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On the orthogonal exact designs for some polynomial responses

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1. Summary and introduction.

The aim of this article is to construct the orthogonal exact designs for the m-th degree polynomial responses in k-factors whose levels are simple and whose scales are less than those of factorial designs. We shall see the symmetric designs in § 3 which satisfy above requirements and also which have levels of each factor with 0, ± 1 for $m=2$ and with $\pm 1/3$, ± 1 for $m=3$. In § 4 we shall see the orthogonal or the nearly orthogonal designs whose levels are 0, ± 1 and are equal to or nearly equal to 0.5, 1.5 and 2 for $m=2$. We do not consider any optimality of those designs but we think those designs are useful for easy setting of their levels and are desirable to test the hypothesis that each coefficient of polynomial is equal to 0.

Many works of J. Kiefer and other authors [3], [5] have concerned with the problem of constructing D-optimal designs. They have treated the optimal design problems with probablity measures ξ called "continuous" designs, whose allocation ratios n_i/N of observations for levels have often been irrational. Their approach is very beautiful and gives many interesting results, but is purely theoretical. On the other direction, G. E. P. Box, and K. B. Wilson [1], V. V. Nalimov, T. I. Golikova and N. G. Mikeshina [6], H. O. Hartley [2] and A. C. Atkinson [8], [9] showed "integer" or "exact" designs—their number N of observations was integer and n_i takes on only multiples of $1/N$—with some optimal properties. Their optimal properties are "rotatable", "D-optimal" or "quasi D-optimal". Our aim is also to construct the exact designs.

2. Definitions, notations and restrictions.

Suppose that the response function is at most the m-th degree polynomial on k-dimensional Euclidean space $\mathscr{X}$ and $y_{\mathbf{x}}$ is an observation on $\boldsymbol{x}\in\mathscr{X}$ with variance σ^2 equal to 1.

$$E[y_{\mathbf{x}}]=\sum_{i_1+i_2+\cdots+i_k\leq m}\gamma_{i_1i_2\cdots i_k}x_1^{i_1}x_2^{i_2}\cdots x_k^{i_k}, \qquad [2.1]$$

where m is equal to 2 or 3. [2.1] is able to drive other representation without loss of generality,

$$E[y_{\mathbf{x}}]=\sum_{i_1+i_2+\cdots+i_k\leq m}\beta_{i_1i_2\cdots i_k}\phi_{i_1i_2\cdots i_k}(\boldsymbol{x}), \qquad [2.2]$$

where $\phi_{i_1i_2\cdots i_k}(\boldsymbol{x})$ is the $(i_1+i_2+\cdots+i_k)$-th degree polynomial and satisfies two requirements;

(i) $\phi_{i_1i_2\cdots i_k}(\boldsymbol{x})$ contains only one of the highest degree terms and the other terms, if any, are of lower degree, and

(ii) $\phi_{i_1i_2\cdots i_k}(\boldsymbol{x})$ is invariant under permutation of factors.

For example, they are satisfied by $x_1^{i_1}x_2^{i_2}\cdots x_k^{i_k}$ in [2.1] and also, when k and m are both equal to 2, we see $\phi_{00}(\boldsymbol{x})=1$, $\phi_{10}(\boldsymbol{x})=x_1+\alpha$, $\phi_{01}(\boldsymbol{x})=x_2+\alpha$, $\phi_{20}(\boldsymbol{x})=x_1^2+\beta(x_1+x_2)+\gamma$, $\phi_{02}(\boldsymbol{x})=x_2^2+\beta(x_1+x_2)+\gamma$, $\phi_{11}(\boldsymbol{x})=x_1x_2+\beta'(x_1+x_2)+\delta$.

If we can construct some design which makes $\{\phi_{i_1i_2\cdots i_k}(\boldsymbol{x})\}$ be orthogonal, then we can carry out to test the hypothesis $H_{i_1i_2\cdots i_k}$: $\beta_{i_1i_2\cdots i_k}=0$ without confounding. In practice, many experimenters use the 3^k-or 4^k-factorial designs not for the reason of their optimality but for their orthogonality. We shall find designs ξ which make some series $\{\phi_{i_1i_2\cdots i_k}(\boldsymbol{x})\}$ to be orthogonal and whose numbers of observations are less than that of factorial design.

Similar to factorial designs, we restrict to the m-th order symmetric designs whose levels are 0 or ±1 for $m=2$ and are $\pm1/3$ or ±1 for $m=3$.

DEFINITION. *A design ξ is called the m-th order symmetric if*

$$\int_{\mathscr{X}}x_1^{m_1}x_2^{m_2}\cdots x_k^{m_k}\xi(d\boldsymbol{x})=\begin{cases}0, & \text{when some } m_i \text{ is odd,}\\ \text{constant,} & \text{otherwise.}\end{cases}$$

where constants are independent of factors, $m=m_1+m_2+\cdots+m_k$ and m_i, $1\leqq i\leqq k$, is non-negative integer.

We denote the moments by

$$\lambda_2=\int_{\mathscr{X}}x_i^2\xi(d\boldsymbol{x}),\quad \lambda_4=\int_{\mathscr{X}}x_i^4\xi(d\boldsymbol{x}),\quad \lambda_8=\int_{\mathscr{X}}x_i^6\xi(d\boldsymbol{x}),\quad i=1,2,\cdots,k,$$

$$\lambda_3=\int_{\mathscr{X}}x_i^2x_j^2\xi(d\boldsymbol{x}),\quad \lambda_7=\int_{\mathscr{X}}x_i^4x_j^2\xi(d\boldsymbol{x}),$$

$$\lambda_6=\int_{\mathscr{X}} x_i^2 x_j^2 x_h^2 \xi(dx),\ 1\leqq i<j<h\leqq k. \qquad [2.3]$$

Let E_r be the set of points of $\mathscr{X}$ with r coordinates whose absolute values are equal to 1. Then, the other k-r coordinates are zero for $m=2$ and are $\pm 1/3$ for $m=3$. Let E be

$$E=E_1\cup E_2\cup\ldots\cup E_k.$$

E_r contains $\binom{k}{r}2^r$ points for $m=2$ and $\binom{k}{r}2^k$ points for $m=3$.

Setting n_r to E_r, we can construct the symmetric design where n_r observations are allocated on each point of E_r and we have

$$N=\begin{cases}\sum_{r=0}^{k}\binom{k}{r}n_r 2^r, & \text{for } m=2,\\ \sum_{r=0}^{k}\binom{k}{r}n_r 2^k, & \text{for } m=3.\end{cases} \qquad [2.5]$$

3. Construction of orthogonal designs.

Case 1. $m=2$. We have the following contributions from E_r's to the moments;

$$\lambda_2=\lambda_4=\frac{1}{N}\sum_{r=1}^{k}\binom{k-1}{r-1}n_r 2^{r-1},$$

$$\lambda_3=\frac{1}{N}\sum_{r=2}^{k}\binom{k-2}{r-2}n_r 2^{r-2}. \qquad [3.1]$$

Putting

$$\begin{aligned}&\phi_{00\cdots0}(\boldsymbol{x})=1,\ \phi_{10\cdots0}(\boldsymbol{x})=x_1,\cdots\cdots,\ \phi_{00\cdots1}(\boldsymbol{x})=x_k,\\ &\phi_{20\cdots0}(\boldsymbol{x})=x_1^2-\lambda_2,\cdots\cdots,\ \phi_{00\cdots2}(\boldsymbol{x})=x_k^2-\lambda_2,\\ &\phi_{11\cdots0}(\boldsymbol{x})=x_1x_2,\cdots\cdots,\ \phi_{0\cdots11}(\boldsymbol{x})=x_{k-1}x_k,\end{aligned} \qquad [3.2]$$

then the information matrix of the M. L. E.'s of coefficients in [2.2] is

$$M_2(\xi)=\left[\begin{array}{cc|cccc|c} 1 & 0 & & & & & \\ 0 & \lambda_2\cdot I_k & & 0 & & & 0 \\ \hline & & \lambda_4-\lambda_2^2 & \lambda_3-\lambda_2^2 & \cdots\cdots & \lambda_3-\lambda_2^2 & \\ & 0 & \lambda_3-\lambda_2^2 & \ddots & & \vdots & 0 \\ & & \vdots & & & & \\ & & \lambda_3-\lambda_2^2 & \cdots\cdots & & \lambda_4-\lambda_2^2 & \\ \hline & 0 & & 0 & & & \lambda_3\cdot I_{\frac{k(k-1)}{2}} \end{array}\right] \qquad [3.3]$$

The series of polynomials $\{\phi_{i_1 i_2 \cdots i_k}(\boldsymbol{x}); i_1+i_2+\cdots+i_k \leqq 2,\ i_j$ is non-negative integer, $j=1,2,\ldots,k\}$ satisfies the requirements (i) and (ii). It is orthogonal w.r.to $\xi=\xi(n_1;E_1,\ldots,n_k;E_k)$ if and only if

$$\lambda_3=\lambda_2^2. \qquad [3.4]$$

The existence of designs satisfying [3.4] is obvious. For example, the factorial design of 3^k satisfies it with $\lambda_2=2/3$ and $\lambda_3=4/9$. We tabulate all of such designs for $k\leqq 6$ on Table I.

Case 2. $\boldsymbol{m=3}$. We have the moments;

$$\begin{aligned}
\lambda_2 &= \frac{1}{N}\sum_{r=1}^{k}\binom{k-1}{r-1}[n_r+(\tfrac{1}{3})^2 n_{r-1}]2^k,\\
\lambda_3 &= \frac{1}{N}\sum_{r=2}^{k}\binom{k-2}{r-2}[n_r+2(\tfrac{1}{3})^2 n_{r-1}+(\tfrac{1}{3})^4 n_{r-2}]2^k,\\
\lambda_4 &= \frac{1}{N}\sum_{r=1}^{k}\binom{k-1}{r-1}[n_r+3(\tfrac{1}{3})^4 n_{r-1}]2^k,\\
\lambda_6 &= \frac{1}{N}\sum_{r=3}^{k}\binom{k-3}{r-3}[n_r+3(\tfrac{1}{3})^2 n_{r-1}+3(\tfrac{1}{3})^4 n_{r-2}+(\tfrac{1}{3})^6 n_{r-3}]2^k,\\
\lambda_7 &= \frac{1}{N}\sum_{r=2}^{k}\binom{k-2}{r-2}[n_r+(\tfrac{1}{3})^2 n_{r-1}+(\tfrac{1}{3})^4 n_{r-1}+(\tfrac{1}{3})^6 n_{r-2}]2^k,\\
\lambda_8 &= \frac{1}{N}\sum_{r=1}^{k}\binom{k-1}{r-1}[n_r+(\tfrac{1}{3})^6 n_{r-1}]2^k.
\end{aligned} \qquad [3.5]$$

Adding following [3.6] to [3.2],

$$\begin{aligned}
&\phi_{30\cdots 0}(\boldsymbol{x})=x_1^3-\frac{\lambda_4}{\lambda_2}x_1,\ \ldots,\ \phi_{0\cdots 03}(\boldsymbol{x})=x_k^3-\frac{\lambda_4}{\lambda_2}x_k,\\
&\phi_{21\cdots 0}(\boldsymbol{x})=x_1^2x_2-\frac{\lambda_3}{\lambda_2}x_2,\ \ldots,\ \phi_{0\cdots 12}(\boldsymbol{x})=x_k^2x_{k-1}-\frac{\lambda_3}{\lambda_2}x_{k-1},\\
&\phi_{111\cdots 0}(\boldsymbol{x})=x_1x_2x_3,\ \ldots,\ \phi_{0\cdots 111}(\boldsymbol{x})=x_{k-2}x_{k-1}x_k,
\end{aligned} \qquad [3.6]$$

we have a series of polynomials up to third degree satisfying (i) and (ii).

The information matrix in this case is

$$M_3(\xi)=\left[\begin{array}{c|cc|c}
M_2(\xi) & 0 & & 0\\
\hline
0 & (\lambda_8-\frac{\lambda_4^2}{\lambda_2})\cdot I_k & {}^t\Delta & 0\\
 & \Delta & D & \\
\hline
0 & 0 & & \lambda_6\cdot I_{\frac{k(k-1)(k-2)}{6}}
\end{array}\right] \qquad [3.7]$$

where $\triangle=(\delta_{ij,h})$ is $k(k-1)\times k$ matrix and $\delta_{ij,h}$ is an inner product of $x_h^3-\frac{\lambda_4}{\lambda_2}x_h$ and $x_i^2 x_j-\frac{\lambda_3}{\lambda_2}x_j$,

$$\delta_{ij,h}=\begin{cases}\lambda_7-\frac{\lambda_3\lambda_4}{\lambda_2}, & \text{when } j=h,\\ 0, & \text{when } j\neq h,\end{cases}$$

and $D=(d_{ij,st})$ is $k(k-1)\times k(k-1)$ matrix and $d_{ij,st}$ is an inner product of $x_i^2x_j-\frac{\lambda_3}{\lambda_2}x_j$ and $x_s^2x_t-\frac{\lambda_3}{\lambda_2}x_t$,

$$d_{ij,st}=\begin{cases}\lambda_7-\frac{\lambda_3^2}{\lambda_2}, & \text{when } (i,j)=(s,t),\\ \lambda_6-\frac{\lambda_3^2}{\lambda_2}, & \text{when } i\neq s,\ j=t,\\ 0, & \text{otherwise.}\end{cases}$$

Therefore, the series $\{\phi_{i_1i_2\cdots i_k}(x);i_1+i_2+\cdots+i_k\leqq 3\}$ is orthogonal w.r. to ξ if and only if

$$\lambda_3=\lambda_2^2,\quad \lambda_3\lambda_4=\lambda_2\lambda_7,\quad \lambda_2\lambda_6=\lambda_3^2. \qquad [3.8]$$

We tabulate all designs that satisfy above condition [3.8] for $k\leqq 5$ on Table II.

4. Some orthogonal designs and nearly orthogonal designs for $m=2$.

Let $E_1(\pm\alpha)$ be the set of points with one non-zero coordinate whose absolute value is equal to α. In this section, we consider the symmetric designs which allocate n_0 to E_0, 1 to each point of $E_1(\pm\alpha)$ and 1 to each point of E_s for some s, $2\leqq s\leqq k$, then we have

$$N=n_0+2k+\binom{k}{s}2^s,$$

$$\lambda_2=\frac{1}{N}[2\alpha^2+\binom{k-1}{s-1}2^s],$$

$$\lambda_3=\frac{1}{N}\binom{k-2}{s-2}2^s,$$

$$\lambda_4=\frac{1}{N}[2\alpha^4+\binom{k-1}{s-1}2^s]. \qquad [4.1]$$

For orthogonality, it must hold that

$$\alpha^2=\sqrt{\binom{k-2}{s-2}2^{s-2}[n_0+2k+\binom{k}{s}2^s]}-\binom{k-1}{s-1}2^{s-1}, \qquad [4.2]$$

by means of [3.4].

We shall pick up the designs with $\alpha=2$ and with α being nearly equal to 0.5, 1.0 and 1.5 and tabulate them on Table III. In those cases, all α's are irrational numbers except the case $\alpha=2$.

We consider two cases; the first we approximate α by three decimals, and the second, we approximate α by 0.5, 1.0 or 1.5. In each case, its value yields non-zero covariance between M.L.E.'s of coefficients of the second degree terms,

$$Cov. = \frac{\lambda_2^2-\lambda_3}{(\lambda_4+(k-1)\lambda_3-k\lambda_2^2)(\lambda_4-\lambda_3)}, \qquad [4.3]$$

and their variance are equal to

$$V = \frac{(\lambda_4+(k-2)\lambda_3-(k-1)\lambda_2^2)}{(\lambda_4+(k-1)\lambda_3-k\lambda_2^2)(\lambda_4-\lambda_3)}, \qquad [4.4]$$

Case number i means of the first case and i' means of the second case on Table III.

5. Some explanations for the tables.

When we plan an experiment, it is desirable to have requirements; (*a*) the number of observations is small, and (*b*) the power of test is large.

The non-centrality of the F-test for the hypothesis that the corresponding coefficient of response function is zero depends on each diagonal element of information matrix. $|M|^{1/p}$ is the geometric mean of those values where p is equal to $(k+1)(k+2)/2$ for $m=2$ and equal to $(k+1)(k+2)(k+3)/6$ for $m=3$, and the D-optimal criterion is to make $|M|$ maximum.

We may choose some design fitted our aim from Table I. II and III.

On the other hand, we note designs with comparatively large n_0, for example, the case number 1 of $k=6$ for $m=2$, etc, on Table I. When the point $(0,0,\dots,0)$ of E_0 means the standard level of production process, we may obtain large part of observations of the experiment without disturbing usual running of process.

Finally, we omit the cases that designs satisfy our requirement but are the merely replication of some other design in Table I or II. For example, $(n_0, n_1, n_3, n_4)=(24,0,2,0,0)$ is the twice replication of $(12,0,1,0,0)$, which is the case number 1 of $k=4$ on Table I, and is omitted.

Table I. The orthogonal designs for quadratic response and diagonal elements of their information matrices.

k	case No.	n_0	n_1	n_2	n_3	n_4	n_5	n_6	N	λ_2	λ_3	λ_4-λ_2^2	$\lvert M\rvert^{1/p}$
3	1	4	0	1	0				16	0.500	0.250	0.250	0.354
	2	7	1	1	0				25	0.400	0.240	0.160	0.286
	3	1	1	1	1				27	0.667	0.222	0.444	0.442
4	1	12	0	1	0	0			36	0.333	0.222	0.111	0.207
	2	4	0	0	1	0			36	0.667	0.222	0.444	0.434
	3	17	1	1	0	0			49	0.286	0.204	0.082	0.172
	4	1	2	0	1	0			49	0.571	0.245	0.327	0.378
	5	2	0	0	1	1			50	0.800	0.160	0.640	0.483
	6	24	2	1	0	0			64	0.250	0.188	0.063	0.146
	7	0	4	0	1	0			64	0.500	0.250	0.250	0.330
	8	8	2	1	1	0			80	0.500	0.250	0.250	0.330
	9	33	3	1	0	0			81	0.222	0.173	0.049	0.126
	10	1	6	0	1	0			81	0.444	0.247	0.198	0.290
	11	1	0	0	1	3			81	0.889	0.099	0.790	0.476
	12	1	1	1	1	1			81	0.667	0.222	0.444	0.434
5	1	24	0	1	0	0	0		64	0.250	0.188	0.063	0.129
	2	31	1	1	0	0	0		81	0.222	0.173	0.049	0.110
	3	16	0	0	1	0	0		96	0.500	0.250	0.250	0.315
	4	40	2	1	0	0	0		100	0.200	0.160	0.040	0.095
	5	51	3	1	0	0	0		121	0.182	0.149	0.033	0.083
	6	64	4	1	0	0	0		144	0.167	0.139	0.028	0.074
	7	10	6	0	1	0	0		150	0.400	0.240	0.160	0.239
	8	79	5	1	0	0	0		169	0.154	0.130	0.024	0.066
	9	25	3	1	1	0	0		175	0.400	0.240	0.160	0.239
	10	0	0	4	0	0	1		192	0.500	0.250	0.250	0.315
	11	96	6	1	0	0	0		196	0.143	0.122	0.020	0.060
	12	4	0	2	1	0	1		196	0.571	0.245	0.327	0.367
	13	40	0	2	1	0	0		200	0.400	0.240	0.160	0.239
	14	8	0	0	0	2	1		200	0.800	0.160	0.640	0.496
	15	0	4	0	1	1	0		200	0.600	0.240	0.360	0.388
	16	16	12	0	1	0	0		216	0.333	0.222	0.111	0.189
	17	115	7	1	0	0	0		225	0.133	0.116	0.018	0.054
	18	0	4	3	0	1	0		240	0.500	0.250	0.250	0.315
	19	42	4	2	1	0	0		242	0.364	0.231	0.132	0.212
	20	23	6	0	2	0	0		243	0.444	0.247	0.198	0.273
	21	1	1	1	1	1	1		243	0.667	0.222	0.444	0.431
6	1	40	0	1	0	0	0	0	100	0.200	0.160	0.040	0.085
	2	49	1	1	0	0	0	0	121	0.182	0.149	0.033	0.074
	3	60	2	1	0	0	0	0	144	0.167	0.139	0.028	0.065
	4	73	3	1	0	0	0	0	169	0.154	0.130	0.024	0.058
	5	88	4	1	0	0	0	0	196	0.143	0.122	0.020	0.052
	6	40	0	0	1	0	0	0	200	0.400	0.240	0.160	0.227
	7	8	0	0	0	0	1	0	200	0.800	0.160	0.640	0.507
	8	105	5	1	0	0	0	0	225	0.133	0.116	0.018	0.047
	9	34	4	0	1	0	0	0	242	0.364	0.231	0.132	0.199
	10	124	6	1	0	0	0	0	256	0.125	0.109	0.016	0.043
	11	32	8	0	1	0	0	0	288	0.333	0.222	0.111	0.176
	12	145	7	1	0	0	0	0	289	0.118	0.104	0.014	0.039
	13	57	1	1	1	0	0	0	289	0.353	0.228	0.125	0.191
	14	6	4	0	0	1	0	0	294	0.571	0.245	0.327	0.360
	15	168	8	1	0	0	0	0	324	0.111	0.099	0.012	0.036
	16	56	4	1	1	0	0	0	324	0.333	0.222	0.111	0.176
	17	24	0	1	0	1	0	0	324	0.556	0.247	0.309	0.348
	18	4	0	0	0	0	1	2	324	0.889	0.099	0.790	0.523

19	34	12	0	1	0	0	0	338	0.308	0.213	0.095	0.158
20	8	0	0	1	0	1	0	360	0.667	0.222	0.444	0.430
21	80	0	2	1	0	0	0	360	0.333	0.222	0.111	0.176
22	193	9	1	0	0	0	0	361	0.105	0.094	0.011	0.033
23	57	7	1	1	0	0	0	361	0.316	0.216	0.100	0.164
24	1	5	1	0	1	0	0	361	0.526	0.249	0.277	0.325
25	40	16	0	1	0	0	0	392	0.286	0.204	0.082	0.142
26	8	0	0	2	0	0	1	392	0.571	0.245	0.327	0.360
27	220	10	1	0	0	0	0	400	0.100	0.090	0.010	0.031
28	60	10	1	1	0	0	0	400	0.300	0.210	0.090	0.152
29	8	4	2	0	1	0	0	416	0.500	0.250	0.250	0.305
30	249	11	1	0	0	0	0	441	0.095	0.086	0.009	0.029
31	177	2	4	0	0	0	0	441	0.190	0.154	0.036	0.079
32	65	13	1	1	0	0	0	441	0.286	0.204	0.082	0.142
33	73	4	0	2	0	0	0	441	0.381	0.236	0.145	0.212
34	50	20	0	1	0	0	0	450	0.267	0.196	0.071	0.129
35	50	0	0	1	1	0	0	450	0.533	0.249	0.284	0.331
36	280	12	1	0	0	0	0	484	0.091	0.083	0.008	0.027
37	72	16	1	1	0	0	0	484	0.273	0.198	0.074	0.133
38	90	10	2	1	0	0	0	490	0.286	0.204	0.082	0.142
39	3	7	3	0	1	0	0	507	0.462	0.249	0.213	0.275
40	64	24	0	1	0	0	0	512	0.250	0.188	0.063	0.118
41	16	8	0	1	1	0	0	512	0.500	0.250	0.250	0.305
42	16	0	0	0	1	1	1	512	0.750	0.188	0.563	0.483
43	9	1	1	0	1	1	0	513	0.667	0.222	0.444	0.430
44	44	2	1	1	1	0	0	528	0.500	0.250	0.250	0.305
45	313	13	1	0	0	0	0	529	0.087	0.079	0.008	0.025
46	217	6	4	0	0	0	0	529	0.174	0.144	0.030	0.070
47	81	19	1	1	0	0	0	529	0.261	0.193	0.068	0.125
48	65	12	0	2	0	0	0	529	0.348	0.227	0.121	0.187
49	1	0	0	1	1	0	2	529	0.696	0.212	0.484	0.450
50	115	7	3	1	0	0	0	539	0.286	0.204	0.082	0.142
51	15	6	4	0	1	0	0	567	0.444	0.247	0.198	0.261
52	348	14	1	0	0	0	0	576	0.083	0.076	0.007	0.024
53	92	22	1	1	0	0	0	576	0.250	0.188	0.063	0.118
54	32	0	0	0	2	0	1	576	0.667	0.222	0.444	0.430
55	0	0	0	2	0	1	1	576	0.667	0.222	0.444	0.430
56	82	28	0	1	0	0	0	578	0.235	0.180	0.055	0.108
57	2	0	0	0	0	1	6	578	0.941	0.055	0.886	0.498
58	140	4	4	1	0	0	0	588	0.286	0.204	0.082	0.142
59	88	12	1	2	0	0	0	612	0.333	0.222	0.111	0.176
60	385	15	1	0	0	0	0	625	0.080	0.074	0.006	0.022
61	265	10	4	0	0	0	0	625	0.160	0.134	0.026	0.062
62	105	25	1	1	0	0	0	625	0.240	0.182	0.058	0.111
63	65	20	0	2	0	0	0	625	0.320	0.218	0.102	0.167
64	25	5	1	0	2	0	0	625	0.560	0.246	0.314	0.351
65	33	0	0	1	1	1	0	625	0.640	0.230	0.410	0.411
66	165	1	5	1	0	0	0	637	0.286	0.204	0.082	0.142
67	120	20	2	1	0	0	0	640	0.250	0.188	0.063	0.118
68	80	0	0	2	1	0	0	640	0.500	0.250	0.250	0.305
69	0	8	0	3	0	0	1	640	0.500	0.250	0.250	0.305
70	104	32	0	1	0	0	0	648	0.222	0.173	0.049	0.099
71	16	0	2	2	0	1	0	648	0.556	0.247	0.309	0.348
72	28	2	1	3	0	0	1	656	0.500	0.250	0.250	0.305
73	155	10	4	1	0	0	0	675	0.267	0.196	0.071	0.129
74	424	16	1	0	0	0	0	676	0.077	0.071	0.006	0.021
75	120	28	1	1	0	0	0	676	0.231	0.178	0.053	0.105
76	136	0	1	3	0	0	0	676	0.385	0.237	0.148	0.215
77	4	4	4	2	0	0	1	676	0.462	0.249	0.213	0.275
78	8	0	1	0	2	0	2	676	0.692	0.213	0.479	0.447
79	136	4	3	2	0	0	0	684	0.333	0.222	0.111	0.176
80	14	4	0	2	1	0	1	686	0.571	0.245	0.327	0.360
81	148	18	3	1	0	0	0	704	0.250	0.188	0.063	0.118
82	48	0	0	0	2	1	0	720	0.667	0.222	0.444	0.430

83	130	36	0	1	0	0	0	722	0.211	0.166	0.044	0.092
84	18	0	8	1	0	0	1	722	0.421	0.244	0.177	0.243
85	18	0	0	0	0	3	2	722	0.842	0.133	0.709	0.520
86	5	15	5	0	1	0	0	725	0.400	0.240	0.160	0.227
87	6	0	0	1	1	1	2	726	0.727	0.198	0.529	0.469
88	465	17	1	0	0	0	0	729	0.074	0.069	0.005	0.020
89	321	14	4	0	0	0	0	729	0.148	0.126	0.022	0.055
90	137	31	1	1	0	0	0	729	0.222	0.173	0.049	0.099
91	73	28	0	2	0	0	0	729	0.296	0.209	0.088	0.150
92	129	5	1	3	0	0	0	729	0.370	0.233	0.137	0.204
93	65	2	4	1	1	0	0	729	0.444	0.247	0.198	0.261
94	1	1	1	1	1	1	1	729	0.667	0.222	0.444	0.430

* $p=(k+1)(k+2)/2$.

** Values of $|M|^{1/p}$ of D-optimal designs are 0.474, 0.489, 0.507 and 0.526, respectively, for $k=3$, 4, 5 and 6.

Table II. **The orthogonal designs for cubic response and diagonal elements of their information matrices.**

k	case No.	n_0	n_1	n_2	n_3	n_4	n_5	N	λ_2	$\lambda_4-\lambda_2^2$	λ_3	$\lambda_8-\frac{\lambda_4^2}{\lambda_2}$	$\lambda_7-\frac{\lambda_3^2}{\lambda_2}$	λ_6	$\lvert M\rvert^{1/p}$
3	1	1	1	1	1			64	0.556	0.198	0.309	0.040	0.110	0.171	0.175
4	1	0	1	0	1	0		128	0.556	0.198	0.309	0.040	0.110	0.171	0.168
	2	1	0	1	0	1		128	0.556	0.198	0.309	0.040	0.110	0.171	0.168
	3	1	1	1	1	1		256	0.556	0.198	0.309	0.040	0.110	0.171	0.168
5	1	0	1	0	1	0	1	512	0.556	0.198	0.309	0.040	0.110	0.171	0.165
	2	1	0	1	0	1	0	512	0.556	0.198	0.309	0.040	0.110	0.171	0.165
	3	2	0	1	1	0	2	768	0.556	0.198	0.309	0.040	0.110	0.171	0.165
	4	2	3	0	1	0	0	864	0.704	0.176	0.495	0.028	0.124	0.348	0.209
	5	6	0	2	0	0	1	864	0.704	0.176	0.495	0.028	0.124	0.348	0.209
	6	0	0	1	0	3	2	864	0.407	0.176	0.166	0.048	0.072	0.068	0.105
	7	1	0	0	2	0	6	864	0.407	0.176	0.166	0.048	0.072	0.068	0.105
	8	1	1	1	1	1	1	1024	0.556	0.198	0.309	0.040	0.110	0.171	0.165

*$p=(k+1)(k+2)(k+3)/6$.

Table III. Some orthogonal designs and corresponding nearly-orthogonal designs explained in § 4.

		n_0	n_1	n_2	n_3	n_4	n_5	n_6	α	N	λ_2	λ_3	λ_4-λ_2^2	*Cov.*	V	$\|M\|^{1/p}$
$k=3$	1	0	1	1	0				0.499	18	0.471	0.222	0.229	0.000	4.372	0.326
	1′								0.5		0.472	0.222	0.228	0.015	4.378	0.326
	2	6	1	0	1				1.525	20	0.632	0.4	0.541	0.000	1.849	0.551
	2′								1.5		0.625	0.4	0.543	−0.035	1.941	0.543
$k=4$	1	2	1	0	1	0			0.981	42	0.617	0.381	0.234	0.000	4.264	0.406
	1′								1		0.619	0.381	0.236	0.042	4.242	0.406
	2	24	1	0	1	0			2	62	0.5	0.25	0.625	0.000	1.6	0.625
	3	3	1	0	0	1			1.547	27	0.770	0.593	0.424	0.000	2.358	0.602
	3′								1.5		0.759	0.593	0.391	−0.098	2.569	0.591
	4	12	1	0	0	1			2	36	0.667	0.444	0.889	0.000	1.125	0.629
$k=5$	1	8	1	0	1	0	0		0.499	98	0.495	0.245	0.246	0.000	4.063	0.310
	1′								0.5		0.495	0.245	0.246	0.000	4.063	0.310
	2	1	1	0	0	1	0		1.023	91	0.726	0.527	0.200	0.001	5.004	0.466
	2′								1		0.725	0.527	0.199	−0.036	5.020	0.466
	3	0	1	0	0	0	1		1.527	42	0.726	0.527	0.259	0.002	3.864	0.616
	3′								1.5		0.725	0.527	0.248	−0.101	4.047	0.612
	4	8	1	0	0	0	1		2	50	0.8	0.640	0.640	0.000	1.563	0.689
$k=6$	1	22	1	0	0	1	0	0	1.093	274	0.592	0.350	0.243	0.015	4.114	0.376
	1′								1		0.591	0.350	0.243	−0.013	4.138	0.376
	2	42	1	0	0	1	0	0	2	294	0.571	0.327	0.327	0.000	3.062	0.383
	3	2	1	0	0	0	1	0	1.091	206	0.788	0.621	0.169	0.000	5.914	0.504
	3′								1		0.786	0.621	0.168	−0.097	5.962	0.504
	4	5	1	0	0	0	0	1	2	81	0.889	0.790	0.394	0.000	2.531	0.705

$*p=(k+1)(k+2)/2$.

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