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蛯原, 幸義
九州大学教養部数学教室

南部, 徳盛
九州大学教養部数学教室

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A remark on the initial-value problems for some quasi-linear parabolic equations

Yukiyoshi EBIHARA and Tokumori NANBU

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Let (x, t) be a variable in $\boldsymbol{R}^n\times[0,\infty)$ $(n\geqq 3)$.

Consider the Initial-Value Problem for the equations of the form:

(1) $u_t-\Delta u+gu+G(u, u_{x_i}, u_{x_i x_j})+F(u, u_{x_i})=f(x, t),$
$(x, t)\in\boldsymbol{R}^n\times(0, \infty),$

with the initial condition

(2) $u(x, 0)=u_0(x),\ x\in\boldsymbol{R}^n,$

where $u_t=\dfrac{\partial u(x,t)}{\partial t}$, $u_{x_i}=\dfrac{\partial u}{\partial x_i}$, $u_{x_i x_j}=\dfrac{\partial^2 u}{\partial x_i\partial x_j}$, Δ is the n-dimensional Laplacian, g is a nonnegative constant and G, F, f are the followings:

(3) $G(u, u_{x_i}, u_{x_i x_j}) = C_{10}u^{p_1}(1+C_{11}|\nabla u|^{2q_1}+C_{12}u^{r_1}e^u)\sum_{i,j}^{n} a_{ij}u_{x_i x_j},$

(4) $F(u, u_{x_i})=C_{20}u^{p_2}(1+C_{21}u^{r_2}e^u)+C_{22}|\nabla u|^{2q_2}(1+C_{23}u^{r_3}+C_{24}u^{r_4}e^u)$, where a_{ij}, C_{ij} are constants and p_i, q_i, r_i are positive integers with $p_2>1$,

(5) $f=f(x, t)$ is a given function satisfying some smoothness and smallness conditions described below.

Previously, in [1] the authors obtained a global smooth solution for the problem (1)-(2) when n: arbitrary, $g>0$ and $f\equiv 0$ and showed the exponential stability of the trivial solution $u\equiv 0$. In [1], the positivity of the operator $(g-\Delta)^k$ on $L^2(\mathbf{R}^n)$ played the essential role.

And quite recently in [2], the authors established to obtain such solutions in the cases involving n: arbitrary, $g=0$, $f\not\equiv 0$ and $F_0(u)\cdot u\geqq 0$ where $F_0(u)=C_{20}u^{p_2}(1+C_{21}u^{r_2}e^u)$ which is a part of $F(u, u_{x_i})$ in (4). The technical point is the application of the positivity of the operator $1+(-\Delta)^k$ on $L^2(\boldsymbol{R}^n)$.

However, we did not succeed to cases without such monotonicity condition on $F_0(u)$. But we can report here that in the special cases, using the general version of Sobolev Lemma by L. Nirenberg the problem (1)-(2) will admit global solutions with small data.

We use the usual notations concerning the function spaces as L^p, H^k,

$L^p(0, T; H)$, $\mathscr{E}^p_{[0,T]}(H)$, etc.

LEMMA 1. (L. Nirenberg [6]) *Suppose* $u \in H^1$ *with* $n \geqq 3$. *Then* $u \in L^r$ *with* $r = \frac{2n}{n-2}$ *and*

(6) $$[\int_{R^n} |u|^r dx]^{\frac{1}{r}} \leqq \frac{1}{\sqrt{n}} \cdot \frac{2(n-1)}{n-2} [\int_{R^n} |\nabla u|^2 dx]^{\frac{1}{2}}$$

where ∇u *is a gradient of* u.

Now, as in the previous manner (cf. [1], [2]) we use:

$(.,..)_l \equiv \langle (-\Delta)^l .,..\rangle$, $|\cdot|_l^2 \equiv (.,.)_l$,

$((.,..))_l \equiv \langle (1+(-\Delta)^l).,..\rangle$, $\|\cdot\|_l^2 \equiv ((.,.))_l$

for each positive integer l.

LEMMA 2. *Suppose* $u \in H^l$ $(l \geqq [\frac{n}{2}]+1, n \geqq 3)$. *If* $p \geqq 1 + \frac{4}{n}$, *then we have*

(7) $$\int_{R^n} |u|^{p+1} dx \leqq Const. |u|_1^2 \|u\|_l^{p-1}$$

where the constant depends only on n, l, p.

PROOF. We have by Hölder Inequality that

$$\int_{R^n} |u|^{p+1} dx = \int_{R^n} |u|^2 \cdot |u|^{p-1} dx$$

$$\leqq [\int_{R^n} |u|^r dx]^{\frac{2}{r}} [\int_{R^n} |u|^{(p-1)\frac{r}{r-2}} dx]^{1-\frac{2}{r}}, \quad \text{where } r = \frac{2n}{n-2}.$$

Here we can see that

$$(p-1)\frac{r}{r-2} \geqq \frac{4}{n} \cdot \frac{\frac{2n}{n-2}}{\frac{4}{n-2}} = 2,$$

and therefore we can put,

$$(p-1)\frac{r}{r-2} = 2 + \alpha \quad (\alpha \geqq 0).$$

Thus we get,

$$\int_{R^n} |u|^{p+1} dx \leqq [\int_{R^n} |u|^r dx]^{\frac{2}{r}} [\int_{R^n} |u|^2 |u|^\alpha dx]^{1-\frac{2}{r}}.$$

With the use of Lemma 1 and the standard Sobolev Lemma we should have,

$$\int_{R^n}|u|^{p+1}dx \leqq C|u|_1^2|u|_\infty^{\alpha(1-\frac{2}{r})}[\int_{R^n}|u|^2dx]^{1-\frac{2}{r}}$$

and further we have,

$$\int_{R^n}|u|^{p+1}dx \leqq C|u|_1^2\|u\|_l^{\alpha(1-\frac{2}{r})}\|u\|_l^{2(1-\frac{2}{r})}$$

$$=C|u|_1^2\|u\|_l^{p-1}.$$

This completes the proof. (q. e. d.)

LEMMA 3. *Suppose that the number p_2 appeared in $F_0(u)$ in* (4) *satisfies*

$$p_2 \geqq 1+\frac{4}{n},$$

then there exists an increasing continuous function $\phi_k(s)$ $(s \geqq 0)$ with $\phi_k(0)=0$ such that for $u \in H^k$ $(k \geqq [\frac{n}{2}]+3)$,

$$(8) \qquad |(G(u, u_{x_i}, u_{x_ix_j}), u)_0+(F(u, u_{x_i}), u)_0| \leqq |u|_1^2\phi_k(\|u\|_k).$$

PROOF. For the first term, we have,

$$(G(u, u_{x_i}, u_{x_ix_j}), u)_0$$

$$=C_{10}\int_{R^n}u^{p_1+1}(1+C_{11}|\nabla|^{2q_1}+C_{12}u^{r_1}e^u)\sum_{i,j}^n a_{ij}u_{x_ix_j}dx$$

$$=C_{10}\sum_{i,j}^n -a_{ij}\int_{R^n}u_{x_i}\cdot[u^{p_1+1}(1+C_{11}|\nabla u|^{2q_1}+C_{12}u^{r_1}e^u)]_{x_j}dx,$$

and hence it suffices to know that

$$|\int_{R^n}u_{x_i}u^p\cdot u_{x_j}dx| \leqq Const.\ |u|_1^2\|u\|_k^p,$$

$$|\int_{R^n}u_{x_i}(u_{x_j})^q u_{x_jx_i}dx| \leqq Const.\ |u|_1^2\|u\|_k^q,$$

$$|\int_{R^n}u_{x_i}(u^pe^u)u_{x_j}dx| \leqq Const.\ |u|_1^2\|u\|_k^p e^{C\|u\|_k}.$$

These are true because that $|u|_\infty$, $|u_{x_i}|_\infty$, $|u_{x_jx_i}|_\infty$ are bounded by $\|u\|_k$. For the second term, we also have,

$$|\int_{R^n}F(u, u_{x_i})udx|$$

$$\leqq|\int_{R^n}C_{20}u^{p_2+1}(1+C_{21}u^{r_2}e^u)dx|+|\int_{R^n}C_{22}|\nabla u|^{2q_2}u(1+C_{23}u^{r_3}+C_{24}u^{r_4}e^u)dx|$$

$$\leqq |C_{20}|\,|1+C_{21}u^{r_2}e^u|_\infty \int_{R^n} |u|^{p_2+1}dx$$

$$+|C_{22}|\,|u(1+C_{23}u^{r_3}+C_{24}u^{r_4}e^u)|_\infty|\,|\nabla u|^{2q_2-2}|_\infty \int_{R^n} |\nabla u|^2 dx.$$

Thus applying Lemma 2 we should have

$$\left|\int_{R^n} F(u, u_{x_i})u dx\right|$$

$$\leqq Const.\ |u|_1^2\{\|u\|_k^{p_2-1}(1+\|u\|_k^{r_2}e^{C\|u\|_k})+\|u\|_k^{2q_2-1}(1+\|u\|_k^{r_3}$$

$$+\|u\|_k^{r_4}e^{C\|u\|_k})\}.$$

Thus, adding these estimates we will get a function $\phi_k(s)$ which satisfies the properties of this lemma. (q. e. d.)

Then we have our statement in this note.

THEOREM. *Suppose that k is larger than k_0+3 ($k_0=[\frac{n}{2}]+2$), $u_0 \in H^{k+4}$, and*

$$f(x,t) \in \bigcap_{i=0}^{2} \mathscr{E}^i_{[0,\infty]}(H^{k+3-i}) \cap L^1(0,\infty; H^{k+3}).$$

Then there exists a positive constant δ_0 such that if,

$$\int_0^\infty \|f(t)\|_{k+3}dt+\|u_0\|_{k+3}<\delta\leqq\delta_0,$$

the problem (1)-(2) *will have one and only one solution in the class*

$$\mathscr{E}^0_{[0,\infty]}(H^k) \cap \mathscr{E}^1_{[0,\infty]}(H^{k-2}).$$

SKETCH OF THE PROOF.

The details will follow from the arguments in § 2 in [2].

A priori, we have two equalities,

$$\frac{1}{2}(|u|_0^2)'+|u|_1^2+(G+F,u)_0=(f,u)_0,$$

$$\frac{1}{2}(|u|_k^2)'+|u|_{k+1}^2+(G+F,u)_k=(f,u)_k,$$

by multiplying u' to both sides of the equation (1). And adding these equalities we get,

$$\frac{1}{2}(\|u\|_k^2)'+(|u|_1^2+|u|_{k+1}^2)$$

$$\leqq \|f\|_k\|u\|_k+|(G+F,u)_0|+|(G+F,u)_k|.$$

Therefore, using Lemma 2. 3 in [2] and Lemma 3 in this note, we will have an increasing continuous function $\bar{\phi}_k(s)$ ($s\geqq 0$) with $\bar{\phi}_k(0)=0$ such that

$$\frac{1}{2}(\|u\|_k^2)' \leqq \{-1+\bar{\phi}_k(\|u\|_k)\}(|u|_1^2+|u|_{k+1}^2)+\|f\|_k\|u\|_k.$$

Thus, by integrating, it follows that

$$\|u(t)\|_k^2 \leqq \|u_0\|_k^2 + 2\int_0^t (|u|_1^2+|u|_{k+1}^2)\{-1+\bar{\phi}_k(\|u\|_k)\}dt + 2\int_0^t \|f\|_k\|u\|_k dt.$$

From this inequality, the conclusion will follow. (q. e. d.)

REMARK 1. Though we treat the equations in the cases $n \geqq 3$, $p_2 \geqq 1+\frac{4}{n}$ in this note, one can see that there exist bounded smooth solutions in the semi-linear cases when $n=3$, $p_2=2$. (This is well known by H. Fujita [**3**].)

REMARK 2. When $p_2=1$ and $C_{20}<0$, we get a global solution without such restrictions to n and nonlinearities. (See [**1**], [**2**].)

REMARK 3. For equations (1) we can not always obtain a global solution when the norm of the initial value is growing. (cf. H. Fujita [**3**], S. Kaplan [**4**], H. Levine [**5**], etc.) In particular, S. Portnoy [**7**] showed the instability of $u \equiv 0$ to the equation

$$u_t = \Delta u + u^2 \quad (n=2).$$

This means that we can not obtain the global solution for any non-zero initial values. However, in the case of initial-boundary value problem in the bounded domain for these equations we can show the stability of $u \equiv 0$ for any n. (See [**1**].)

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