gamma(z)\Gamma(c-a-b+z)}{\Gamma(c-a+z)\Gamma(c-b+z)}, \tag{3.3}$$

$\operatorname{Re} c>0,\ \operatorname{Re} z>\max\{0,\ \operatorname{Re}(a+b-c)\}$, [10, (I. 15. 2)]

$$\begin{aligned}\mathscr{M}\{(x-1)^{c-1}&F(a,b;c;1-x)H(x-1);z\}\\ &=\frac{\Gamma(c)\Gamma(1+a-c-z)\Gamma(1+b-c-z)}{\Gamma(1-z)\Gamma(1+a+b-c-z)},\end{aligned} \tag{3.4}$$

$\operatorname{Re} c>0,\ \operatorname{Re} z>\max\{\operatorname{Re}(1+a-c),\ \operatorname{Re}(1+b-c)\}$ [10, (I. 15. 4)]

into

$$\mathscr{M}\{x^a\int_0^\infty t^b\varphi_1(xt)\varphi_2(t)dt;z\}=\Phi_1(z+a)\Phi_2(1-z-a+b), \tag{3.5}$$

[10, (I. 1. 14)]

change the parameters suitably and set $x=1-w$, then we have (3.2) after certain arrangements, where $H(t)$ is the Heaviside function, $\mathscr{M}\{\varphi_1;z\}=\Phi_1(z)$ and $\mathscr{M}\{\varphi_2;z\}=\Phi_2(z)$.

COROLLARY. *Set*

$$\begin{aligned}I(\alpha,\beta,\eta;\gamma,\delta,\zeta;w)=\frac{w^{\alpha+\gamma-1}}{\Gamma(\alpha)\Gamma(\gamma)}&\int_0^1 v^{\alpha-1}(1-v)^{\gamma-1}(1-wv)^{-\gamma-\delta}\\ &F(\alpha+\beta,-\eta;\alpha;wv)F(\gamma+\delta,-\zeta;\gamma;\frac{w(1-v)}{1-wv})\ dv,\end{aligned} \tag{3.6}$$

then we have

$$\mathscr{M}\{I(\alpha,\beta,\eta;\gamma,\delta,\zeta;w);z\}=\frac{\Gamma(z)\Gamma(z-\beta+\eta-\delta)\Gamma(z-\delta+\zeta)}{\Gamma(z-\beta-\delta)\Gamma(z+\alpha+\eta-\delta)\Gamma(z+\gamma+\zeta)} \tag{3.7}$$

for $\alpha>0,\ \gamma>0,\ \operatorname{Re} z>\max\{0,\ \delta-\zeta,\ \beta-\eta+\delta\}$.

NOTE 5. In the right hand side of (3.7), there are 9 combinations of the set of the parameters to be cancelled each one of the denominator and the numerator. Then in any case the side is equal to the Mellin transform of some Gauss function by the formulas (3.3) and

(3.8) $\mathscr{M}\{x^a\varphi(x);z\}=\Phi(z+a)$, where $\mathscr{M}\{\varphi;z\}=\Phi(z)$. [**10**, (I.1.3)]

But no formula is produced except (2.20) and (2.21). So any new product rule except (2.22) and (2.23) can not be known. Similar discussions have been made by Buschman [**2**] for (3.2) in order to obtain Erdélyi type formulas, where he pointed out 9 special combinations. There are, however, ${}_4P_2\times{}_4P_2\div 2=72$ possibilities of combinations. Examining all cases by using (1.8), (1.13) and

$$F(a,b;c;z)=(1-z)^{-a}F\left(a,c-b;c;\frac{z}{z-1}\right), \tag{3.9}$$

we find that, in addition to the Erdélyi formulas, a new one is constructed no more than the following:

$$\begin{aligned}(3.10)\quad F(a,b;c;z)=&\frac{\Gamma(c)}{\Gamma(\lambda)\Gamma(c-\lambda)}(1-z)^{-a+a'+\lambda}\int_0^1 t^{\lambda-1}(1-t)^{c-\lambda-1}\\ &(1-tz)^{a-a'-c}F(a-a',b-c+a'+\lambda;\lambda;tz)\,F\left(a',-a+b+a';c-\lambda;\frac{z(1-t)}{1-tz}\right)dt,\\ &\operatorname{Re}c>\operatorname{Re}\lambda>0,\quad |\arg(1-z)|<\pi.\end{aligned}$$

From the fact (3.7), we have

LEMMA 3. *Let* $\alpha>0$, $\gamma>0$ and $0<\lambda<\alpha+\gamma$. *There hold the identities*

$$\begin{aligned}(3.11)\quad I(\alpha,\beta,\eta;\gamma,\delta,\zeta;w)&=I(\lambda,-\lambda+\alpha+\beta,\eta;-\lambda+\alpha+\gamma,\lambda-\alpha+\delta,\lambda-\alpha+\zeta;w)\\ &=I(\lambda,-\lambda+\beta-\eta+\gamma+\delta+\zeta,\eta;-\lambda+\alpha+\gamma,\lambda+\eta-\gamma-\zeta,\lambda+\eta-\gamma-\delta;w)\\ &=I(\lambda,-\lambda+\alpha+\eta-\zeta,\beta+\zeta;-\lambda+\alpha+\gamma,\lambda-\alpha+\beta-\eta+\delta+\zeta,\lambda-\alpha+\zeta;w)\\ &=I(\lambda,-\lambda+\gamma+\delta,\beta+\zeta;-\lambda+\alpha+\gamma,\lambda+\beta-\gamma,\lambda+\eta-\gamma-\delta;w).\end{aligned}$$

By Theorem 3 and Lemma 3 the following is clear.

THEOREM 6. *Let* $1\leq p\leq r_j\leq q\leq\infty\,(j=1,2)$, $\alpha>0$, $\gamma>0$ and $0<\lambda<\alpha+\gamma$. *Assume the conditions* $A_1(\gamma,\delta,\zeta;a;p,r_1)$, $A_1(\alpha,\beta,\eta;\frac{1}{p}-\frac{1}{r_1}+\delta+a;r_1,q)$, $A_1(-\lambda+\alpha+\gamma,\lambda-\alpha+\delta,\lambda-\alpha+\zeta;a;p,r_2)$ *and* $A_1(\lambda,-\lambda+\alpha+\beta,\eta;\frac{1}{p}-\frac{1}{r_2}+\lambda-\alpha+\delta+a;r_2,q)$. *If* $x^af\in L_p$, *then* $x^{\frac{1}{p}-\frac{1}{q}+\beta+\delta+a}I^{\alpha,\beta,\eta}I^{\gamma,\delta,\zeta}f$ *and* $x^{\frac{1}{p}-\frac{1}{q}+\beta+\delta+a}I^{\lambda,-\lambda+\alpha+\beta,\eta}I^{-\lambda+\alpha+\gamma,\lambda-\alpha+\delta,\lambda-\alpha+\zeta}f$ *belong to* L_q *and there holds the identity*

$$I^{\alpha,\beta,\eta}I^{\gamma,\delta,\zeta}f=I^{\lambda,-\lambda+\alpha+\beta,\eta}I^{-\lambda+\alpha+\gamma,\lambda-\alpha+\delta,\lambda-\alpha+\zeta}f. \tag{3.12}$$

NOTE 6. From the rest of formulas in Lemma 3, we obtain the following identities by supposing respective conditions:

$$
\begin{aligned}
(3.13)\quad & I^{\alpha,\beta,\eta} I^{\gamma,\delta,\zeta} f \\
&= I^{\lambda,-\lambda+\beta-\eta+\gamma+\delta+\zeta,\eta} I^{-\lambda+\alpha+\gamma,\lambda+\eta-\gamma-\zeta,\lambda+\eta-\gamma-\delta} f \\
&= I^{\lambda,-\lambda+\alpha+\eta-\zeta,\beta+\zeta} I^{-\lambda+\alpha+\gamma,\lambda-\alpha+\beta-\eta+\delta+\zeta,\lambda-\alpha+\zeta} f \\
&= I^{\lambda,-\lambda+\gamma+\delta,\beta+\zeta} I^{-\lambda+\alpha+\gamma,\lambda+\beta-\gamma,\lambda+\eta-\gamma-\delta} f.
\end{aligned}
$$

NOTE 7. Replace the condition A_1 by A_2 in Theorem 6 and in Note 6, we have

$$
\begin{aligned}
(3.14)\quad & J^{\alpha,\beta,\eta} J^{\gamma,\delta,\zeta} f = J^{\lambda,-\lambda+\alpha+\beta,-\lambda+\alpha+\eta} J^{-\lambda+\alpha+\gamma,\lambda-\alpha+\delta,\zeta} f \\
&= J^{\lambda,-\lambda-\eta+\gamma+\zeta,-\lambda-\beta+\gamma+\zeta} J^{-\lambda+\alpha+\gamma,\lambda+\beta+\eta-\gamma+\delta-\zeta,\zeta} f \\
&= J^{\lambda,-\lambda+\alpha+\beta+\eta+\delta-\zeta,-\lambda+\alpha+\eta} J^{-\lambda+\alpha+\gamma,\lambda-\alpha-\eta+\zeta,\eta+\delta} f \\
&= J^{\lambda,-\lambda+\gamma+\delta,-\lambda-\beta+\gamma+\zeta} J^{-\lambda+\alpha+\gamma,\lambda+\beta-\gamma,\eta+\delta} f.
\end{aligned}
$$

NOTE 8. Set $\beta=\delta=0$, $\lambda=\gamma$ in the last identities in (3.13) and (3.14), then

$$(3.15)\quad E^{\alpha,\eta} E^{\gamma,\zeta} f = E^{\gamma,\zeta} E^{\alpha,\eta} f,\quad K^{\alpha,\eta} K^{\gamma,\zeta} f = K^{\gamma,\zeta} K^{\alpha,\eta} f$$

are valid, which are just ones obtained by Buschman [1].

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