Existence and decay of solutions of a beam equation with a nonlinear damping term

中尾, 慎宏
九州大学教養部数学教室

南部, 徳盛
九州大学教養部数学教室

https://doi.org/10.15017/1449006

出版情報：九州大学教養部数学雑誌. 11 (2), pp.103-116, 1978-10. 九州大学教養部数学教室

Math. Rep.
XI-2, 1978.

Existence and decay of solutions of a beam equation with a nonlinear damping term

Mitsuhiro NAKAO and Tokumori NANBU
(Received April 18, 1978)

Introduction

In this paper we consider the initial-boundary value problem for the nonlinear beam equation

$$\frac{\partial^2}{\partial t^2}u+\alpha\frac{\partial^4}{\partial x^4}u+\nu(x)\frac{\partial}{\partial t}u+\rho(x,\frac{\partial}{\partial t}u)$$
$$-[\beta+k\int_0^l u_\xi(\xi,t)^2 d\xi]\frac{\partial^2 u}{\partial x^2}=f(x,t) \qquad (0.1)$$

on $[0,l]\times R^+$ with conditions

$$u(x,0)=u_0(x),\quad \frac{\partial}{\partial t}u(x,0)=u_1(x), \qquad (0.2)$$

$$u(0,t)=u(l,t)=u_{xx}(0,t)=u_{xx}(l,t)=0 \qquad (0.3)$$

where l, α and k are positive constants and R^+ denotes $[0,\infty)$.

This problem has an important physical meaning and has been investigated by many authors (see Woinowsky-Krieger [**11**], Reiss and Matkowsky [**9**], Dickey [3], Ball [1] etc.). Most of earlier papers, however, have treated the non damping equation (i.e. the case $\nu\equiv\rho\equiv0$). In particular, recently Ball [1] has proved the existence and uniqueness of both weak and smooth solutions for arbitrary $\beta\in R$.

The equation with $\nu>0$, $\rho\equiv0$, $\beta<0$ has been discussed by Reiss and Matokowsky [**9**]. See also Ball [2] where the existence and stability of solutions of some generalized equations has been studied.

In the present paper we restrict ourselves to the case $\beta>-\frac{2\alpha}{l^2}$, $\nu>0$ and $\rho(x,s)$ being monotonic in s to prove the existence and exponential decay of classical solutions for the problem (0.1)-(0.3). Our method and result here are closely related to those of our previous paper [7] where the non linear wave equations have been treated.

We consider only hinged boundary problem (0.3), but the results will be valid (without essential changes) for clamped boundary problem:

$$u(0, t)=u(l, t)=u_x(0, t)=u_x(l, t)=0.$$

1. Preliminaries

We use standard notation for function spaces and norms, and their precise definitions are omitted (see e.g. Lions [4]). For brevity in notation derivatives with respect to time t are denoted by the dots above symbols or D_t, while derivatives with respect to distance x are denoted by $\frac{\partial^m}{\partial x^m}u=u^{(m)}$ or $D_x^m u$. Following Ball [1], let us introduce the subspaces of the hilbert space $L^2(\Omega)$ $(\Omega\equiv[0, l])$:

$$S_0=\{u\in H^8(\Omega)\,|\,u, u^{(2)}, u^{(4)}, u^{(6)}\in H_0^1(\Omega)\}$$
$$S_1=\{u\in H^6(\Omega)\,|\,u, u^{(2)}, u^{(4)}\in H_0^1(\Omega)\}$$
$$S_2=\{u\in H^4(\Omega)\,|\,u, u^{(2)}\in H_0^1(\Omega)\}$$
$$S_3=H^2(\Omega)\cap H_0^1(\Omega).$$

(In [1] the space S_0 is not introduced and the notation is slightly diffrent).

The boundary condition (0.3) will be realized requiring the solution u belongs to the space S_2. The following two lemmas can be proved in the same way as in [1].

LEMMA 1.1. *If* $u\in S_{3-j}$, $0<j<3$, *then we have the inequalities*

$$\|f^{(i)}\|_{L^2(\Omega)}\leq(l/\sqrt{2})\|f^{(i+1)}\|_{L^2(\Omega)} \tag{1.1}$$

for $i=0, \ldots, 2j+1$.

LEMMA 1.2. $\phi_n=\sin(n\pi x/l)$, $n=1, 2, \ldots,$ *is a basis of the spaces* S_j, $0\leq j\leq 3$.

Here we state our hypotheses on ν, ρ and f:

A$_1$. $\nu(x)$ is in $C^4(\Omega)$ and $0<\nu_0\leq\nu(x)\leq\nu_1$ for some positive constants ν_0 and ν_1.

A$_2$. $\rho(x, s)$ is in $C^4(\Omega\times R)$ and satisfies the conditions

$$k_0\sum_{j=1}^{2}|s|^{r_i+2}\leq\rho(x, s)s\leq k_1(1+\sum_{i=1}^{2}|s|^{r_i})|s|^2$$

for $s\in R$ and

$$k_0\sum_{i=1}^{2}|s|^{r_i}\leq\frac{\partial}{\partial s}\rho(x, s)\leq k_1(1+\sum_{i=1}^{2}|s|^{r_i})$$

for $s\in R$ where k_0, k_1, r_1, r_2 are some positive constants. Moreover we assume

$$\max_{x\in\Omega}\left|\frac{\partial}{\partial s}\rho(x, s)\right|\to 0 \text{ as } s\to 0, \text{ and for } 0\leq i+j\leq 2$$

$$\left(\frac{\partial}{\partial x}\right)^i\left(\frac{\partial}{\partial s}\right)^j\rho(x, s)=0 \text{ at } (x, s)=(0, 0) \text{ and } (l, 0).$$

A$_3$. $f(t)\in C^1(R^+; H^2\cap H_0^1)\cap C^3(R^+; L^2)$ and

$$\max_{x\in\Omega}|f(x,t)|+\delta_i(t)\leq k_2e^{-\theta t}$$

for $i=0,1,2,3$, where k_2, θ are some positive constants and

$$\delta_i(t)=(\int_t^{t+1}\|D_t^i f(s)\|_{L^2}^2 ds)^{1/2}.$$

Regarding initial data we assume

A$_4$. $(u_0, u_1)\in S_0\times S_1$.

We conclude this section by giving a lemma which is useful for the investigation of the asymptotic behaviour of solutions of evolution equations ([**5**], [**6**], [**8**]).

LEMMA 1.3. *Let $\phi(t)$ be a bounded nonnegative function on $R^+=[0,\infty)$ such that*

$$\max_{s\in[t,\,t+1]}\phi(s)^{1+r}\leq C(\phi(t)-\phi(t+1))+\delta(t)$$

for $t\in R^+$, where $r\geq 0$, $C>1$ are constants and $\delta(t)$ is a nonnegative function. Then it holds that

(i) *if $\lim_{t\to\infty}\delta(t)=0$ then $\lim_{t\to\infty}\phi(t)=0$*

(ii) *if $r>0$ and $\delta(t)\leq C't^{-\theta-1}$ with $\theta>\frac{1}{r}$ then $\phi(t)<C''t^{-1/r}$ for $t>0$*

and

(iii) *if $r=0$ and $\delta(t)\leq C''e^{-\theta t}$ with $\theta>0$ then*

$$\phi(t)\leq C''e^{-\theta' t}$$

where C', C'' are positive constants depending on the bound $\max_{t\in[0,\,1]}\phi(t)$ and other known constants and

$$\theta'=\min(\theta,\ \log\frac{C}{C-1}).$$

PROOF. For the proof see [**6**].

In this paper we use the above lemma mainly in the case $r=0$.

2. Approximate Solutions

For our purpose we employ Galerkin's method. First note that $\phi_n(x)=\sin(n/l\pi x)$, $n=1,2,\dots$, is a basis of S_j, $j=0,1,2,3$ and $L^2(\Omega)$. We assume as m-th approximate solution

$$u_m(x,t)=\sum_{i=1}^{m}\alpha_{i,m}(t)\phi_i(x) \tag{2.1}$$

where $\alpha_{i,m}(t)$ should be determined by the system of ordinary differential equations

$$(\ddot{u}_m(t), \phi_i)+(\alpha u_m^{(4)}, \phi_i)+(\nu\dot{u}_m(t), \phi_i)+(\rho(\cdot, \dot{u}_m(t)), \phi_i)$$
$$-([\beta+k\|u_m^{(1)}\|_{L^2}^2]u_m^{(2)}(t), \phi_i)=(f(\cdot, t), \phi_i),\ i=1, 2, \dots, m, \tag{2.2}$$

and

$$\begin{aligned} u_m(0) &= \sum_{i=1}^m \alpha_{i,m}(0)\phi_i(x)\to u_0 \quad \text{in} \quad S_0 \\ \dot{u}_m(0) &= \sum_{i=1}^m \dot{\alpha}_{i,m}(0)\phi_i(x)\to u_1 \quad \text{in} \quad S_1. \end{aligned} \tag{2.3}$$

The equation (2.2) can be written in the form

$$\begin{aligned} &\ddot{u}_m(t)+\alpha u_m^{(4)}(t)+\nu\dot{u}_m(t)+P_m\rho(\cdot, \dot{u}_m(t)) \\ &\quad -[\beta+k\|u_m^{(1)}(t)\|_{L^2}^2]u_m^{(2)}=P_m f(t) \end{aligned} \tag{2.2$'$}$$

where P_m is the projection of L^2-space to the subspace spanned by $\phi_1, \dots, \phi_m$.

It is easy to see that $u_m(t)$ exists on R^+. In the following we shall estimate $u_m(t)$ in various norms. Constants are denoted by C_i or λ_i, $i=1, 2, \dots$, their dependence on relevant parameters being mentioned where necessary. For simplicity we use the notation $\|\cdot\|$ for $\|\cdot\|_{L^2}$.

LEMMA 2.1 *We have*

$$\begin{aligned} &E(u_m(t))+\int_t^{t+1}\nu_0\|\dot{u}_m(s)\|^2+k_0\sum_{i=1}^2\|\dot{u}_m(s)\|_{L^{r_i+2}}^{r_i+2}\,ds \\ &\quad \le C_1 e^{-\lambda_1 t}, \quad m=1, 2, 3, \dots, \quad t\in R^+ \end{aligned} \tag{2.4}$$

where we set

$E(u(t))\equiv\|\dot{u}(t)\|^2+\alpha\|u^{(2)}(t)\|^2+\beta\|u^{(1)}(t)\|^2+\frac{k}{2}\|u^{(1)}(t)\|^4$ *and* C_1, λ_1 *are positive constants depending on* $\|u_0\|_{H^2}$, $\|u_1\|$, k_2, θ *and other known constants.*

PROOF. First note that $\alpha\|u^{(2)}(t)\|^2+\beta\|u^{(1)}(t)\|^2\geqq\text{const.}\|u^{(2)}(t)\|^2\geqq 0$ by our assumption on β. Multiplying (2.2) by $\dot{\alpha}_{i,m}(t)$, summing over i and integrating over $[t, t+1]$ we have

$$\begin{aligned} &\frac{1}{2}E(u_m(t+1))+\int_t^{t+1}\{\nu\dot{u}_m(s), \dot{u}_m(s))+(\rho(\dot{u}_m(s)), \dot{u}_m(s))\}\,ds \\ &\quad =\frac{1}{2}E(u_m(t))+\int_t^{t+1}(f(s), \dot{u}_m(s))\,ds \end{aligned} \tag{2.5}$$

and, using the assumptions A_1, A_2, A_3,

$$\begin{aligned} &\int_t^{t+1}\nu_0\|\dot{u}_m(s)\|^2+2k_0\sum_{i=1}^2\|\dot{u}_m(s)\|_{L^{r_i+2}}^{r_i+2}\,ds \\ &\quad \le E(u_m(t))-E(u_m(t+1))+\delta_0(t)^2 \quad (\equiv D_0(t)^2). \end{aligned} \tag{2.6}$$

It is seen from (2.6) that there exists two points $t_1\in[t, t+\frac{1}{4}]$, $t_2\in[t+\frac{3}{4}, t+1]$ such that $\nu_0\|\dot{u}_m(t_i)\|^2\le 4D_0(t)^2$ $i=1, 2$.

Next, multiplying (2.2) by $\alpha_{i,m}(t)$, summing over i and integrating over $[t_1, t_2]$ we have

$$\int_{t_2}^{t_2}(\alpha\|u_m^{(2)}(s)\|^2+\beta\|u_m^{(1)}(s)\|^2+k\|u_m^{(1)}(s)\|^4)\,ds$$
$$=\int_{t_1}^{t_2}\|\dot{u}_m(s)\|^2ds+(\dot{u}_m(t_1),u_m(t_1))-(\dot{u}_m(t_2),u_m(t_2))$$
$$+\int_{t_1}^{t_2}\{-(\nu\dot{u}_m,u_m)+(f(s),u_m(s))\}\,ds$$
$$+k_1\int_{t_1}^{t_2}\int_\Omega(1+\sum_{i=1}^{2}|\dot{u}_m(x,s)|^{r_i})|\dot{u}_m(x,s)|\ |u_m(x,s)|dxds$$
$$\le C_2(D_0^2(t)+D_0(t)\max_{s\in[t,t+1]}\|u_m(s)\|+\delta_0(t)\max_{s\in[t,t+1]}\|u_m(s)\|$$
$$+D_0(t)\max_{s\in[t,t+1]}\|u_m(s)\|_{H_0^1}) \qquad (2.7)$$

where we have used the fact

$$\|u\|_{L^p(\Omega)}\le C_p\|u\|_{H_0^1(\Omega)},\quad 1<p<\infty,$$

for $u\in H_0^1=H_0^1(\Omega)$, C_p being constant.

From (2.6) and (2.7) we see that there exists a point $t^*\in[t_1,t_2]$ such that

$$\nu_0\|\dot{u}_m(t^*)\|^2+2k_0\sum_{i=1}^{2}\|\dot{u}_m(t^*)\|_{L^{r_i+2}}^{r_i+2}+\alpha\|u_m^{(2)}(t^*)\|^2$$
$$+\beta\|u_m^{(1)}(t^*)\|^2+k\|u_m^{(1)}(t^*)\|^4$$
$$\le 2D_0(t)^2+2C_2\{D_0(t)^2+(D_0(t)+\delta_0(t))\max_{s\in[t,t+1]}\|u_m(s)\|$$
$$+D_0(t)\max_{s\in[t,t+1]}\|u_m(s)\|_{H^1}\}$$

and hence, with the aid of Lemma 1.1,

$$E(u_m(t^*))\le C_3\{D_0(t)^2+(D_0(t)+\delta_0(t))\max_{s\in[t,t+1]}E(u_m(s))^{\frac{1}{2}}\} \qquad (2.8)$$

Therefore a similar equation to (2.5) implies

$$\max_{s\in[t,t+1]}E(u_m(s))\le E(u_m(t^*))+\int_t^{t+1}\nu_1\|\dot{u}_m(s)\|^2\,ds$$
$$+k_1\int_t^{t+1}\int_\Omega(1+\sum_{i=1}^{2}|u_m(x,s)|^{r_i})|\dot{u}_m(x,s)|^2dxds+\int_t^{t+1}|(f(s),\ \dot{u}_m(s))|ds$$
$$\le C_4\{D_0(t)^2+(D_0(t)+\delta_0(t))\max_{s\in[t,t+1]}E(u_m(s))^{\frac{1}{2}}\}$$

and recalling the definition of $D_0(t)$

$$\max_{s\in[t,t+1]}E(u_m(s))\le C_5(E(u_m(t))-E(u_m(t+1))+\delta_0(t)^2). \qquad (2.9)$$

The inequality (2.9) implies readily

$$E(u_m(t))\le\max\ (\max_{s\in[0,1]}E(u_m(s)),\ C_5\delta_0(t)^2).$$

Therefore to prove our lemma it suffices by Lemma 1.3 to show that $\max_{s\in[0,1]}$

$E(u_m(s))$ is bounded by a constant independint of m. But by a familiar argument we see easily

$$E(u_m(t))\leq C_6(E(u_m(0))\leq C_7(\|u_0\|_{H^2}, \|u_1\|_{H^1})$$

for $t\in[0,1]$, which completes the proof. q.e.d.

LEMMA 2.2. *We have*

$$\|\dot u_m(t)\|_E^2\leq C_8 e^{-\lambda_2 t} \quad \textit{for} \quad t\in R^+ \tag{2.10}$$

where we set

$$\|u(t)\|_E^2=\|\dot u(t)\|^2+\alpha\|u^{(2)}(t)\|^2+\beta\|u^{(1)}\|^2$$
$$(\geqq\|\dot u(t)\|^2+(\alpha+\frac{\beta l^2}{2})\|u^{(2)}(t)\|^2>0)$$

and C_8, λ_2 *are positivec onstants depending on* $\|u_0\|_{H^4}, \|u_1\|_{H^2}, k_2, \theta, C_1, \lambda_1$ *etc. but independent of m.*

PROOF. Differentiating (2.2) with respect to t we have

$$(\dddot u_m(t), \phi_i)+(\alpha\dot u_m{}^{(4)}(t), \phi_i)+\nu(\ddot u_m(t), \phi_i)+\rho'(\cdot, \dot u_m(t))\ddot u_m(t), \phi_i)$$
$$-(2k(u_m{}^{(1)}, \dot u_m{}^{(1)})u_m{}^{(2)}(t)+[\beta+k(u_m{}^{(1)}(t))^2]\dot u_m{}^{(2)}(t), \phi_i)$$
$$=(\dot f(t), \phi_i). \qquad (\rho'(x, s)\equiv\frac{\partial}{\partial s}\rho(x, s)) \tag{2.11}$$

From (2.11) we have

$$\frac{1}{2}\|\dot u_m(t+1)\|_E^2+\nu_0\int_t^{t+1}\|\ddot u_m(s)\|^2 ds+\int_t^{t+1}\int_\Omega|\rho'(x, \dot u_m(x, s))\|\ddot u_m(x, s)|^2 dxds.$$
$$<\frac{1}{2}\|\dot u_m(t)\|_E^2+2k\int_t^{t+1}\int_\Omega|(u_m{}^{(1)}, \dot u_m{}^{(1)})||u_m{}^{(2)}||\ddot u_m|dxds$$
$$+k\int_t^{t+1}\|u_m{}^{(1)}\|^2\int_\Omega|\dot u_m{}^{(2)}\ddot u_m|\, dxds+\frac{1}{\nu_0}\int_t^{t+1}\|\dot f(s)\|^2 ds$$
$$+\frac{\nu_0}{2}\int_t^{t+1}\|\ddot u_m(s)\|^2 ds$$

and hence

$$\nu_0\int_t^{t+1}\|\ddot u_m(s)\|^2 ds+2\int_t^{t+1}\int_\Omega k_0\sum_{i=1}^2|\dot u_m(x, s)|^{r_i}|\ddot u_m(x, s)|^2 dxds$$
$$\leq 4k\int_t^{t+1}\|u_m{}^{(1)}(s)\|\,\|\dot u_m{}^{(1)}(s)\|\,\|u_m{}^{(2)}(s)\|\,\|\ddot u_m(s)\|ds$$
$$+2k\int_t^{t+1}\|u_m{}^{(1)}(s)\|^2\|\dot u_m{}^{(2)}(s)\|\,\|\ddot u_m(s)\|ds$$
$$+\frac{2}{\nu_0}\delta_1^2(t)+\|\dot u_m(t)\|_E^2-\|\dot u_m(t+1)\|_E^2. \tag{2.12}$$

Using Lemmas 1.1 and 2.1 the first and second terms of the right hand side of (2.12) are estimated as follows:

$$\text{(the first and the second terms)}\leq C_9 e^{\frac{-\lambda_1}{2}t}\max_{s\in[t,t+1]}\|\dot u_m(s)\|_E^2$$

and we have

$$\int_t^{t+1}\|\ddot{u}_m(s)\|^2 ds+\int_t^{t+1}\int_\Omega\sum_{i=1}^2|\dot{u}_m(x,s)|^{r_i}|\ddot{u}_m(x,s)|^2 dxds\leq D_1^2(t) \quad (2.13)$$

where

$$D_1^2(t)\equiv C_{10}(\|\dot{u}_m(t)\|_E^2-\|\dot{u}_m(t+1)\|_E^2+e^{\frac{-\lambda_1}{2}t}\max_{s\in[t,t+1]}\|\dot{u}_m(s)\|_E^2+\delta_1^2(t)).$$

Next, by (2.11) we have

$$\begin{aligned}
&\int_t^{t+1}(\alpha\|\dot{u}_m{}^{(2)}(s)\|^2+\beta\|\dot{u}_m{}^{(1)}(s)\|^2)ds\\
&\leq|(\ddot{u}_m(t),\dot{u}_m(t)|+|(\ddot{u}_m(t+1),\dot{u}_m(t+1))|\\
&\quad+\int_t^{t+1}\{\nu_1\|\ddot{u}_m(s)\|\|\dot{u}_m(s)\|+\|\dot{f}(s)\|\|\dot{u}_m(s)\|\}ds\\
&\quad+\int_t^{t+1}\int_\Omega\{k_1(1+\sum_{i=1}^2|\dot{u}_m(x,s)|^{r_i})|\ddot{u}_m(x,s)|\\
&\qquad\qquad+2k|(u_m{}^{(1)}(s),\dot{u}_m{}^{(1)}(s))||u_m{}^{(2)}(x,s)|\\
&\qquad\qquad+k\|u_m{}^{(1)}(s)\|^2|\dot{u}_m{}^{(2)}(x,s)|\}|\dot{u}_m(x,s)|dxds\\
&\leq 2\max_{s\in[t,t+1]}\|\ddot{u}_m(s)\|\|\dot{u}_m(s)\|+\{(\nu_1+k_1)D_1(t)+\delta_1(t)\}\\
&\quad\times\max_{s\in[t,t+1]}\|\dot{u}_m(s)\|+k_1\sum_{i=1}^2\int_t^{t+1}\int_\Omega(|\dot{u}_m|^{r_i}|\ddot{u}_m|^2+|\dot{u}_m|^{r_i+2})dxds\\
&\quad+2k\max_{s\in[t,t+1]}\|u_m{}^{(1)}(s)\|\|\dot{u}_m{}^{(1)}(s)\|\|u_m{}^{(2)}(s)\|\|\dot{u}_m(s)\|\\
&\quad+k\max_{s\in[t,t+1]}\|u_m{}^{(1)}(s)\|^2\|\dot{u}_m{}^{(2)}(s)\|\|\dot{u}_m(s)\|\\
&\leq C_{11}(e^{-\lambda_1 t}+D_1^2(t)+\delta_1^2(t)+e^{\frac{-\lambda_1}{2}t}\max_{s\in[t,t+1]}\|\dot{u}_m(s)\|_E) \qquad (2.14)
\end{aligned}$$

From (2.13) and (2.14) we see that there exists a point $t^*\in[t,t+1]$ such that

$$\|\dot{u}_m(t^*)\|_E^2\leq D_1^2(t)+\text{ (the right hand side of (2.14))}$$

and hence, using similar inequality to (2.12),

$$\begin{aligned}
\max_{s\in[t,t+1]}\|\dot{u}_m(s)\|_E^2&\leq C_{12}\{e^{-\lambda_1 t}+D_1^2(t)+e^{\frac{-\lambda_1}{2}t}\max_{s\in[t,t+1]}\|\dot{u}_m(s)\|_E\\
&\leq C_{13}\{e^{-\lambda_1 t}+\delta_1^2(t)+e^{\frac{-\lambda_1}{2}t}\max_{s\in[t,t+1]}\|\dot{u}_m(s)\|_E^2+\|\dot{u}_m(t)\|_E^2\\
&\qquad-\|\dot{u}_m(t+1)\|_E^2\}+\frac{1}{2}\max_{s\in[t,t+1]}\|\dot{u}_m(s)\|_E^2.
\end{aligned}$$

Therefore there existr a time T_0 such that if $t\geqq T_0$, then

$$\begin{aligned}
\max_{s\in[t,t-1]}\|\dot{u}_m(s)\|_E^2&\leq 3C_{13}\{e^{-\lambda_1 t}+\delta_1^2(t)+\|\dot{u}_m(t)\|_E^2\\
&\qquad-\|\dot{u}_m(t+1)\|_E^2\}. \qquad (2.15)
\end{aligned}$$

Thus to prove Lemma 2.2 it suffices (by Lemma 1.3) to show that $\|\dot{u}_m(t)\|_E^2$ is bounded on $[0,T_0+1]$ by a constant independent of m. For this it suffices also to show $\|u_m(0)\|_E^2$ is bounded by such constant. However we have by (2.2)′

$$\|\dot u_m(0)\|_E^2=\|\ddot u_m(0)\|^2+\alpha\|\dot u_m^{(2)}(0)\|^2+\beta\|\dot u_m^{(1)}(0)\|^2$$
$$\le\|-\alpha u_m^{(4)}(0)-\nu\dot u_m(0)-P_m\rho(\cdot,\dot u_m(0))$$
$$+[\beta+k\|u_m^{(1)}(0)\|^2]u_m^{(2)}(0)+P_mf(0)\|^2+\alpha\|\dot u_m^{(2)}(0)\|^2+\beta\|\dot u_m^{(1)}(0)\|^2$$
$$\le C_{14}(\|u_0\|_{H^4}^2+\|u_1\|_{H^2}^2+\sum_{i=1}^{2}\|u_1\|_{H^1}^{2r_i+2}+\|f(0)\|^2)$$
$$<+\infty \qquad (\text{by } (2.3))$$

which completes the proof. q.e.d.

We proceed to further estimations of $u_m(t)$. To do so we consider here the linear equation:

$$\ddot u(t)+u^{(4)}(t)+\alpha\dot u(t)-\beta u^{(2)}(t)=F(t)\quad \text{on } R^+. \qquad (2.16)$$

For the solutions of (2.16) we obtain the following lemma.

LEMMA 2.3. *Let* $F(t)\in L^2_{loc}(R^+;\ L^2)$ *and* $u(t)$ *be a solution of* (2.16) *with*

$$u(t)\in L^2_{loc}(R^+;S_2) \text{ and } u(t)\in L^2_{loc}(R^+;L^2).$$

Then we have

$$\max_{s\in[t,t+1]}\|u(s)\|_E^2\le C_{15}(\|u(t)\|_E^2-\|u(t+1)\|_E^2+\Delta(t)^2)$$

where C_{15} *is a positive constant independent of* u *and*

$$\Delta^2(t)=\int_t^{t+1}\|F(s)\|^2\,ds.$$

PROOF. The proof is easy (cf. Lemma 2.1) and omitted.

Using above Lemma we obtain first

LEMMA 2.4. *Let* $u_m(t)$ *be m-th approximate solution. Then we have*

$$\max_{s\in[t,t+1]}\|\ddot u_m(s)\|_E^2\le C_{16}e^{-\lambda_3 t}\quad \text{for } t\in R^+$$

where C_{16}, λ_3 *are positive constants depending on other known constants, in particular,* $\|u_0\|_{H^6}$ *and* $\|u_1\|_{H^4}$, *but independent of m.*

PROOF. Differentiating (2.2) two times we have

$$(D_t^4u_m(t),\phi_i)+(\alpha\ddot u_m^{(4)}(t),\phi_i)+(\nu\dddot u_m(t),\phi_i)$$
$$-(\beta\ddot u_m^{(2)}(t),\phi_i)+(F_m(\cdot,t),\phi_i)=(\ddot f(t),\phi_i) \qquad (2.17)$$

$i=1,2,3,\dots,$ m, where

$$F_m(x,t)=-k\{2(u_m^{(1)},\dot u_m^{(1)})\dot u_m^{(2)}+\|u_m^{(1)}\|\ddot u_m^{(2)}+2\|\dot u_m^{(1)}\|^2u_m^{(2)}$$
$$+2(u_m^{(1)},\ddot u_m^{(1)})u_m^{(2)}\}+\rho''(x,\dot u_m)(\ddot u_m)^2+\rho'(x,\dot u_m)\dddot u_m.$$
$$(\rho''(x,s)\equiv\frac{\partial^2}{\partial s^2}\rho(x,s)).$$

Applying Lemma 2.3 to the equation (2.17) we have

$$\max_{s\in[t,t+1]}\|u_m(s)\|_E^2\le C_{15}\{\|\ddot{u}_m(t)\|_E^2-\|\ddot{u}_m(t+1)\|_E^2$$
$$+2\int_t^{t+1}(\|F_m(\cdot,s)\|^2ds+\|\tilde{f}(t)\|^2)ds\}. \tag{2.18}$$

Now we estimate $\int_t^{t+1}\|F_m(s)\|^2\,ds$:

$$\int_t^{t+1}\|(u_m{}^{(1)},\dot{u}_m{}^{(1)})\dot{u}_m{}^{(2)}\|^2ds$$
$$\le\max_{s\in[t,t+1]}\|u_m{}^{(1)}(s)\|^2\|\dot{u}_m{}^{(1)}(s)\|^2\|\dot{u}_m{}^{(2)}(s)\|^2$$
$$\le C_{17}e^{-(\lambda_1+\lambda_2)t}\quad\text{(by Lemmas 2.1, 2.2)}, \tag{2.19}$$
$$\int_t^{t+1}\|u_m{}^{(1)}(s)\|^4\|\ddot{u}_m{}^{(2)}\|^2ds+\int_t^{t+1}\|\dot{u}_m{}^{(1)}\|^4\|u_m{}^{(2)}\|^2ds$$
$$\le C_{18}e^{-\lambda_1t}\max_{s\in[t,t+1]}\|\ddot{u}_m(s)\|_E^2\quad\text{(by Lemmas 2.1 and 2.2)},$$
$$\int_t^{t+1}|(u_m{}^{(1)},\ddot{u}_m{}^{(1)})|^2\|u_m{}^{(2)}(s)\|^2ds$$
$$\le\int_t^{t+1}\|u_m{}^{(1)}(s)\|^2\|u_m{}^{(2)}(s)\|^2\|\ddot{u}_m{}^{(1)}(s)\|^2ds$$
$$\le C_{19}e^{-\lambda_1t}\max_{s\in[t,t+1]}\|\ddot{u}_m(s)\|_E^2\ \text{(by Lemmas 1.1 and 2.1)} \tag{2.21}$$

and moreover

$$\int_t^{t+1}\int_\Omega|\rho''(x,\dot{u}_m(x,s))|^2|\ddot{u}_m(x,s)|^4\,dxds$$
$$\le\max_{x\in\Omega,s\in[t,t+1]}|\rho''(x,\dot{u}_m(x,s))|^2\int_t^{t+1}\|\ddot{u}_m(s)\|_{L^\infty}^2\|\ddot{u}_m(s)\|^2\,ds$$
$$\le C_{20}{}^-e^{\lambda_2t}\max_{s\in[t,t+1]}\|\ddot{u}_m\|_E^2\quad\text{(by Lemma 2.2)} \tag{2.22}$$

where we note again

$$\|u\|_{L^\infty}\le\text{const.}\,\|u_m{}^{(1)}\|\ \text{ if }\ u\in H_0^1.$$

For the last term $\rho'(\cdot,\dot{u}_m)\ddot{u}_m$ we have

$$\int_t^{t+1}\|\rho'(\cdot,\dot{u}_m(s))\ddot{u}_m(s)\ddot{u}_m(s)\|^2ds\le\max_{x\in\Omega,s\in[t,t+1]}|\rho'(x,\dot{u}_m(x,s))|^2$$
$$\times\max_{s\in[t,t+1]}\|\ddot{u}_m(s)\|_E^2. \tag{2.23}$$

Since $\|\rho'(\cdot,s)\|_{L^\infty}\to0$ as $s\to0$ and $\|\dot{u}_m(x,s)\|\le(l^2/2)\cdot\|\dot{u}_m{}^{(2)}(s)\|\le(l^2/2a)\cdot C_8e^{-\lambda_2s}$, we have

$$\max_{x\in\Omega,s\in[t,t+1]}|\rho'(x,\dot{u}_m(x,s))|\le\varepsilon(t) \tag{2.24}$$

where $\varepsilon(t)$ is a function tending to 0 as $t\to\infty$. From above estimates (2.18)-(2.24) we obtain

$$\max_{s\in[t,t+1]}\|\ddot{u}_m(s)\|_E^2\le C_{21}\{\|\ddot{u}_m(t)\|_E^2-\|\ddot{u}_m(t+1)\|_E^2$$
$$+(e^{-\lambda_1t}+e^{-\lambda_2t}+\varepsilon(t))\max_{s\in[t,t+1]}\|\ddot{u}_m(s)\|_E^2+e^{-\lambda_1t-\lambda_2t}+\delta_2(t)^2\}$$

and hence there exists a time $T_1>0$ such that it $t\geqq T_1$ then

$$\max_{s\in[t,t+1]}\|\ddot{u}_m(s)\|_E^2\leq C_{22}(\|\ddot{u}_m(t)\|_E^2-\ddot{u}_m\|\ddot{u}_m(t+1)\|_E^2 + e^{-(\lambda_1+\lambda_2)t}+e^{-2\theta t}). \tag{2.25}$$

Next, we shall show the boundeness of $\|\ddot{u}_m(0)\|_E$.

$$\begin{aligned}\|\ddot{u}_m(0)\|_E^2&=\|\dddot{u}_m(0)\|^2+\alpha\|\ddot{u}_m{}^{(2)}(0)\|^2+\beta\|\ddot{u}_m{}^{(1)}(0)\|^2\\ &\leq\|D_t\{-\alpha u_m{}^{(4)}-\nu\dot{u}_m(t)-P_m\rho(\cdot,\dot{u}_m(t))+\beta u_m{}^{(2)}(t)\\ &\qquad+k\|u_m{}^{(1)}\|^2u_m{}^{(2)}+f\}_{t=0}\|^2\\ &\quad+(\alpha+\frac{l^2}{2}\beta)\|\{-\alpha u_m{}^{(6)}-\nu\dot{u}_m{}^{(2)}-D_x^2P_m\rho(\cdot,\dot{u}_m)\\ &\qquad+\beta u_m{}^{(4)}+k\|u_m{}^{(1)}\|^2u_m{}^{(4)}+f^{(2)}\}_{t=0}\|^2.\end{aligned} \tag{2.26}$$

Since P_m and D_x^2 are commutative on $H^2\cap H_0^1$, we see $D_x^2P_m\rho(\cdot,\dot{u}_m(t))=P_mD_x^2\rho(\cdot,\dot{u}_m(t))$ because $\rho(\cdot,\dot{u}_m(t))\in H^2\cap H_0^1$ by the assumption $\rho(x,0)=0$. Thus by (2.26) we have easily

$$\|\ddot{u}_m(0)\|_E^2\leq C_{23}(\|u_0\|_{H^6},\|u_1\|_{H^4},\|\dot{f}(0)\|_{H^2})<+\infty. \tag{2.27}$$

From (2.25) and (2.27) we can conclude the estimate in Lemma 2.4 is valid. q.e.d.

Finally we shall show:

LEMMA 2.5. *We have*

$$\|\dddot{u}_m(t)\|_E^2\leq C_{24}e^{-\lambda_4t}\quad for\quad t\in R^+,$$

where C_{24}, λ_4 *are positive constants depending, in particular,* $\|u_0\|_{H^8}$ *and* $\|u_1\|_{H^6}$.

PROOF. Differentiating (2.2)′ three times with respect to t, using similar arguments as in the proof of Lemma 2.4 we can show easily the following inequality

$$\max_{s\in[t,t+1]}\|\dddot{u}_m(s)\|_E^2\leq C_{25}(\|\dddot{u}_m(t)\|_E^2-\|\dddot{u}_m(t+1)\|_E^2+e^{-\lambda_5t}+\delta_3^2(t))\quad\text{for } t\geqq T_2$$

where λ_5 and T_2 are some positive constants. Therefore to complete the proof it suffices to show the boundedness of $\|\dddot{u}_m(0)\|_E^2$. Now,

$$\begin{aligned}\|\dddot{u}_m(0)\|_E^2&=\|D_t\{u_m{}^{(4)}+P_m\rho(\dot{u}_m)-(\beta+k\|u_m{}^{(1)}\|^2)u_m{}^{(2)}+f(t)\}_{t=0}\|_E^2\\ &\leq\|\dot{u}_m{}^{(4)}(0)\|_E^2+\|P_m\rho'(\dot{u}_m)\ddot{u}_m)_{t=0}\|_E^2+\|\beta\dot{u}_m{}^{(2)}(0)\|_E^2\\ &\qquad+k^2\|(\|u_m{}^{(1)}(0)\|^2\dot{u}_m{}^{(2)}+2(u_m{}^{(1)})u_m{}^{(2)}(0))\|_E^2+\|\dot{f}(0)\|_E^2\\ &\leq\|u_m{}^{(8)}(0)+\alpha\dot{u}_m{}^{(4)}(0)-(\beta+k\|u_m{}^{(1)}(0)\|^2)u_m{}^{(6)}(0)\\ &\qquad\qquad+D_x^4P_m\rho(\dot{u}_m(0))\|^2+\alpha\|\dot{u}_m{}^{(6)}(0)\|^2+\beta\|\dot{u}_m{}^{(5)}(0)\|^2\\ &\qquad+\|\rho''(\cdot,\dot{u}_m(0))|\ddot{u}_m(0)|^2\|^2+\alpha\|D_x^2P_m[\rho'(\cdot,\dot{u}_m(0))\ddot{u}_m(0)]\|^2\\ &\qquad+|\beta|\|D_xP_m[\rho'(\cdot,\dot{u}_m(0))\ddot{u}_m(0)]\|^2+\beta^2\|\ddot{u}_m(0)\|^2+\alpha\beta^2\|\dot{u}_m{}^{(4)}(0)\|^2\\ &\qquad+|\beta|^3\|\dot{u}_m{}^{(3)}(0)\|^2+k^2\|\{4(u_m{}^{(1)}(0),\dot{u}_m{}^{(1)}(0))\dot{u}_m{}^{(2)}(0)\\ &\qquad\qquad+\|u_m{}^{(1)}(0)\|^2\ddot{u}_m{}^{(2)}(0)+2\|\dot{u}_m{}^{(1)}(0)\|^2u_m{}^{(2)}(0)\end{aligned}$$

$$+2(u_m^{(1)}(0), \dot{u}_m^{(1)}(0))\dot{u}_m^{(2)}(0)\}\|^2$$
$$+\alpha k^2\|\{\|u_m^{(1)}(0)\|^2\dot{u}_m^{(4)}(0)+2(u_m^{(1)}(0), \dot{u}_m^{(1)}(0))u_m^{(4)}(0)\}\|^2$$
$$+|\beta|k^2\|\{u_m^{(1)}(0)\|^2\dot{u}_m^{(3)}(0)+2(u_m^{(1)}(0), \dot{u}_m^{(1)}(0))u_m^{(3)}(0)\}\|^2$$
$$+\|\ddot{f}(0)\|^2+\alpha\|D_x^2 P_m\dot{f}(0)\|^2+\frac{|\beta|l}{2}\|D_x^2 P_m f(0)\|^2$$
$$\leq C_{23}(\|u_0\|_{H^8}, \|u_1\|_{H^6}, \|\dot{f}(0)\|_{H^2})$$

where we have used the fact that $\rho'(\dot{u}_m(0)\ddot{u}_m(0), D_x^2\rho(\cdot, \dot{u}_m(0))$ and $\dot{f}(0)$ are in $H^2\cap H_0^1$ (by A_2 and A_3). The proof is completed. q. e. d.

3. Passage to the limit

In this section we shall prove several convergency properties of the approximate solutions $\{u_m(x, t)\}$, the limit function being the required classical solution. As is proved in the previous section we know

$$\|D_t^i u_m(t)\|_E\leq C_{24}e^{-\lambda t} \qquad t\in R^+, \quad i=0, 1, 2, 3, \tag{3.1}$$

for a certain $C_{24}, \lambda>0$.

Therefore, noting that the space $H^1(\Omega)$ is imbedded into $C(\Omega)$ and the imbedding map is compact, we can apply Ascoli-Arzela's lemma to the sequence $\{u_m(t)\}$ to obtain that there exists a subsequence, which will be denoted by the same symbol for simplicity, and a function $u(x, t)$ such that

$D_t^i u_m(t)\rightarrow D_t^i u(t)$ in $C^1(\Omega)$ uniformly on each compact interval $[0, T]$ $(T>0)$ for $i=0, 1, 2$, and consequently

$D_t^i\rho(x, \dot{u}_m(x, t))\longrightarrow D_t^i\rho(x, \dot{u}(x, t))$ uniformly on $\Omega\times[0, T]$ for $i=0, 1$, and moreover

$D_t^i\|u_m^{(1)}(t)\|^2\longrightarrow D_t^i\|u^{(1)}(t)\|^2$ uniformly on $[0, T]$ for $i=0, 1, 2$.

Furthermore by a standard compactness argument we may assume

$$D_t^i u_m(t)\longrightarrow D_t^i u(t) \quad \text{weakly star in } L^\infty(R^+; L^2)$$

and

$$D_t^{i-1}u_m(t)\longrightarrow D_t^{i-1}u(t) \quad \text{weakly star in } L^\infty(R^+; H^2)$$

for $i=0, 1, 2, 3, 4$, and

$$\|D_t^i u(t)\|_E\leq C_{25}e^{-\lambda t}, \quad t\in R^+, \quad i=0, 1, 2, 3. \tag{3.1$'$}$$

The inequality (3.1)$'$ implies in particular

$$D_t^i u\in C(R^+; L^2), D_t^{i-1}u\in C(R^+; H^2)\subset C(R^+; C^1(\Omega))$$

for $i=0, 1, 2, 3$. (3.2)

From the above convergency properties of $u_m(t)$, we can conclude that $u(t)$ is a generalized solution of the equation (0.1). However we know

$$u^{(4)}(x,t)=-\ddot{u}-\rho(\cdot,\dot{u})+[\beta+k\|u^{(1)}(t)\|^2]u^{(2)}(t)+f(t)$$
$$\in C^1(R^+;L^2) \tag{3.3}$$

and hence we have easily

$$u(x,t)\in C^1(R^+;H^4)\subset C^1(R^+;C^3(\Omega)). \tag{3.4}$$

The relations (3.2) and (3.4) imply

(the right hand side of (3.3)) $\in C^1(R^+;C^1(\Omega))$

and consequently

$$u\in C^1(R^+;C^5(\Omega)). \tag{3.5}$$

Thus we conclude that

$$u\in C^1(R^+;C^5(\Omega))\cap C^2(R^+;C^1(\Omega)) \tag{3.6}$$

and u is a classical solution of the equation (0.1).

We must show that this solution u satifies the initial-boundary condition (0.2)-(0.3). From the equations

$$u_m(x,t)=\int_0^t \dot{u}_m(x,s)ds+u_m(x,0)$$

and

$$\dot{u}_m(x,t)=\int_0^t \ddot{u}_m(x,s)ds+\dot{u}_m(x,0)$$

we have, by taking the limits as $m\to\infty$,

$$u(x,t)=\int_0^t \dot{u}(x,t)ds+u_0(x)$$

$$\dot{u}(x,t)=\int_0^t \ddot{u}(x,t)ds+u_1(x)$$

which yield (0.2).

To see (0.3) we note that

$u_m(t)\in S_2$ and $\|u_m(t)\|_{H^4}\leq$const. $\|u_m^{(4)}\|\leq$const. $<+\infty$,

which allows us to assume

$u_m(t)\longrightarrow u(t)$ weakly star in $L^\infty(R^+;S_2)$,

that is,

$$u(t)\in L^\infty(R^+;S_2). \tag{3.7}$$

The relations (3.6) and (3.7) imply immediately (0.3).

Finally to see the decay of solutions we note that the right hand side of (3.3) decays exponentially in the L^2-norm and consequently, taking account of the boundary condition (0.3),

$$\|u(t)\|_{H^4}\leq C_{26}e^{-\lambda_5 t}$$

for some $C_{26}, \lambda_5>0$.

Thus the right hand side of (3.3) decays exponentially in the norm corresponding $C(\Omega)$ (in fact, $C^1(\Omega)$-norm if $f(t)$ decays exponentially in

$C^1(\Omega)$), and we have easily

$$|u(t)|_{C^4(\Omega)}\leq C_{27}e^{-\lambda_6 t}$$

for some $C_{27}, \lambda_6>0$.

The uniqueness of classical solution of the problem (0.1)-(0.3) is clear.

Now, we summarize above results in the following:

THEOREM 3.1. *Under the assumptions* A_1)-A_4) *the problem* (0.1)-(0.3) *admits a unique classical solution* $u(x,t)$, *satisfying*

$$\sum_{i=0}^{3}\|D_t^i u(t)\|_E+|u(t)|_{C^4(\Omega)}\leq C_{28}e^{-\lambda t},\quad t\in R^+ \tag{3.8}$$

for some positive constants C_{28}, λ *which depend, in particular, on* $\|u_0\|_{H^8}$ *and* $\|u_1\|_{H^6}$.

COROLLARY. *Let* u *be the solution in Theorem* 3.1. *Then we have*

$$\sum_{i=1}^{3}|D_t^i u(t)|_{C^1(\Omega)}+|u(t)|_{C^4(\Omega)}\leq C_{30}e^{-\lambda t}.$$

4. Final remarks

In this section we mention the case $\nu\equiv 0$ briefly. At this time we cannot expect the decay of classical solutions in the classical norm, i.e., in $C^4(\Omega)$ for this case, and we consider the generalized solutions.

Now we assume

A_1'). $\nu\equiv 0$ in (0.1).

A_2'). $\rho(x,s)$ is measurable in x and continuous in s on $\Omega\times R^+$, and satisfies, for simplicity,

$$k_0|s|^{r+2}\leq\rho(x,s)s\leq k_1(|s|^{r+2}+|s|^2)$$

for some $k_0, k_1, r>0$.

A_3'). $f(t)\in L^{\frac{r+1}{r+2}}(R^+; L^{\frac{r+1}{r+2}}(\Omega))$ and

$$\delta(t)\equiv\left(\int_t^{t+1}\|f(s)\|_{L^{r+1/r+2}}^{(r+2)/(r+1)}ds\right)^{\frac{r+1}{r+2}}\leq\text{const. } t^{-\frac{1}{r'}-1}$$

for some $r'>0$ with $0<r'<r$.

A_4'). $(u_0, u_1)\in S_2\times S_3$.

Let $u_m(x,t)$ be the m-th approximate solution constructed in § 2. Then by familar energy estimate we can obtain

$$\|\dot{u}_m(t)\|^2+\|u_m^{(2)}(t)\|^2+\|u_m^{(1)}(t)\|^2+\int_t^{t+1}\|\dot{u}_m(s)\|_{L^{r+2}}^{r+2}ds$$
$$+\|\ddot{u}_m(t)\|^2+\|\dot{u}_m^{(2)}(t)\|^2+\|\dot{u}_m^{(1)}(t)\|^2\leq C_T<+\infty \text{ on } [0, T] \tag{4.1}$$

for each $T>0$. Therefore by a standard compactness argument we can obtain a generalized solution u of the problem (0.1)-(0.3). Moreover the solution u satisfies the following energy equalities (cf. Strauss [10], Ball [1]):

$$\frac{1}{2}E(u(t+1))+\int_t^{t+1}(\rho(\cdot,\dot{u}(s)),\dot{u}(s))ds=\frac{1}{2}E(u(t))+\int_t^{t+1}(f(s),\dot{u}(s))ds$$

and

$$\int_{t_1}^{t_2}(\alpha\|u^{(2)}(s)\|^2+\beta\|u^{(1)}(s)\|^2+k\|u^{(1)}(s)\|^4)ds$$
$$=\int_{t_1}^{t_2}\|\dot{u}(s)\|^2ds+(\dot{u}(t_1),u(t_1))-(\dot{u}(t_2),u(t_2))$$
$$+\int_{t_1}^{t_2}\{(f(s),u(s))-(\rho(\cdot,\dot{u}(s)),u(s))\}ds$$

for $t, t_1, t_2\geqq 0$.

Therefore we can apply the result in [5] to obtain the inequality

$$\max_{s\in[t,t+1]}E(u(s))^{1+\frac{r}{2}}\leq C_{31}(E(u(t))-E(u(t+1))+\delta(t)^{r+2/r+1}),$$

and with the aid of Lemma 1.3 we obtain as in [5]

THEOREM 4.1 *Under the assumption* $A_1')$-$A_4')$, *the problem* (0.1)-(0.3) *admits a unique generalized solution u, satisfying*

$$\|\dot{u}(t)\|^2+\|u^{(2)}(t)\|^2+\|u^{(1)}(t)\|^4+\int_t^{t+1}\|\dot{u}(s)\|_{L^{r+2}}^{r+2}ds\leq\text{const. }t^{-\frac{2}{r}}$$

for $t\in R^+$.

References

[1] J.M. BALL: *Initial-boundary value problems for an extensible beam,* J. Math. Anal. Appl. 42(1973), 61-90.

[2] J.M. BALL: *Stability theory for an extensible beam,* J. Differential Equations 14(1973), 399-418.

[3] R.W. DICKEY: *Free vibrations and dynamic buckling of the extensible beam,* J. Math. Anal. Appl. 29(1970), 443-454.

[4] J.L. LIONS: Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires, Dunod Gauthier-Villas, Paris, 1969.

[5] M. NAKAO: *Decay of solutions of some nonlinear evolution equations,* J. Math. Anal. Appl. 60(1977), 542-549.

[6] M. NAKAO: *Convergence of solutions of the wave equation with a nonlinear dissipative term to the steady state,* Mem. Fac. Science, Kyushu Univ. Ser A 32(1976), 257-265.

[7] M. NAKAO: *Decay of solutions of some nonlinear wave equations in one dimension,* Funk. Ekvacioj 20(1977), 223-236.

[8] M. NAKAO and T. NANBU: *Existence and decay of solutions for a semilinear parabolic equation,* Math. Rep. College Gen. Ed., Kyushu Univ. 10(1976), 157-162.

[9] E.L. REISS and B.J. MATKOWSKY: *Nonlinear dynamic buckling of a compressed elastic column,* Quart. Appl. Math. 29(1971), 245-260.

[10] W. A. STRAUSS: *On continuity of functions with valued in various Banach spaces,* Pacific J. Math. 19(1966), 543-551.

[11] S. WOINOWSKY-KRIEGER: *The effect of axial force on the vibration of hinged bars,* J. Appl. Mech. 17(1950), 35-36.