Note on the Morrey-Sobolev type imbedding theorems in the strong L^(p, λ) spaces

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Note on the Morrey-Sobolev type imbedding theorems in the strong $\mathscr{L}^{(p,\lambda)}$ spaces

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Introduction.

The Morrey-Sobolev type imbedding theorems in the $\mathscr{L}^{(p,\lambda)}$ spaces have played one of the most important roles in the study of partial differential equations.[1] These theorems were first studied by C. B. Morrey [8] and afterwards by various authors including S. Campanato [1], [2], [3], [4], [5] and G. Stampacchia [14], [15] who established the theorems.

While, Stampacchia introduced the $\mathscr{L}^{(p,\lambda)}$ spaces of strong type which are more general than the $\mathscr{L}^{(p,\lambda)}$ spaces and the imbedding theorems in these spaces were proved by him [14], L. C. Piccinini [13], the author [10], [11], [12] and others. In [12] we have proved imbedding theorems analogous to Stampacchia's theorem for the $\mathscr{L}^{(p,\lambda)}$ spaces assuming that the strong $\mathscr{L}^{(p,\lambda)}$ spaces are imbedded in the Lipschitz spaces or their limiting spaces. In this paper, we shall prove a "genuine" Morrey-Sobolev type imbedding theorem concerned with the strong Hölder spaces under suitable conditions. That is, the theorem is quite analogous to the Stampacchia's theorem and improves partially the theorem [12]. The main tools for the proof are theorems due to S. M. Nikol'skii [9] and the author [10], [12] (with Y. Furushō).

§ 1. Preliminaries

In this section relevant definitions which may be found in [10] and [14], theorems due to Stampacchia, the author, Ono-Furushō and main theorem in this article are stated.

We shall always consider real-valued integrable function $u(x)=u(x_1, \cdots$

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1) For the theory of $\mathscr{L}^{(p,\lambda)}$ spaces see [13], [15] as bibliography.

$\cdots, x_n)$ defined on the n dimensional Euclidean space E^n with compact supports. We consider an arbitrary parallel subcube Q of a fixed bounded cube Q_0. (From now on by a subcube Q of Q_0 we mean parallel subcube of Q_0.) We denote the measure of Q by $|Q|$ and mean value of a function u over Q by u_Q: $u_Q=|Q|^{-1}\int_Q u(x)\,dx$.

DEFINITION 1. A function u is said to belong to the space $\mathscr{L}_r^{(p,\lambda)}=\mathscr{L}_r^{(p,\lambda)}(Q_0)$ (the $\mathscr{L}^{(p,\lambda)}$ space of strong type r), where $1\leqq p<\infty$, $-\infty<\lambda<\infty$, $1\leqq r<\infty$, if for any system of subcubes $S=\{Q_j: \cup Q_j\subset Q_0\}$, no two of which have common interior points, the relation

$$\sup_{Q\subset Q_j}\{|Q|^{\frac{\lambda}{n}-1}\int_Q|u(x)-u_Q|^p dx\}^{\frac{1}{p}}=[u]_{\mathscr{L}^{(p,\lambda)}}(Q_j)=K(Q_j)<\infty \tag{1.1}$$

holds and, furthermore, there exists a constant $L=L(u)$ such that

$$\sup_{\{Q_j\}=S\in\bar{S}}[\sum_j|K(Q_j)|^r]^{\frac{1}{r}}=L \tag{1.2}$$

where $\bar{S}$ denotes the family of all systems of subcubes considered above. We denote L by $[u]_{\mathscr{L}_r^{(p,\lambda)}}(Q^0)$ and define a norm of the space

$\mathscr{L}_r^{(p,\lambda)}(Q_0)$ by $[u]_{\mathscr{L}_r^{(p,\lambda)}}(Q^0)+\|u\|_{L^p(Q_0)}$.

This norm renders the space $\mathscr{L}_r^{(p,\lambda)}(Q_0)$ a Banach space.

DEFINTION 2. A function u is said to be Hölder continuous of strong type $1\leqq r<\infty$ with exponent $0<\alpha<1$ on Q_0, if the following two conditions are satisfied:

(i) u is Hölder continuous with exponent α in Q_0:

(ii) there exists a constant $L=L(u)$ such that, for any system of subcubes Q_j belonging to $\bar{S}$ as in Definition 1 one has

$$\sup_{\{Q_j\}=S\in\bar{S}}[\sum_j|K(Q_j)|^r]^{\frac{1}{r}}=L \tag{1.3}$$

where $K(Q_j)$ denotes the Hölder coefficient with exponent α of $u|_{Q_j}$, the restriction u to the subcube Q_j, We denote L by $[u]_{\mathscr{H}_r^\alpha}(Q_0)$.

DEFINITION 3. A function u is said to belong to the space Lip(a,p) on E^n, where $0<a<\infty$ and $1\leqq p\leqq\infty$, that is u is said to satisfy a Lipschitz condition of order a in $L^p=L^p(E^n)$, if there exists a constant $K=K(u)$ such that

(1.4) $$\sup_{h\in E^n} |h|^{-a+\bar{a}}\Big[\int_{E^n}|D^{\bar{a}}u(x+h)-D^{\bar{a}}u(x)|^p dx\Big]^{\frac{1}{p}}=K$$

where $\bar{a}$ is the greatest integer less than a. We denote K by $[u]_{\mathrm{Lip}(a,p)}$ and define the norm $\|u\|_{\mathrm{Lip}(a,p)}$ by $[u]_{\mathrm{Lip}(a,p)}+\|u\|_{L^p(E^n)}$, endowed with which the space $\mathrm{Lip}(a,p)$ is a Banach space.

Now, as was stated in the introduction, the isomorphism and imbedding theorems of Morrey-Sobolev type due to Stampacchia [**14**] and the author [**10**], [**11**], [**12**] (with Furushō) are the following:

THEOREM A.

1. [**14**] *The Sobolev space* $\mathrm{H}^{1,\frac{n}{1-\alpha}}$ *is isomorphic to the space* $\mathscr{H}^{\alpha}_{\frac{n}{1-\alpha}}$ *with their corresponding norms, where* $0<\alpha<1$.

2. [**12**] *The Sobolev space* $\mathrm{H}^{1,p}$ *is isomorphic to the space* $\mathscr{L}_r^{(p,p(\frac{n}{r}-1))}$ *with their corresponding norms, where* $\frac{1}{r}<\mathrm{Min}(1,\frac{1}{p}+\frac{1}{n})$.

3. [**10**] *The Lipschitz space* $\mathrm{Lip}(a,\frac{n}{a-\alpha})$ *is isomorphic to the space* $\mathscr{H}^{\alpha}_{\frac{n}{a-\alpha}}$ *with their corresponding norms, where* $0<\alpha<1$ *and* $\frac{n+\alpha}{n+1}\leqq a \leqq 1$.

REMARK 1.1, 2, 3 of this theorem are generalization of 1.

THEOREM B.

1. [**14**] *Let u be a function such that u_x belongs to the space* $\mathscr{L}^{(p,\lambda)}(Q_0)$, *where* $1\leqq p<\infty$ *and* $0\leqq\lambda\leqq n$. *Then the following estimates hold for* u.

(i) *If $p<\lambda$ then the function u belongs to the space* $\mathscr{M}^{(\bar{p},\lambda)}$ *(for the definition cf.* [**14**]*) and*

(1.5) $$[u]_{\mathscr{M}^{(\bar{p},\lambda)}}\leqq C\|u_x\|_{\mathscr{L}^{(p,\lambda)}}$$

where $\frac{1}{\bar{p}}=\frac{1}{p}-\frac{1}{\lambda}$ *and C is a constant independent of u.*[2)]

(ii) *If $p=\lambda$ then u belongs to the space* $\mathscr{L}^{(1,0)}=\mathscr{E}_0$

2) Throughout the remainder of this paper C denotes always a constant independent of function u.

(John-Nirenberg space [**6**]) *and*

(1.6) $[u]_{\mathscr{L}^{(1,0)}}\leqq C\|u_x\|_{\mathscr{L}^{(p,\lambda)}}$

(iii) *If* $p>\lambda$ *then u belongs to the space* $\mathscr{L}^{(1,\frac{\lambda}{p}-1)}=C^{0,1-\frac{\lambda}{p}}$ *and*

(1.7) $[u]_{C^{0,1-\frac{\lambda}{p}}}\leqq C\|u_x\|_{\mathscr{L}^{(p,\lambda)}}$

2. [11], [12] *Let u be a function such that* u_x *belongs to the space* $\mathscr{L}_r^{(p,\lambda)}(Q_0)$, *where* $1\leqq p<\infty$, $0<\lambda<n$ and $1\leqq r<\frac{n}{\lambda}p$.
Then the following estimates hold for u.

(1) *If* $\frac{\lambda}{p}<\frac{n}{r}<\frac{\lambda}{p}+1$ *and*

(i) $p\leqq\lambda$, *then the function u belongs to the space* $\mathscr{L}_{r_1}^{(\hat{p},\hat{\lambda})}$ *and*

(1.8) $[u]_{\mathscr{L}_{r_1}^{(\hat{p},\hat{\lambda})}}\leqq C\|u_x\|_{\mathscr{L}_r^{(p,\lambda)}}$

where $\hat{p},\hat{\lambda},r_1$ *are arbitrary constants satisfying* $\frac{n}{\lambda}p<\hat{p}<\frac{n}{\lambda}\tilde{p}$, $\frac{\hat{\lambda}}{\hat{p}}>\frac{\lambda}{\tilde{p}}$

and $r_1\geqq\frac{n}{\lambda}p$ *respectively.*

(ii) $p>\lambda$, *then u belongs to the space* $\mathscr{L}_{r_1}^{(1,\frac{\lambda_1}{p}-1)}=\mathscr{H}_{r_1}^{1-\frac{\lambda_1}{p}}$ *and*

(1.9) $[u]_{\mathscr{H}_{r_1}^{1-\frac{\lambda_1}{p}}}\leqq C\|u_x\|_{\mathscr{L}_r^{(p,\lambda)}}$

where λ_1, r_1 *are arbitrary constants satisfying* $\lambda<\lambda_1<p$ *and* $r_1\geqq\frac{n}{\lambda_1}p$ *respectively.*

(2) *If* $\frac{n}{r}-\frac{\lambda}{p}=1$ *and*

(i) $p<\lambda$, *then u belongs to the space* $\mathscr{L}_{p_\lambda}^{(p_\lambda,n-p_\lambda)}$ $(p_\lambda=\frac{n}{\lambda}p)$ *and*

(1.10) $[u]_{\mathscr{L}_{p_\lambda}^{(p_\lambda,n-p_\lambda)}}\leqq C\|u_x\|_{\mathscr{L}_r^{(p,\lambda)}}$

(ii) $p\geqq\lambda$, *then u belongs to the space* $\mathscr{L}_{p_\lambda}^{(1,\frac{\lambda}{p}-1)}$
(Hölder or John-Nirenberg space of strong type p_λ*) and*

(1.11) $[u]_{\mathscr{L}_{p_\lambda}^{(1,\frac{\lambda}{p}-1)}}\leqq C\|u_x\|_{\mathscr{L}_r^{(p,\lambda)}}$

(3) *If* $\frac{n}{r}-\frac{\lambda}{p}>1$, *then* u_x *is a constant and therefore u is a linear*

function.

Now, our main result which is an analogue to Theorem B. 1(iii), 2. (2). (ii) and an improvement of 2. (1). (ii) reads as follows:

THEOREM. *Let u be a function such that u_x belongs to the space $\mathscr{L}_r^{(p,\lambda)}(Q_0)$, where $1\leqq p<\infty$, $0<\lambda<n$, $1\leqq r<\infty$, $\frac{\lambda}{p}<\frac{n}{r_1}<\frac{\lambda}{p}+1$ and furthermore p is greater than λ. Then u belongs to the space $\mathscr{L}_{r_1}^{(1,\frac{\lambda}{p}-1)}=\mathscr{H}_{r_1}^{1-\frac{\lambda}{p}}$ and*

$$(1.12)\qquad [u]_{\mathscr{H}_{r_1}^{1-\frac{\lambda}{p}}}\leqq C\|u_x\|_{\mathscr{L}_r^{(p,\lambda)}}$$

where r_1 is an arbitrary constant greater than $\frac{n}{\lambda}p$.

§ 2. Proof of the theorem

In this section the proof of the theorem and additional remark are given. Before proceeding to give the proof, we need some lemmas.

LEMMA 1. *This is same as* Theorem A. 3. *taking* $1-\frac{\lambda}{p}$ *in place of* a.

LEMMA 2. [9] *For any positive a and p, q such that $1\leqq p<q\leqq\infty$ and $a-(\frac{1}{p}-\frac{1}{q})n$ is a positive non-integer, we have*

$$(2.1)\qquad \mathrm{Lip}(a,p)\subset\mathrm{Lip}(a-(\frac{1}{p}-\frac{1}{q})n,q)$$

with their corresponding norms.

LEMMA 3. [12] *After a suitable extention to E^n of functions belonging to $\mathscr{L}_r^{(p,\lambda)}$ as functions with supports within a fixed concentric and parallel cube containing Q_0, we have*

$$(2.2)\qquad \mathscr{L}_r^{(p,\lambda)}\subset\mathrm{Lip}(\frac{n}{r}-\frac{\lambda}{p},r)$$

with their corresponding norms, where p, λ, r are constants satisfying $1\leqq p<\infty$, $-p<\lambda\leqq n$, $1\leqq r<\infty$ and $0<\frac{n}{r}-\frac{\lambda}{p}\leqq 1$.

REMARK 2.1. This lemma is deduced by combining Hölder's inequality and the following:

THEOREM C. [1], [7] $(-p\leqq\lambda<0)$ [6] $(\lambda=0)$ [12] $(0<\lambda<n)$

The space $\mathscr{L}_r^{(p,\lambda)}$ *is isomorphic to the space* $\mathscr{L}_r^{(1,\frac{\lambda}{p})}$, *where* $1<p<\infty$, $-p\leqq\lambda<n$, $1\leqq r\leqq\infty$ *and* $\frac{\lambda}{p}\leqq\frac{n}{r}$.

Now, we are going to give the

PROOF OF THE THEOREM.

As the space Lip $\left(a, \frac{n}{a-1+\frac{\lambda}{p}}\right)$ is isomorphic to the space $\mathscr{H}^{1-\frac{\lambda}{p}}_{\frac{n}{a-1+\frac{\lambda}{p}}}$ by Lemma 1, if the constant a is less than and sufficiently close to unity, we have

$$[u]_{\mathscr{H}^{\alpha}_{r_1}}\leqq C[u]_{\mathrm{Lip}(a,r_1)}$$

where $\alpha=1-\frac{\lambda}{p}$ and $r_1=\frac{n}{a-1+\frac{\lambda}{p}}$. Next, applying Lemma 2 this is

$$\leqq C\|u\|_{\mathrm{Lip}(1+\frac{n}{r}-\frac{\lambda}{p},r)}$$

that is,

$$=C\|u_x\|_{\mathrm{Lip}(\frac{n}{r}-\frac{\lambda}{p},r)}$$

and by use of Lemma 3 we obtain

$$\leqq C\|u_x\|_{\mathscr{L}_r^{(p,\lambda)}}$$

Finally, by taking the constant a almost equal to unity we may suppose that r_1 is an arbitrary constant greater than $\frac{n}{\lambda}p$.

Hence, the proof of theorem is complete.

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