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Existence and decay of solutions of some nonlinear wave equations in noncylindrical domains

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Introduction

In this paper we are concerned with the existence and decay of weak solutions of the following nonlinear wave equations with nonlinear dissipative terms

$$\frac{\partial^2}{\partial t^2}u-\Delta u+\rho\left(\frac{\partial}{\partial t}u\right)+\beta(u)=f \text{ in } Q \tag{0.1}$$

with the initial-boundary conditions

$$u(0,x)=u_0(x),\ \frac{\partial}{\partial t}u(x,0)=u_1(x) \text{ and } u|_{\Sigma}=0 \tag{0.2}$$

where $Q=\bigcup_{0<t<\infty}\Omega_t\times\{t\}$ is a bounded increasing (in t) domain in $R^n\times[0,\infty)$ and Σ is the lateral boundary of Q.

Recently, linear and nonlinear wave equations in noncylindrical domains have been treated by many authors. Lions [4] introduced the so-called penalty method to solve existence problem. Using this method, Medeiros [5] proved the existence of weak solution to the problem (0.1)-(0.2) with $\rho\left(\frac{\partial}{\partial t}u\right)=0$ for a wide class of $\beta(u)$ such that $\beta(u)u\geqq 0$. Cooper & Bardos [2] proved the existence and uniqueness of weak solution for the case $\rho=0$, $\beta(u)=|u|^{\alpha}u$ ($\alpha\geqq 0$) and Σ is globaly "time-like" without the increasingness condition on Q. Cooper [1] considered the local decay property of solutions of linear equations (in an exterior domain), assuming the boundary is time-like at each point. Inoue [3] succeeded in proving the existence of classical solutions for the case $n=3$, $\beta=0$ and $\beta(u)=u^3$ when the body is time-like at each point.

In this note we restrict ourselves to the case of the domain being monotone increasing to investigate decay property as well as the existence of weak solutions. We utilize fully the existence of the dissipative term, which is different essentially from earlier papers. It should be noted also that we make no monotonicity conditions on $\beta(u)$. In a cylindrical case, our problem has been treated by one of the authors [6] [7] [8] [9] and our result here can be regarded as partial extensions of those works.

1. Preliminaries

We use some familiar notations of function spaces without definitions. Points of $R^n\times[0,\infty)$ are denoted as (x,t).

First we state our assumptions on Q, ρ, β and the forcing term f;

A$_1$. (i) Let $Q=\bigcup_{0\leq t<\infty}\Omega_t\times\{t\}$ $(R^n\times[0,\infty)$. Then Ω_t is monotone increasing, that is, $\Omega_{t_1}\subset\Omega_{t_2}$ if $t_1\leqq t_2$.

(ii) Ω_t and $\Omega=\bigcup_{0\leq t<\infty}\Omega_t$ is regular and bounded.

A$_2$. $\beta(s)$ is a function on R, satisfying the conditions

$$|\beta(s)|\leqq K_0|s|^{\alpha+1} \tag{1.1}$$

where α is a constant such that

$$0<\alpha\leqq\frac{2}{n-2} \text{ if } n\geqq 3 \text{ and } <\alpha<\infty \text{ if } n=1,2. \tag{1.2}$$

A$_3$. $\rho(s)$ is a function on R, satisfying

$$\begin{aligned}&K_1|s|^{r+2}\leqq\rho(s)s\leqq K_2(1+|s|)^{r+1}\,s \text{ and}\\&(\rho(s_1)-\rho(s_2))(s_1-s_2)\geqq 0\end{aligned} \tag{1.3}$$

where r is a constant satisfying (1.2), admitting $r=0$.

A$_4$. f belongs to $S_{(r+2)/(r+1)}([0,\infty);\ L^{(r+2)/(r+1)}(\Omega_t))$, i.e.,

$$\delta(t)=\left(\int_t^{t+1}\|f(s)\|^{(r+2)/(r+1)}_{L^{(r+2)/(r+1)}(\Omega_s)}ds\right)^{(r+1)/(r+2)}<\text{const.}<\infty.$$

Finally in this section we give our definition of solution. We say a function u on $\bigcup_{0\leq t<\infty}\Omega_t\times\{t\}$ is a weak solution to the problem (0.1)-(0.2) if $u'\in L^\infty_{loc}([0,\infty);L^2(\Omega_t))$, $u\in L^\infty_{loc}([0,\infty);\mathring{H}_1(\Omega_t))$ and $t\to\tilde{u}(\cdot,t)$ (some extension of $u(\cdot,t)$ on Ω), $t\to\tilde{u}'(\cdot,t)$ are continuous with values $\mathring{H}_1(\Omega)$ and $L^2(\Omega)$, respectively. The equations (0.1), (0.2) are satisfied in the distribution sense.

2. Some lemmas on difference inequalities

Here we prepare some lemmas concerning difference inequalities, which will be needed for the proof of decay property of solutions.

LEMMA 2.1 *Let $\{a_m\}_{m=0}^\infty$ be a sequence of positive numbers such that*

$$a_{m+1}-a_m\geqq d_0>0 \quad \textit{or} \quad a_{m+1}\geqq(1+d_1)a_m \tag{2.1}$$

for $m=0,1,2,\ldots,$ *where* d_0, d_1 *are some positive constants. Then we have*

$$a_m \geqq a_{m_0}+d_0(m-m_0) \quad \text{for } m \geqq m_0 \tag{2.2}$$

where m_0 *is the smallest integer such that*

$$m_0 \geqq \max\{\log\left(\frac{d_0}{a_0 d_1}\right)(\log(1+d_1))^{-1}, 0\}.$$

PROOF. First, suppose that $m_0=0$. In this case we have

$$a_0 \geqq d_0 d_1^{-1}$$

and hence (note that $\{a_m\}$ is an increasing sequence)

$$a_m \geqq \min\{a_{m-1}+d_0, (1+d_1)a_{m-1}\} = a_{m-1}+d_0 \quad \text{for } m=1,2,\ldots,$$

which implies immediately (2.2) with $m_0=0$.

Next, we assume $m_0>0$. Then we have

$$a_{m_0} \geqq d_0 d_1^{-1}. \tag{2.3}$$

Indeed, if this was false, we see easily

$$(d_0+a_k)>(1+d_1)a_k \quad \text{for } k=0,1,\ldots,m_0-1$$

and hence by the assumption (2.1)

$$a_{m_0}>(1+d_1)^{m_0}a_0 \tag{2.4}$$

which is a contradiction because $a_{m_0}<d_0 d_1^{-1}$ implies $(1+d_1)^{m_0}a_0>d_0 d_1^{-1}$. Thus by (2.2) we obtain, as in the case $m_0=0$,

$$a_{m_0+k} \geqq a_{m_0}+d_0 k \quad (k=1,2,3,\ldots).$$

LEMMA 2.2. *Let* $\phi(t)$ *be a nonnegative decreasing function on* $R^+ =[0,\infty)$, *satisfying*

$$\phi(t+1)^{1+\alpha}-d_2\phi(t)^{1+\alpha} \leqq d_3(\phi(t)-\phi(t+1)) \tag{2.5}$$

with some constants $\alpha>0, 0<d_2<1, d_3>0$. *Then we have*

$$\phi(t) \leqq c_0(1+t)^{-1/\alpha} \quad \text{for } t\in R^+, \tag{2.6}$$

where c_0 *is a constant depending on* $\phi(0)$ *and other known constants.*

PROOF. Setting $y(t)=\phi(t)^{-\alpha}$, we see easily

$$\begin{aligned} y(t+1)-y(t) &= \int_0^1 \frac{d}{d\theta}\{\theta\phi(t+1)+(1-\theta)\phi(t)\}^{-\alpha}d\theta \\ &\geqq \alpha(\phi(t)-\phi(t+1))\phi(t)^{-(1+\alpha)}. \end{aligned} \tag{2.7}$$

Therefore, if $\phi(t+1)^{1+\alpha}>\frac{1+d_2}{2}\phi(t)^{1+\alpha}$, we have by (2.5)

$$\frac{(1-d_2)}{2}\phi(t)^{1+\alpha} \geqq d_3(\phi(t)-\phi(t+1))$$

and hence, using (2.7),

$$y(t+1)-y(t) \geqq (2d_3)^{-1}(1-d_2)\alpha>0. \tag{2.8}$$

On the other hand, if $\phi(t+1) \leqq \frac{1}{2}(1+d_2)\phi(t)^{1+\alpha}$, we have

$$y(t+1)\geqq\left(\frac{2}{1+d_2}\right)^{\alpha/(1+\alpha)}y(t). \tag{2.9}$$

Thus applying Lemma 2.1 to the sequence

$$\{y(m+\theta)\}_{m=0}^{\infty} \qquad (0\leqq\theta<1)$$

we obtain (2.6) immediately.

REMARK 1. If (2.5) is valid for $\alpha=0$, then it is easy to see that

$$\phi(t)\leqq\text{const.}\, e^{-kt}$$

for some $k>0$.

LEMMA 2.3. *Let $\phi(t)$ be a decreasing nonnegative function on R^+, satisfying*

$$\phi(t+1)\leqq d_4\{A^2(t)+(A(t)+g_1(t))\sqrt{\phi(t)}\}+g_2(t) \qquad \text{for } t\geqq 0 \tag{2.10}$$

where $d_4>0$ and we set

$$A(t)=\{\phi(t)\phi(t+1)\}^{1/(r+2)} \quad (r>0).$$

Moreover, let us assume

$$g_1^2(t)+g_2(t)\leqq d_5(1+t)^{-\theta} \tag{2.11}$$

for some $d_5>0$ and $\theta>2/r$. Then we have

$$\phi(t)\leqq c_1(1+t)^{-2/r} \quad \text{for } t\geqq 0 \tag{2.12}$$

where c_1 is a constant depending on $\phi(0)$ and other known constants.

PROOF. Using Young inequality, we obtain easily from (2.10), for any $\varepsilon>0$,

$$\phi(t+1)^{(r+2)/2}\leqq c_\varepsilon(\phi(t)-\phi(t+1)+g_3(t)^{(r+2)/2})+\varepsilon\phi(t)^{(r+2)/2}$$

where we set $g_3(t)=g_1^2(t)+g_2(t)$. Hereafter c_ε denotes various constants depending on ε. Putting $\psi(t)=\phi(t)+\nu t^{-\theta}$ $(\nu>0)$ we have

$$\psi(t+1)^{(r+2)/2}\leqq c_\varepsilon(\psi(t)-\psi(t+1))+\varepsilon\psi(t)^{(r+2)/2}+I_\varepsilon(t)$$

where

$$I_\varepsilon(t)=c_\varepsilon(-\nu t^{-\theta}+\nu(t+1)^{-\theta})+2^{(r+2)/2}\nu^{(r+2)/2}t^{-\theta(r+2)/2}+g_3(t)^{(r+2)/2}.$$

By the same arguments as in [8] we can show $I_\varepsilon(t)<0$ for large t if (2.11) is valid. Therefore by Lemma 2.2 we obtain (2.12).

REMARK 2. In [8], the following difference inequality is treated:

$$\max_{s\in[t,t+1]}\phi(t)^{1+\alpha}\leqq\text{const.}\,(\phi(t)-\phi(t+1))+g(t). \quad (\alpha\geqq 0).$$

3. Existence and decay of solution

To state our theorems we must recall some functionals on $\mathring{H}_1(\Omega)$ introduced in [6];

$$J_0(u)=\frac{1}{2}\|u\|^2_{\mathring{H}_1(\Omega)}+\int_\Omega\int_0^{u(x)}\beta(s)dsdx,$$

$$J_1(u)=\|u\|^2_{\mathring{H}_1(\Omega)}+\int_\Omega\beta(u)udx,$$

$$\tilde{J}_1(u)=\frac{1}{2}\|u\|^2_{\mathring{H}_1(\Omega)}-\left(K_0S^{\alpha+2}_{\alpha+2}/(\alpha+2)\right)\|u\|^{\alpha+2}_{\mathring{H}_1(\Omega)},$$

and

$$\tilde{J}_1(u)=\|u\|^2_{\mathring{H}_1(\Omega)}-K_0S^{\alpha+2}_{\alpha+2}\|u\|^{\alpha+2}_{\mathring{H}_1},$$

where $S_{\alpha+2}$ is a constant such that

$$\|u\|_{L^{\alpha+2}(\Omega)}\leqq S_{\alpha+2}\|u\|_{\mathring{H}_1}.$$

Note that $J_0(u)\geqq\tilde{J}_0(u)$ and $J_1(u)\geqq\tilde{J}_1(u)$, which will be used later. Moreover we set

$$\tilde{J}_0(x)=\frac{1}{2}x^2-(K_0S^{\alpha+2}_{\alpha+2}/(\alpha+2))x^{\alpha+2}$$

and

$$\tilde{J}_1(x)=x^2-K_0S^{\alpha+2}_{\alpha+2}x^{\alpha+2}$$

for $x\geqq 0$. Let x_0 be a maximal point of $\tilde{J}_1(x)$, i.e.,

$$x_0=\left\{\frac{2}{K_0S^{\alpha+2}_{\alpha+2}(\alpha+2)}\right\}^{1/\alpha}$$

and set $D_0=\tilde{J}_0(x_0)$. Note that $\tilde{J}_1$ and $\tilde{J}_0$ are both increasing on $[0,x_0]$.

The stable set W_0 is defined as follows.

$$W_0=\{(u_0,u_1)\in\mathring{H}_1(\Omega_0)\times L^2(\Omega_0)\mid\|u_0\|_{\mathring{H}_1(\Omega)}<x_0 \text{ and }$$
$$\|(u_0,u_1)\|_{W_0}\equiv\frac{1}{2}\|u_1\|^2_{L^2(\Omega_0)}+J(\tilde{u}_0)<D_0\}$$

Hereafter, for a function $u(x)$ defined on a subset Ω_0 of Ω we set

$$\tilde{u}(x)=\begin{cases}u(x) & \text{if} \quad x\in\Omega_0\\ 0 & \text{if} \quad x\notin\Omega_0\end{cases}$$

for $x\in\Omega$.

Now we are ready to state our results.

THEOREM 1. *Let* $(u_0,u_1)\in W_0$. *Then there exists a constant* $M_0=M_0(D_0-\|(u_0,u_1)\|_{W_0})>0$ *such that if*

$$M=\sup_t\left(\int_t^{t+1}\|f(s)\|^{(r+2)/(r+1)}_{L^{(r+2)/(r+1)}(\Omega)}ds\right)^{(r+1)/(r+2)}<M_0$$

then the problem (0.1)-(0.2) *has a weak solution* $u(t)$ *satisfying*

$$\|u'(t)\|_{L^2(\Omega_t)}<\sqrt{2D_0} \quad \textit{and} \quad \|u(t)\|_{\mathring{H}_1(\Omega_t)}<x_0.$$

THEOREM 2. *Let $M<M_0$. Then, the solution u in Theorem 1 satisfies the following decay property*:

(i) *if $r=0$ and $\delta(t)\leqq d_0e^{-\lambda t}$ $(d_0, \lambda_0>0)$, then*

$$\|u(t)\|_{E(t)}\leqq c_2e^{-\lambda t} \quad \textit{for } t\geqq 0$$

with some $\lambda>0$, and

(ii) *if $r>0$ and $\delta(t)\leqq d_0(1+t)^{-(1+\eta)-1/r}$ for some $\eta>0$, then*

$$\|u(t)\|_{E(t)}\leqq c_3(1+t)^{-1/r} \quad \textit{for } t>0$$

where we recall

$$\delta(t)=\left(\int_t^{t+1}\|\tilde{f}(s)\|^{(r+2)/(r+1)}_{L^{(r+2)/(r+1)}(\Omega)}ds\right)^{(r+1)/(r+2)}$$

and we set

$$\|u(t)\|_{E(t)}=\|u'(t)\|_{L^2(\Omega_t)}+\|u(t)\|_{\mathring{H}_1(\Omega_t)}.$$

Of course the constant c_2, c_3 depend on the initial data (u_0, u_1).

REMARK 3. If we assume $\beta(u)u\geqq 0$, then the assertions of above Theorems are valid for all $(u_0, u_1)\in\mathring{H}_1(\Omega_0)\times L^2(\Omega_0)$ without the restriction on $\|(u_0, u_1)\|_{W_0}$.

Let us proceed to the proofs of above Theorems.

PROOF OF THEOREM 1. We employ a penalty method. Let $\chi(x, t)$ be the characteristic function of $\Omega\times R^+-Q$ and consider the equation

$$\frac{\partial^2}{\partial t^2}u_\varepsilon-\Delta u_\varepsilon+\rho\left(\frac{\partial}{\partial t}u_\varepsilon\right)+\beta(u_\varepsilon)+\frac{\chi}{\varepsilon}u_\varepsilon=\tilde{f} \text{ on } \Omega\times R^+ \tag{3.1}$$

$$\text{with} \quad u_\varepsilon(x, 0)=\tilde{u}_0(x),\ \frac{\partial}{\partial x}u_\varepsilon(x, 0)=\tilde{u}_1(x) \text{ and } u_\varepsilon(t)|_{\partial\Omega}=0. \tag{3.2}$$

Since Ω_0 is regular we may assume $\tilde{u}_0(x)\in\mathring{H}_1(\Omega)$. We solve (3.1) by Galerkin method. Let $\{w_j(x)\}$ be a basis of $\mathring{H}_1(\Omega)$ and let

$$u_{m,\varepsilon}(x, t)=\sum_{j=1}^m\alpha_{j,m,\varepsilon}(t)w_j(x)$$

where $\alpha_{j,m,\varepsilon}(t)$ are determined by the system of ordinary differential equation

$$(u''_{m,\varepsilon}(t), w_j)+(\nabla u_{m,\varepsilon}(t), \nabla w_j)+(\rho(u'_{m,\varepsilon}), w_j)$$
$$+(\beta(u_{m,\varepsilon}(t)), w_j)+\frac{1}{\varepsilon}(\chi(t)u_{m,\varepsilon}(t), w_j)=(\tilde{f}(t), w_j),$$

and

$$u_{m,\varepsilon}(0)\longrightarrow\tilde{u}_0 \quad \text{strongly in } \mathring{H}_1(\Omega),$$
$$u'_{m,\varepsilon}(0)\longrightarrow\tilde{u}_1 \quad \text{strongly in } L^2(\Omega).$$

Since $\chi(x, t)$ is monotonically increasing with respect to t, we obtain, as long as $u_{m,\varepsilon}(t)$ exists,

$$\frac{1}{2}\|u'_{m,\varepsilon}(t_2)\|^2_{L^2(\Omega)}+J_0(u_{m,\varepsilon}(t_2);\ \varepsilon)+\int_{t_1}^{t_2}(\rho(u'_{m,\varepsilon}(t)),u'_{m,\varepsilon}(t))dt$$

$$\leqq\frac{1}{2}u'_{m,\varepsilon}(t_1)\|_{L^2(\Omega)}+J_0(u_{m,\varepsilon}(t_1);\ \varepsilon)+\int_{t_1}^{t_2}(\tilde{f}(t),u'_{m,\varepsilon}(t))dt \tag{3.3}$$

for $t_2\geqq t_1\geqq 0$, where

$$J_0(u;\varepsilon)=J_0(u)+\frac{1}{2\varepsilon}\|\chi u\|^2_{L^2(\Omega)}.$$

Also we have

$$\int_{t_1}^{t_2}\{\frac{1}{\varepsilon}\|\chi u_{m,\varepsilon}\|^2_{L^2(\Omega)}+J_1(u_{m,\varepsilon}(t))\}dt$$
$$\leqq(u'_{m,\varepsilon}(t_1),u_{m,\varepsilon}(t_1))+(u'_{m,\varepsilon}(t_2),u_{m,\varepsilon}(t_2)) \tag{3.4}$$
$$+\int_{t_1}^{t_2}(\|u'_{m,\varepsilon}(t)\|^2_{L^2(\Omega)}+(f(t),u_{m,\varepsilon}(t)))dt.$$

Since $\frac{1}{2}\|u'_{m,\varepsilon}(0)\|_{L^2(\Omega)}+J_0(u_{m,\varepsilon}(0);\varepsilon)$ tends to $\|(u_0,u_1)\|_{W_0}$ as $m\to\infty$, we see that for $\eta>0$ there exists $m_0>0$ such that

$$\frac{1}{2}\|u'_{m\varepsilon}(0)\|_{L^2(\Omega)}+J(u_{m,\varepsilon}(0);\varepsilon)<D_0-\|(u_0,u_1)\|_{W_0}+\eta \tag{3.5}$$

and

$$\|u_{m,\varepsilon}(0)\|_{\mathring{H}_1}<x_0 \qquad \text{for } m>m_0.$$

Thus combining (3.3)-(3.5) we can conclude by almost the same argument as in [**6**] that there exists a constant $M_0=M_0(D_0-\|(u_0,u_1)\|_{W_0})$ such that if $M<M_0$, $u_{m,\varepsilon}(t)$ exists on $[0,\infty)$ and the following estimates hold:

$$\frac{1}{2}\|u'_{m,\varepsilon}(t)\|^2_{L^2(\Omega)}+J_0(u_{m,\varepsilon}(t);\varepsilon)<D_0 \quad \text{for } t\in[0,\infty) \tag{3.6}$$

From (3.6) it follows that

$$\|u'_{m,\varepsilon}(t)\|_{L^2(\Omega)}<\sqrt{2D_0},\quad \|u_{m,\varepsilon}(t)\|_{\mathring{H}_1(\Omega)}<x_0,$$
$$\frac{1}{\varepsilon}\|\chi(t)u_{m,\varepsilon}(t)\|_{L^2(\Omega)}<\sqrt{2D_0}\ \text{and}\ \int_t^{t+1}\|u'_{m,\varepsilon}(s)\|^{r+2}_{L^{r+2}(\Omega)}ds\leqq c(D_0,M_0)$$
$$<\infty \tag{3.7}$$

for large m.

Thus by standard compactness and monotonicity arguments (see Lions, Strauss [**4, 10**]) we see that there exist a subsequence $\{u_{m_j,\varepsilon}\}$ of $\{u_{m,\varepsilon}(t)\}$ and a function $u_\varepsilon(x,t)$ such that

$$u_{m_j,\varepsilon}\longrightarrow u_\varepsilon \text{ weakly* in } L^\infty([0,\infty);\ \mathring{H}_1(\Omega)) \text{ and a.e. in } \Omega\times R^+,$$
$$\beta(u_{m_j,\varepsilon})\longrightarrow\beta(u_\varepsilon)\quad \text{weakly* in } L^\infty([0,\infty);\ L^2(\Omega))$$

$$\left.\begin{aligned} &u'_{m_j,\varepsilon} \longrightarrow u'_\varepsilon \quad \text{weakly* in } L^\infty([0,\infty);\ L^2(\Omega)) \\ &\rho(u'_{m_j,\varepsilon}) \longrightarrow \rho(u'_\varepsilon) \quad \text{weakly in } L^{(r+2)/(r+1)}_{loc}(R^+;\ L^{(r+2)/(r+1)}(\Omega)) \end{aligned}\right\} \quad (3.8)$$

and

$$\frac{1}{\sqrt{\varepsilon}}\chi u_{m_j,\varepsilon} \longrightarrow \frac{1}{\sqrt{\varepsilon}}\chi u_\varepsilon \quad \text{weakly* in } L^\infty([0,\infty);\ L^2(\Omega)).$$

Thus $u_\varepsilon(t)$ is a weak solution of the problem (3.1)-(3.2) and the estimates (3.7) still hold for u_ε. Repeated use of compactness and monotonicity arguments for $\{u_\varepsilon\}$ show that a subsequence $\{u_{\varepsilon_j}\}$ of $\{u_\varepsilon\}$ $(\varepsilon_j \to 0)$ satisfies the convergence properties as in (3.8). In particular we obtain that $\frac{1}{\sqrt{\varepsilon_j}}\chi u_{\varepsilon_j}$ is convergent with respect to weak* topology of $L^\infty([0,\infty);L^2(\Omega))$. Hence the limit function $u(t)$ of $\{u_{\varepsilon_j}\}$ satisfies (note that Ω_t is regular)

$$\chi u=0, \ \text{i.e.,} \ u \in L^\infty([0,\infty);\ \mathring{H}_1(\Omega_t)).$$

It is easy to see $u(x,0)=\tilde{u}_0(x)$ and $u'(x,0)=\tilde{u}_1(x)$. Thus u is a required weak solution of our problem. (the continuity of $u(\cdot,t)$, $u'(\cdot,t)$ are assured by Strauss [**10**])

PROOF OF THEOREM 2. For the proof of Theorem 2 it suffices to show that the approximate solutions $u_{m,\varepsilon}(t)$ (m: large) satisfy the decay estimate independent of m, ε. We know already

$$\tilde{J}_1(u_{m,\varepsilon}(t)) \geqq \tilde{J}_0(u_{m,\varepsilon}(t)) \geqq k_0\|u_{m,\varepsilon}(t)\|^2_{\mathring{H}_1(\Omega)}$$

for all $t \in [0,\infty)$ with some $k_0>0$.

We shall derive a difference inequality concerning the energy of $u_{m,\varepsilon}$ to apply Lemma 2.3. First we observe that, for $t_1 \geqq t_2$,

$$E(u_{m,\varepsilon}(t_2)) - E(u_{m,\varepsilon}(t_1)) + \frac{K_1}{2}\int_{t_1}^{t_2}\|u'_{m,\varepsilon}(s)\|^{r+2}_{L^{r+2}(\Omega)}ds \leqq 0 \qquad (3.9)$$

where we set

$$E(u_{m,\varepsilon}(t)) = \frac{1}{2}\|u'_{m,\varepsilon}(t)\|^2 + J_0(u_{m,\varepsilon}(t);\ \varepsilon) + \left(\frac{r+1}{r+2}\right)\left(\frac{2K_1}{r+2}\right)^{1/(r+1)}\int_t^\infty \|\tilde{f}(s)\|^{(r+2)/(r+1)}_{L^{(r+2)/(r+1)}(\Omega)}ds.$$

By the assumption on $f(t)$ and by (3.9) $E(u(t))$ is a nonnegative monotone decreasing function on R^+. Let t be fixed arbitrarily. Then by (3.9) with $t_1=t$, $t_2=t+1$, we see that there exist two points $t_1 \in [t, t+\frac{1}{4}]$ and $t_2 \in [t+\frac{3}{4}, t+1]$ such that

$$\|u'_{m,\varepsilon}(t_i)\|_{L^{r+2}(\Omega)} \leqq \left(\frac{8}{K_1}\right)^{1/(r+2)} D(t)$$

where $D(t)=\{E(u_{m,\varepsilon}(t))-E(u_{m,\varepsilon}(t+1))\}^{1/(r+2)}$, and hence, as in (3.4), we have

$$\begin{aligned}&\int_{t_1}^{t_2}\left\{\frac{1}{\varepsilon}\|\chi u_{m,\varepsilon}(s)\|_{L^2(\Omega)}^2+\|u_{m,\varepsilon}(s)\|_{\overset{\circ}{H}_1(\Omega)}^2\right\}ds\\&\leqq c(\Omega)\{D(t)\max_{s\in[t,t+1]}\|u_{m,\varepsilon}(s)\|_{L^2(\Omega)}+D(t)^2\\&\qquad+\delta(t)\max_{s\in[t,t+1]}\|u_{m,\varepsilon}(s)\|_{L^2(\Omega)}\}\end{aligned} \tag{3.10}$$

where $c(\Omega)$ denotes constants depending on meas(Ω). From (3.9) and (3.10) we see that there exists a time $t^*\in[t,\ t+1]$ such that

$$\begin{aligned}&\frac{1}{2}\|u_{m,\varepsilon}'(t^*)\|_{L^2(\Omega)}^2+J_0(u_{m,\varepsilon}(t^*);\ \varepsilon)\\&\leqq c(\Omega)\{D(t)^2+(D(t)+\delta(t))\max_{s\in[t,t+1]}\|u_{m,\varepsilon}(s)\|_{L^2(\Omega)}\}\\&\leqq c(\Omega)\ \{D(t)^2+(D(t)+\delta(t))\max_{s\in[t,t+1]}\sqrt{E(u_{m,\varepsilon}(s))}\}.\end{aligned} \tag{3.11}$$

Since $E(u_{m,\varepsilon}(t))$ is monotone decreasing we have by (3.11)

$$\begin{aligned}E(u_{m,\varepsilon}(t+1))\leqq E(u_{m,\varepsilon}(t^*))\leqq c(\Omega)\{D(t)^2+D(t)+\delta(t))\sqrt{E(u_{m,\varepsilon}(t))}\\+\sum_{i=0}^{\infty}\delta(t+i)^{(r+2)(r+1)}\}\end{aligned} \tag{3.12}$$

Applying Lemma 2.3 to (3.12) we obtain immediately the desired result.

References

[1] J. COOPER: *Local decay of solutions of the wave equation in the exterior of a moving body,* J. Math. Anal. Appl. 49 (1975), 130-153.

[2] J. COOPER and C. BARDOS: *A nonlinear wave equation in a time dependent domain,* J. Math. Anal. Appl. 42 (1973), 29-60.

[3] A. INOUE: *Sur $\square u+u^3=f$ dans un domaine noncylindrique,* J. Math. Anal. Appl. 46 (1974), 777-819.

[4] J. L. LIONS: *Une remarque sur les problemes d'evolution nonlineaires dans les domaines non cylindriques,* Rev. Romaine Pures Appl. Math. 9 (1964), 11-18.

[5] L. A. MEDEIROS: *Non-linear wave equations in domains with variable boundary,* Arch. rational mech. Anal. 47 (1972), 47-58.

[6] M. NAKAO: *Bounded, periodic, and almost-periodic solutions of nonlinear hyperbolic partial differential equations,* J. Differential Equations 23 (1977), 368-386.

[7] M. NAKAO: *Asymptotic stability of the bounded or almost periodic solution of the wave equation with nonlinear dissipative term,* J. Math. Anal. Appl. 58 (1977), 336-343.

[8] M. NAKAO: *Convergence of solutions of the wave equation with a nonlinear dissipative term to the steady state,* Mem. Fac. Sci., Kyushu Univ. Ser. A 30 (1976), 257-265.

[9] M. NAKAO: *Decay of solutions of some nonlinear evolution equations,* J. Math. Anal. Appl. 60 (1977), 542-549.

[10] W. A. STRAUSS: *On continuity of functions with values in various Banach spaces,* Pacific J. Math. 19 (1966), 543-551.