

## Note on the Morrey–Sobolev type imbedding theorems in the strong $L^{\lambda}(p, \lambda)$ spaces

Ono, Akira  
Department of Mathematics, College of General Education, Kyushu University

<https://doi.org/10.15017/1448997>

---

出版情報：九州大学教養部数学雑誌. 11 (1), pp.31-37, 1977-10. 九州大学教養部数学教室  
バージョン：  
権利関係：

## Note on the Morrey-Sobolev type imbedding theorems in the strong $\mathcal{L}^{(p,\lambda)}$ spaces

Akira ONO

(Received May 16, 1977)

### Introduction.

The Morrey-Sobolev type imbedding theorems in the  $\mathcal{L}^{(p,\lambda)}$  spaces have played one of the most important roles in the study of partial differential equations.<sup>1)</sup> These theorems were first studied by C.B. Morrey [8] and afterwards by various authors including S. Campanato [1], [2], [3], [4], [5] and G. Stampacchia [14], [15] who established the theorems.

While, Stampacchia introduced the  $\mathcal{L}^{(p,\lambda)}$  spaces of strong type which are more general than the  $\mathcal{L}^{(p,\lambda)}$  spaces and the imbedding theorems in these spaces were proved by him [14], L.C. Piccinini [13], the author [10], [11], [12] and others. In [12] we have proved imbedding theorems analogous to Stampacchia's theorem for the  $\mathcal{L}^{(p,\lambda)}$  spaces assuming that the strong  $\mathcal{L}^{(p,\lambda)}$  spaces are imbedded in the Lipschitz spaces or their limiting spaces. In this paper, we shall prove a "genuine" Morrey-Sobolev type imbedding theorem concerned with the strong Hölder spaces under suitable conditions. That is, the theorem is quite analogous to the Stampacchia's theorem and improves partially the theorem [12]. The main tools for the proof are theorems due to S.M. Nikol'skii [9] and the author [10], [12] (with Y. Furushō).

### §1. Preliminaries

In this section relevant definitions which may be found in [10] and [14], theorems due to Stampacchia, the author, Ono-Furushō and main theorem in this article are stated.

We shall always consider real-valued integrable function  $u(x) = u(x_1, \dots$

---

1) For the theory of  $\mathcal{L}^{(p,\lambda)}$  spaces see [13], [15] as bibliography.

$\dots, x_n$ ) defined on the  $n$  dimensional Euclidean space  $E^n$  with compact supports. We consider an arbitrary parallel subcube  $Q$  of a fixed bounded cube  $Q_0$ . (From now on by a subcube  $Q$  of  $Q_0$  we mean parallel subcube of  $Q_0$ .) We denote the measure of  $Q$  by  $|Q|$  and mean value of a function  $u$  over  $Q$  by  $u_Q$ :  $u_Q = |Q|^{-1} \int_Q u(x) dx$ .

DEFINITION 1. A function  $u$  is said to belong to the space  $\mathcal{L}_r^{(p,\lambda)} = \mathcal{L}_r^{(p,\lambda)}(Q_0)$  (the  $\mathcal{L}_r^{(p,\lambda)}$  space of strong type  $r$ ), where  $1 \leq p < \infty$ ,  $-\infty < \lambda < \infty$ ,  $1 \leq r < \infty$ , if for any system of subcubes  $S = \{Q_j: \cup Q_j \subset Q_0\}$ , no two of which have common interior points, the relation

$$(1.1) \quad \sup_{Q \in Q_j} \{ |Q|^{\frac{1}{p}-1} \int_Q |u(x) - u_Q|^p dx \}^{\frac{1}{p}} = [u]_{\mathcal{L}_r^{(p,\lambda)}(Q_j)} = K(Q_j) < \infty$$

holds and, furthermore, there exists a constant  $L = L(u)$  such that

$$(1.2) \quad \sup_{\{Q_j\} \in \bar{S}} [\sum_j |K(Q_j)|^r]^{\frac{1}{r}} = L$$

where  $\bar{S}$  denotes the family of all systems of subcubes considered above.

We denote  $L$  by  $[u]_{\mathcal{L}_r^{(p,\lambda)}(Q_0)}$  and define a norm of the space

$$\mathcal{L}_r^{(p,\lambda)}(Q_0) \text{ by } [u]_{\mathcal{L}_r^{(p,\lambda)}(Q_0)} + \|u\|_{L^p(Q_0)}.$$

This norm renders the space  $\mathcal{L}_r^{(p,\lambda)}(Q_0)$  a Banach space.

DEFINITION 2. A function  $u$  is said to be Hölder continuous of strong type  $1 \leq r < \infty$  with exponent  $0 < \alpha < 1$  on  $Q_0$ , if the following two conditions are satisfied:

(i)  $u$  is Hölder continuous with exponent  $\alpha$  in  $Q_0$ :

(ii) there exists a constant  $L = L(u)$  such that, for any system of subcubes  $Q_j$  belonging to  $\bar{S}$  as in Definition 1 one has

$$(1.3) \quad \sup_{\{Q_j\} \in \bar{S}} [\sum_j |K(Q_j)|^r]^{\frac{1}{r}} = L$$

where  $K(Q_j)$  denotes the Hölder coefficient with exponent  $\alpha$  of  $u|_{Q_j}$ , the restriction  $u$  to the subcube  $Q_j$ . We denote  $L$  by  $[u]_{\mathcal{H}_r^\alpha(Q_0)}$ .

DEFINITION 3. A function  $u$  is said to belong to the space  $\text{Lip}(a, p)$  on  $E^n$ , where  $0 < a < \infty$  and  $1 \leq p \leq \infty$ , that is  $u$  is said to satisfy a Lipschitz condition of order  $a$  in  $L^p = L^p(E^n)$ , if there exists a constant  $K = K(u)$  such that

$$(1.4) \quad \sup_{h \in E^n} |h|^{-a+\bar{a}} \left[ \int_{E^n} |D^{\bar{a}}u(x+h) - D^{\bar{a}}u(x)|^p dx \right]^{\frac{1}{p}} = K$$

where  $\bar{a}$  is the greatest integer less than  $a$ . We denote  $K$  by  $[u]_{\text{Lip}(a,p)}$  and define the norm  $\|u\|_{\text{Lip}(a,p)}$  by  $[u]_{\text{Lip}(a,p)} + \|u\|_{L^p(E^n)}$ , endowed with which the space  $\text{Lip}(a,p)$  is a Banach space.

Now, as was stated in the introduction, the isomorphism and imbedding theorems of Morrey-Sobolev type due to Stampacchia [14] and the author [10], [11], [12] (with Furushō) are the following:

**THEOREM A.**

1. [14] The Sobolev space  $H^{1, \frac{n}{1-\alpha}}$  is isomorphic to the space  $\mathcal{H}^{\frac{n}{1-\alpha}}$  with their corresponding norms, where  $0 < \alpha < 1$ .

2. [12] The Sobolev space  $H^{1,p}$  is isomorphic to the space  $\mathcal{L}^{(p, \lambda)}(\frac{n}{r}, p(\frac{n}{r}-1))$  with their corresponding norms, where  $\frac{1}{r} < \min(1, \frac{1}{p} + \frac{1}{n})$ .

3. [10] The Lipschitz space  $\text{Lip}(a, \frac{n}{a-\alpha})$  is isomorphic to the space  $\mathcal{H}^{\frac{n}{a-\alpha}}$  with their corresponding norms, where  $0 < \alpha < 1$  and  $\frac{n+\alpha}{n+1} \leq a \leq 1$ .

REMARK 1.1, 2, 3 of this theorem are generalization of 1.

**THEOREM B.**

1. [14] Let  $u$  be a function such that  $u_x$  belongs to the space  $\mathcal{L}^{(p,\lambda)}(Q_0)$ , where  $1 \leq p < \infty$  and  $0 \leq \lambda \leq n$ . Then the following estimates hold for  $u$ .

(i) If  $p < \lambda$  then the function  $u$  belongs to the space  $\mathcal{M}^{(\tilde{p}, \lambda)}$  (for the definition cf. [14]) and

$$(1.5) \quad [u]_{\mathcal{M}^{(\tilde{p}, \lambda)}} \leq C \|u_x\|_{\mathcal{L}^{(p,\lambda)}}$$

where  $\frac{1}{\tilde{p}} = \frac{1}{p} - \frac{1}{\lambda}$  and  $C$  is a constant independent of  $u$ .<sup>2)</sup>

(ii) If  $p = \lambda$  then  $u$  belongs to the space  $\mathcal{L}^{(1,0)} = \mathcal{E}_0$ .

---

2) Throughout the remainder of this paper  $C$  denotes always a constant independent of function  $u$ .

(John-Nirenberg space [6]) and

$$(1.6) \quad [u]_{\mathcal{L}^{(1,0)}} \leq C \|u_x\|_{\mathcal{L}^{(p,\lambda)}}$$

(iii) If  $p > \lambda$  then  $u$  belongs to the space  $\mathcal{L}^{(1, \frac{\lambda}{p}-1)} = C^{0,1-\frac{\lambda}{p}}$  and

$$(1.7) \quad [u]_{C^{0,1-\frac{\lambda}{p}}} \leq C \|u_x\|_{\mathcal{L}^{(p,\lambda)}}$$

2. [11], [12] Let  $u$  be a function such that  $u_x$  belongs to the space  $\mathcal{L}_r^{(p,\lambda)}(Q_0)$ , where  $1 \leq p < \infty$ ,  $0 < \lambda < n$  and  $1 \leq r < \frac{n}{\lambda} p$ .

Then the following estimates hold for  $u$ .

$$(1) \quad \text{If } \frac{\lambda}{p} < \frac{n}{r} < \frac{\lambda}{p} + 1 \quad \text{and}$$

(i)  $p \leq \lambda$ , then the function  $u$  belongs to the space  $\mathcal{L}_{r_1}^{(\hat{p}, \hat{\lambda})}$  and

$$(1.8) \quad [u]_{\mathcal{L}_{r_1}^{(\hat{p}, \hat{\lambda})}} \leq C \|u_x\|_{\mathcal{L}_r^{(p,\lambda)}}$$

where  $\hat{p}, \hat{\lambda}, r_1$  are arbitrary constants satisfying  $\frac{n}{\lambda} p < \hat{p} < \frac{n}{\lambda} \tilde{p}, \frac{\hat{\lambda}}{\hat{p}} > \frac{\lambda}{\tilde{p}}$

and  $r_1 \geq \frac{n}{\lambda} p$  respectively.

$$(ii) \quad p > \lambda, \text{ then } u \text{ belongs to the space } \mathcal{L}_{r_1}^{(1, \frac{\lambda_1}{p}-1)} = \mathcal{H}_{r_1}^{1-\frac{\lambda_1}{p}} \text{ and}$$

$$(1.9) \quad [u]_{\mathcal{H}_{r_1}^{1-\frac{\lambda_1}{p}}} \leq C \|u_x\|_{\mathcal{L}_r^{(p,\lambda)}}$$

where  $\lambda_1, r_1$  are arbitrary constants satisfying  $\lambda < \lambda_1 < p$  and  $r_1 \geq \frac{n}{\lambda_1} p$  respectively.

$$(2) \quad \text{If } \frac{n}{r} - \frac{\lambda}{p} = 1 \quad \text{and}$$

(i)  $p < \lambda$ , then  $u$  belongs to the space  $\mathcal{L}_{p_1}^{(p_1, n-p_1)}$  ( $p_1 = \frac{n}{\lambda} p$ ) and

$$(1.10) \quad [u]_{\mathcal{L}_{p_1}^{(p_1, n-p_1)}} \leq C \|u_x\|_{\mathcal{L}_r^{(p,\lambda)}}$$

(ii)  $p \geq \lambda$ , then  $u$  belongs to the space  $\mathcal{L}_{p_1}^{(1, \frac{\lambda}{p}-1)}$

(Hölder or John-Nirenberg space of strong type  $p_1$ ) and

$$(1.11) \quad [u]_{\mathcal{L}_{p_1}^{(1, \frac{\lambda}{p}-1)}} \leq C \|u_x\|_{\mathcal{L}_r^{(p,\lambda)}}$$

(3) If  $\frac{n}{r} - \frac{\lambda}{p} > 1$ , then  $u_x$  is a constant and therefore  $u$  is a linear

function.

Now, our main result which is an analogue to Theorem B.1(iii), 2. (2). (ii) and an improvement of 2. (1). (ii) reads as follows:

**THEOREM.** *Let  $u$  be a function such that  $u_x$  belongs to the space  $\mathcal{L}_r^{(p,\lambda)}(Q_0)$ , where  $1 \leq p < \infty$ ,  $0 < \lambda < n$ ,  $1 \leq r < \infty$ ,  $\frac{\lambda}{p} < \frac{n}{r} < \frac{\lambda}{p} + 1$  and furthermore  $p$  is greater than  $\lambda$ . Then  $u$  belongs to the space  $\mathcal{L}_{r_1}^{(1, \frac{\lambda}{p}-1)}$   $= \mathcal{H}_{r_1}^{1-\frac{\lambda}{p}}$  and*

$$(1.12) \quad [u]_{\mathcal{H}_{r_1}^{1-\frac{\lambda}{p}}} \leq C \|u_x\|_{\mathcal{L}_r^{(p,\lambda)}}$$

where  $r_1$  is an arbitrary constant greater than  $\frac{n}{\lambda}p$ .

## § 2. Proof of the theorem

In this section the proof of the theorem and additional remark are given. Before proceeding to give the proof, we need some lemmas.

**LEMMA 1.** *This is same as Theorem A. 3. taking  $1 - \frac{\lambda}{p}$  in place of  $\alpha$ .*

**LEMMA 2.** [9] *For any positive  $a$  and  $p, q$  such that  $1 \leq p < q \leq \infty$  and  $a - (\frac{1}{p} - \frac{1}{q})n$  is a positive non-integer, we have*

$$(2.1) \quad \text{Lip}(a, p) \subset \text{Lip}(a - (\frac{1}{p} - \frac{1}{q})n, q)$$

with their corresponding norms.

**LEMMA 3.** [12] *After a suitable extension to  $E^n$  of functions belonging to  $\mathcal{L}_r^{(p,\lambda)}$  as functions with supports within a fixed concentric and parallel cube containing  $Q_0$ , we have*

$$(2.2) \quad \mathcal{L}_r^{(p,\lambda)} \subset \text{Lip}(\frac{n}{r} - \frac{\lambda}{p}, r)$$

with their corresponding norms, where  $p, \lambda, r$  are constants satisfying  $1 \leq p < \infty$ ,  $-p < \lambda \leq n$ ,  $1 \leq r < \infty$  and  $0 < \frac{n}{r} - \frac{\lambda}{p} \leq 1$ .

REMARK 2.1. This lemma is deduced by combining Hölder's inequality and the following:

THEOREM C. [1], [7] ( $-p \leq \lambda < 0$ ) [6] ( $\lambda = 0$ ) [12] ( $0 < \lambda < n$ )

The space  $\mathcal{L}_r^{(p, \lambda)}$  is isomorphic to the space  $\mathcal{L}_r^{(1, \frac{\lambda}{p})}$ ,

where  $1 < p < \infty$ ,  $-p \leq \lambda < n$ ,  $1 \leq r \leq \infty$  and  $\frac{\lambda}{p} \leq \frac{n}{r}$ .

Now, we are going to give the

PROOF OF THE THEOREM.

As the space  $\text{Lip}\left(a, \frac{n}{a-1+\frac{\lambda}{p}}\right)$  is isomorphic to the space

$\mathcal{H}^{1-\frac{\lambda}{p}}_{\frac{n}{a-1+\frac{\lambda}{p}}}$  by Lemma 1, if the constant  $a$  is less than and sufficiently close

to unity, we have

$$[u]_{\mathcal{H}_{r_1}^a} \leq C[u]_{\text{Lip}(a, r_1)}$$

where  $\alpha = 1 - \frac{\lambda}{p}$  and  $r_1 = \frac{n}{a-1+\frac{\lambda}{p}}$ . Next, applying Lemma 2 this is

$$\leq C\|u\|_{\text{Lip}(1+\frac{n}{r}-\frac{\lambda}{p}, r)}$$

that is,

$$= C\|u_x\|_{\text{Lip}(\frac{n}{r}-\frac{\lambda}{p}, r)}$$

and by use of Lemma 3 we obtain

$$\leq C\|u_x\|_{\mathcal{L}_r^{(p, \lambda)}}$$

Finally, by taking the constant  $a$  almost equal to unity we may suppose

that  $r_1$  is an arbitrary constant greater than  $\frac{n}{\lambda}p$ .

Hence, the proof of theorem is complete.

## Bibliography

- [1] S. CAMPANATO: *Proprietà di Hölderianità di alcune classi di funzioni*, Ann. Scuola Norm. Sup. Pisa, **17**(1963), 175-188.
- [2] S. CAMPANATO: *Proprietà di inclusione per spazi di Morrey*, Ricerche Mat., **12**(1963), 67-86.
- [3] S. CAMPANATO: *Proprietà di una famiglia di spazi funzionali*, Ann. Scuola Norm. Sup. Pisa, **18**(1964), 137-160.
- [4] S. CAMPANATO: *Equazioni ellittiche del 11° ordine e spazi  $\mathcal{L}^{(2, \lambda)}$* , Ann. Mat. Pura Appl., **69**(1965), 321-382.
- [5] S. CAMPANATO: *Maggiorazioni interpolatorie negli spazi  $H_1^{r, p}(\Omega)$* , Ann. Mat. Pura Appl., **75**(1967), 261-276.
- [6] F. JOHN-L. NIRENBERG: *On functions of bounded mean oscillation*, Comm. Pure Appl. Math., **14**(1961), 415-426.
- [7] G. N. MEYERS: *Mean oscillation over cubes and Hölder continuity*, Proc. Amer. Math. Soc., **15**(1964), 717-721.
- [8] C. B. MORREY: *On the solution of quasi linear elliptic partial differential equations*, Trans. Amer. Math. Soc., **43**(1938), 126-166.
- [9] S. M. NIKOL'SKII: *On imbedding, continuation and approximation theorems for differentiable functions of several variables*, Russian Math. Surveys, **16**(1961), 55-104.
- [10] A. ONO: *On imbedding between strong Hölder spaces and Lipschitz spaces in  $L^p$* , Mem. Fac. Sci. Kyushu Univ., A, **20**(1970), 76-93.
- [11] A. ONO: *On imbedding theorems in strong  $\mathcal{L}^{(q, p)}$  spaces*, Rend. Ist. Mat. Univ. di Trieste, **4**(1972), 53-65.
- [12] A. ONO-Y. FURUSHO: *On isomorphism and inclusion property for certain  $\mathcal{L}^{(p, \lambda)}$  spaces of strong type*, Ann. Mat. Pura Appl., (to appear).
- [13] L. C. PICCININI: *Proprietà di inclusione e di interpolazione tra spazi di Morrey e loro generalizzazioni*, (1969), Thesis Scuola Norm. Sup. Pisa.
- [14] G. STAMPACCHIA: *The spaces  $\mathcal{L}^{(p, \lambda)}$ ,  $N^{(p, \lambda)}$  and interpolation*, Ann. Scuola Norm. Sup. Pisa, **19**(1965), 443-462.
- [15] G. STAMPACCHIA: *The  $\mathcal{L}^{(p, \lambda)}$  spaces and applications to the theory of partial differential equations*, Proc. Conf. Diff. Equ. and Appl., Bratislava, (1969).