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Introduction

Let H be a real Hilbert space and V, W be real Banach spaces with $V \subset W \subset H$. We assume V is dense in W and H, and the natural injections from V into W and from W into H are both continuous. We identify H with its dual H^* (i.e. $V \subset W \subset H \subset W^* \subset V^*$). Pairing between V^* and V is denoted by (,).

Let us consider the nonlinear evolution equation

$$u''(t)+B(t)u'(t)+Au(t)=0 \quad (t \in \boldsymbol{R}^+=[0,\infty)) \qquad \text{(E)}$$

where A is the Fréchet derivative of a nonnegative functional $F_A(u)$ on V and $B(t)$ is a bounded operator for each t from W to W^*.

Regarding the operators A and $B(t)$, we make the following assumptions:

H_1. For each $d>0$, the set $\{u \in V \mid F_A(u) \leqq d\}$ is bounded, and $(Au, u) \geqq k_0 F_A(u)$ for $u \in V$ with some $k_0>0$.

H_2. $B(t)$ satisfies the inequalities;

$k_1 h(t) \|v\|_W^{r+1} \geqq \|B(t)v\|_{W^*}$ and

$(B(t)v, v) \geqq h(t) \|v\|_W^{r+2}$ for $v \in W$

where $k_1 (\geqq 0)$, $r (\geqq 0)$ are constants and $h(t)$ is a function on $\boldsymbol{R}^+$ with $h(t) \geqq \underline{h} > 0$.

Recently in [5], one of the present authors has investigated the decay property of solutions of (E) in the case of $B(t)$ being independent of t, and subsequently, in [6], the case that both of A and B depend on t has been treated. In [6], however, we are interested mainly in the case $A(t)$ and (or) $B(t)$ tend to 0 as $t \to \infty$ in a certain sense, and little attention is paid to the case that they are unbounded with respect to t.

The object of this paper is to prove that the solutions of (E) approach to 0 as $t \to \infty$ in the energy if we make some restriction on the growth of $h(t)$ appearing in H_2. As simple examples show our result is best possible in a certain sense (see section 3).

Though our method and result are related to those of [5], [6], they are essentially generalizations of a recent work [1] by Artstein and Infante, where the second order ordinary differential equation

$$\ddot{x}(t)+h(t)\dot{x}(t)+kx(t)=0 \quad (k>0)$$

is mainly discussed.

In section 3 we give some typical examples.

1. Preliminaries

Here we state our definition of solutions of (E) and a lemma due to Artstein and Infante [1].

DEFINITION. A V-valued function $u(t)$ on $\boldsymbol{R}^+=[0,\infty)$ is said to be a solution of (E) if $u\in C(\boldsymbol{R}^+; V)$, $u'\in C(\boldsymbol{R}^+; H)\cap L^{r+2}_{loc}(\boldsymbol{R}^+; W)$, $u''\in L^1_{loc}(\boldsymbol{R}^+; V^*)$ and the equation (E) is valid in V^* for a.e. $t\in\boldsymbol{R}^+$.

Let $u(t)$ be a solution of (E). Then we have formally

$$\frac{d}{dt}E(u(t))+(B(t)u'(t),\ u'(t))=0 \text{ (a.e.)} \tag{1.1}$$

where we set

$$E(u(t))=\frac{1}{2}\|u'(t)\|_H^2+F_A(u(t)).$$

Moreover we have formally, by integration by parts,

$$(u'(t_2),\ u'(t_2))-(u'(t_1),\ u'(t_1))+\int_{t_1}^{t_2}\{\|u'(t)\|_H^2+ \\ +(B(t)u'(t),\ u(t))+(Au(t),\ (u(t))\}\ dt=0 \tag{1.2}$$

for $t_1,\ t_2\in\boldsymbol{R}^+$.

Throughout the paper we consider only the solutions satisfying (1.1) and (1.2), which does not seem to be so restrictive in practical problems.

For our argument the following lemma is essential.

LEMMA ([1]). *Let $a_1, a_2, \ldots\ldots$ be a sequence of positive numbers with the property that, for some N_0, $\sum_{i=1}^{n} a_i \leqq N_0 n^2$. Then $\sum_{i=1}^{\infty}\frac{1}{a_i}=\infty$.*

2. Result

Our result is the following:

THEOREM. *In addition to* H_1 *and* H_2, *suppose that the function $h(t)$ in* H_2 *satisfies the growth condition*

$$\frac{1}{T^{2+r}}\int_0^T h(t)dt \leqq N_0 \quad \textit{for any } T>0, \tag{2.1}$$

with some $N_0>0$.

Then, for any solution $u(t)$ *of* (E) *such that* $\|\dot{u}(t)\|_H$ *is uniformly continuous for large* t *we have*

$$\lim_{t\to\infty} E(u(t))=0.$$

PROOF. By (1.1) and the definition of solution, $E(u(t))$ is continuous in t and

$$\begin{aligned}\dot{E}(u(t)) &\equiv \limsup_{h\to+0} \frac{E(u(t+h))-E(u(t))}{h}\\ &= -(B(t)u'(t),\ u'(t))\\ &\leqq -h(t)\|u'(t)\|_W^{r+2} \quad \leqq -\underline{h}\|u'(t)\|_W^{r+2}. \end{aligned} \tag{2.2}$$

Therefore, by a standard argument of stability (see, Lasalle [2]), we have

$$\lim_{t\to\infty} E(u(t))=c_0(\text{const.}) \text{ and } \lim_{t\to\infty}\|u'(t)\|_H=0 \tag{2.3}$$

We claim that there is a monotonic increasing sequence of integers n_1, n_2,, ..., $n_i\to\infty$ as $i\to\infty$, such that

$$\int_{n_i}^{n_i+1}(B(t)u'(t),\ u(t))dt\to 0 \ (n_i\to\infty). \tag{2.4}$$

Suppose that it were false; then there would exist a $\delta>0$ and n_0 such that

$$0<\delta\leqq\left(\int_n^{n+1}(B(t)u'(t),\ u(t))\,dt\right)^2 \text{ for } n\geqq n_0.$$

Then we have, by H_1 and H_2,

$$\begin{aligned}\delta^{(r+2)/2} &\leqq \left(\int_n^{n+1} k_1 h(t)\|u'(t)\|_W^{r+1}\|u(t)\|_W dt\right)^{r+2}\\ &\leqq k_1^{r+2}\left(\int_n^{n+1} h(t)\,dt\right)\left(\int_n^{n+1} h(t)\|u'(t)\|_W^{r+2}\|u(t)\|_W^{\frac{r+1}{r+2}}dt\right)^{r+1}\\ &\leqq \text{const.}\left(\int_n^{n+1} h(t)dt\right)\left(\int_n^{n+1} h(t)\|u'(t)\|_W^{r+2}\,dt\right)^{r+1}.\end{aligned}$$

Thus, setting $a_n=\left(\int_n^{n+1} h(t)dt\right)^{\frac{1}{r+1}}$, we have

$$\begin{aligned}\frac{1}{a_n} &\leqq \text{const.}\int_n^{n+1} h(t)\|u'(t)\|_W^{r+2}\,dt\\ &\leqq \text{const.}\int_n^{n+1}(B(t)u'(t),\ u'(t))\,dt.\end{aligned} \tag{2.5}$$

Hence, again by (1.1),

$$\frac{1}{a_n}\leqq \text{const.}\,(E(u(n))-E(u(n+1))$$

and

$$\sum_{n=n_0}^{\infty}\frac{1}{a_n}\leqq \text{const.}\ E(u(n_0))<\infty. \tag{2.6}$$

However, by our assumption on $h(t)$, we have

$$\sum_{i=1}^{n}a_i=\sum_{i=1}^{n}\left(\int_i^{i+1}h(t)\ dt\right)^{\frac{1}{r+1}}$$
$$\leqq n^{\frac{r}{r+1}}\left(\int_0^{n+1}h(t)\ dt\right)^{\frac{1}{r+1}}\leqq N_0'n^2.$$

which means, by Lemma, (2.6) is a contradiction.

Now, by (1.2) and (2.5), we have

$$\lim_{i\to\infty}\int_{n_i}^{n_i+1}(Au(t),\ u(t))\ dt=0$$

which together with H_1 rules out the possibility $c_0\neq 0$. The proof of Theorem is now completed.

3. Examples

In this section we give two typical examples.

EXAMPLE 1. Consider the ordinary differential equation

$$\ddot{x}(t)+h(t)|\dot{x}(t)|^r\dot{x}(t)+k|x(t)|^\alpha x(t)=0, \tag{3.1}$$
$$t\geqq 0,\ (k>0,\ \alpha,\ r\geqq 0).$$

In this case we can take $V=W=H=\boldsymbol{R}$ (real line) and

$$Au=k|u|^\alpha u,\quad B(t)u=h(t)|u|^r u.$$

The equations (1.1), (1.2) are, of course, valid for any usual solution $x(t)$ and $|\dot{x}(t)|$ is uniformly continuos on $\boldsymbol{R}^+$. Therefore, if $h(t)$ satisfies the growth condition (2.1), we have

$E(x(t))=\frac{1}{2}|\dot{x}(t)|^2+\frac{k}{\alpha+2}|x(t)|^{\alpha+2}\to 0$ as $t\to\infty$, that is, $\lim_{t\to\infty}|\dot{x}(t)|=\lim_{t\to\infty}|x(t)|=0$. This is a direct generalization of the result of [1].

Analogeously as in [1], we consider for any $\varepsilon>0$

$$h(t)=t\frac{(\delta+1)(r+1)}{\delta^{r+1}}\{\delta(\delta+1)t^{-\delta-2}+(1+t^{-\delta})^{\alpha+1}\},\quad \text{where } \delta=\frac{\varepsilon}{r+1}.$$

Then

$$\frac{1}{T^{2+r+\varepsilon}}\int_0^T h(t)\ dt\approx\text{const.}\quad \text{and}\quad x(t)=1+(1+t)^{-\varepsilon/(r+1)}$$

is a solution of (3.1). Since $x(t)$ does not approach to 0 as $t\to\infty$, this shows that condition (2.1) is sharp and can not be replaced by

$$\int_0^T h(t)\ dt\leqq N_0T^{2+r+\varepsilon}.$$

EXAMPLE 2. Consider the nonlinear wave equation

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}u-\Delta u+h(t)\ \rho(x,\dfrac{\partial}{\partial t}u)+\beta(x,u)=0 \quad \text{on } \Omega\times \boldsymbol{R}^+ \\ \text{and} \\ \quad u|_{\partial\Omega}=0, \end{cases} \tag{3.2}$$

where Ω is a bounded domain in the n dimensional Euclidean space $\boldsymbol{R}^n$ and $\partial\Omega$ its boundary.

Regarding ρ and β we assume:

$$0\leqq\beta(x,s)s\leqq\text{const.}\ (1+|s|)^{\alpha+2}$$

and

$$C_0|s|^{r+2}\leqq\rho(x,s)s\leqq C_1|s|^{r+2}\ \ (C_0, C_1\text{: const.})$$

with

$$0\leqq\alpha,\ r\leqq\frac{2}{n-2}\ \text{ if } n>2 \ \text{ and } \ 0\leqq\alpha,\ r<\infty \text{ if } n=1,2.$$

Then, under some additional conditions of measurability, continuity, and monotonicity on $\beta(x, s)$ and $\rho(x, s)$, the existence of (generalized) solutions u satisfying

$$u\in C(\boldsymbol{R}^+;\mathring{H}_1(\Omega)),\quad u'\in C(\boldsymbol{R}^+;L^2(\Omega))\cap L^{r+2}(\boldsymbol{R}^+;L^{r+2}(\Omega))$$

is well known (see e.g. Lions and Strauss [3], Nakao [4], etc.). In this case we can take

$$H=L^2(\Omega),\quad V=\mathring{H}_1(\Omega),\quad W=L^{r+2}(\Omega),$$
$$Au=-\Delta u+\beta(x,u) \text{ and } B(t)v=h(t)\rho(x,v).$$

Moreover, the equations (1.1)–(1.2) are known to be valid (cf. Strauss [7]). Thus if $h(t)$ satisfies (2.1) and $\|u'(t)\|_H$ is uniformly continuous on $\boldsymbol{R}^+$, we have

$$E(u(t))=\frac{1}{2}\|u'(t)\|^2_{L^2(\Omega)}+\frac{1}{2}\|u(t)\|^2_{\mathring{H}_1}+\int_\Omega\int_0^{u(x,t)}\beta(x,s)dsdx$$
$$\to 0 \text{ as } t\to\infty.$$

In particular, let us consider the linear equation i.e. $\rho(x,\frac{\partial}{\partial t}u)=\frac{\partial}{\partial t}u$, and $\beta(x,u)=u$ and let $\phi(x)$ be an eigen function of $-\Delta+I$ in $\mathring{H}_1(\Omega)$. Then $u(t)=(1+(1+t)^{-\varepsilon})\phi(x)\,(\varepsilon>0)$ satisfies the equation (3.2) with

$$h(t)=(\varepsilon+1)(1+t)^{-1}+\frac{1}{\varepsilon}(1+t)+\frac{1}{\varepsilon}(1+t)^{1+\varepsilon}$$

This example (due to essentially [1]) implies that the condition (2.1) with $r=0$ cannot be replaced by

$$\frac{1}{T^{2+\varepsilon}}\int_0^T h(t)dt\leqq N_0 \quad \text{for } T>0.$$

4. Remark on first order equation

In a similar and simpler manner it is easily seen that the assertion of Theorem in § 2 is valid also for the first order equation with $E(u(t))$ replaced by $F_A(u(t))$;

$$B(t)u'(t)+Au=0.$$

This result can be applied, for example, to the equation

$$\begin{cases} h(t)\dfrac{\partial}{\partial t}u-\displaystyle\sum_{i=1}^{n}\dfrac{\partial}{\partial x_i}\left(\left|\dfrac{\partial}{\partial x_i}u\right|^{p-2}\dfrac{\partial}{\partial x_i}u\right)=0 \text{ on } \Omega\times \boldsymbol{R}^+ \quad (p\geqq 2) \\ u|_{\partial\Omega}=0 \end{cases}$$

where Ω is a bounded domain in $\boldsymbol{R}^n$. We omit the details.

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