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§ 1 Introduction

As one of the fundamental theorems playing the important role so as to determine the topological structures of compact Riemannian manifolds of positive curvatures, it is well-known, the so-called Toponogov's comparison theorem. This theorem was asserted by V. A. Toponogov [**6**] and completely proved by Y. Tsukamoto and T. Yamaguchi [8] and by D. Gromoll, W. Klingenberg and W. Meyer [**3**] (and we can find the more generalized form in J. Cheeger and D. G. Ebin [2]). This is expressed as follows;

(A) *Let* M *be an* $m(\geqq 2)$*-dimensional complete Riemannian manifold whose sectional curvatures are not less than the constant* k*. Then for every geodesic triangle* Δ *of* M *there exists a geodesic triangle* $\tilde{\Delta}$ *of* $S^m(k)$ *(or* $S^2(k)$*) isometric to* Δ *such that each angle of* Δ *is not less than the corresponding one of* $\tilde{\Delta}$*, where* $S^m(k)$ *is the m-dimensional space form, i.e., the complete simply-connected Riemannian manifold with constant sectional curvature* k*.*

On the other hand the corresponding case where curvatures are bounded above by a constant was investigated by A. D. Alexandrov [**1**] and he obtained;

In a metric space R_K *with curvature*$\leqq K$*, the angles in every triangle* Δ *are not greater than the corresponding angles in the isometric triangle* Δ^K *of K-plane, where K-plane denotes for* $K=0$ *the euclidean, for* $K<0$ *the hyperbolic plane of curvature* K*, and for* $K>0$ *an open hemisphere of curvature* K*.*

However, in his definitions the notion of metric space R_K with curvature $\leqq K$ does not necessarily equal to that of the sectional curvature$\leqq K$ in the case of Riemannian manifold even though of dimension 2, and it is clear that this proposition is not true in the Riemannian case. For example, consider the triangle whose three sides are equal length and compose a closed geodesic of perimeter π on the real projective space with the standard metric of constant curvature 1. That is to say, in the Riemannian case this proposition is true for "local" but not "global". Considering this point, Y. Tsukamoto [7] proved the following two theorems which relate to the comparison of geodesic triangles on Riemannian manifolds whose sectional curvatures are upper-bounded;

(B) *If the sectional curvature* $K\sigma$ *of compact simply-connected Riemannian manifold* M *satisfies the inequalities* $0<K\sigma\leqq 1$ *for all* σ, *then the following two propositions* (a) *and* (b) *are equivalent*; (a) *The angle of geodesic triangle* $\varDelta$ *on* M *is not larger than the corresponding angle of the corresponding triangle* $\varDelta'$ *on* $S^2(1)$, *where the sum of length of three sides of* $\varDelta$ *is less than* 2π. (b) $d(p, C(p))\geqq\pi$ *for all* p *of* M *where* $C(p)$ *is the cut locus of* p. *In particular, if* (1) $m=even$ *or* (2)(*) $1/4<K\sigma\leqq 1$ *for all* σ, *then* (b) *and hence* (a) *are satisfied by* W. Klingenberg [4] *and* [5].

(C) *If the sectional curvature* $K\sigma$ *of complete simply-connected Riemannian manifold* M *satisfies the inequalities* $K\sigma\leqq k\leqq 0$ *for all* σ, *where* k *is a constant, then the angle of a geodesic triangle on* M *is not larger than that of the corresponding triangle on 2-dimensional hyperbolic space with constant curvature* k.

From these theorems we find that if sectional curvatures are bounded above, the comparison of angles of geodesic triangles is closely related to the estimate of the injective radius or to the simply-connectedness of manifold, that is different from the case of curvatures to be bounded below. From this point of view, in the present paper we generalize the above (B) and (C) and prove the following main theorem corresponding to (A).

MAIN THEOREM *Let* M *be an* $m(\geqq 2)$*-dimensional complete Riemannian manifold such that* $K\sigma\leqq k$ *for all* σ *where* k *is a constant. Then the following conditions* (a), (b) *and* (c) *are equivalent to each other*;
(a) *Given a geodesic triangle* $\varDelta=(c_0, c_1, c_2)$ *on* M *such that the perimeter of* $\varDelta$ *is not greater than* $2\pi/\sqrt{k}$ *if* $k>0$, *there exists a geodesic triangle* $\tilde{\varDelta}=(\tilde{c}_0, \tilde{c}_1, \tilde{c}_2)$ *on* $S^m(k)$ (*or* $S^2(k)$) *such that* $L(c_i)=L(\tilde{c}_i)$ *and* $\gamma_i\leqq\tilde{\gamma}_i$ $(i=0, 1, 2)$.

(*) In his paper [5] he claims that the condition (2) may be replaced by (2) $1/4\leqq K_\sigma\leqq 1$ for all σ.

(b) *There exists no non-trivial geodesic loop c on M such that* $L(c) < 2\pi/\sqrt{k}$ *if* $k>0$.

(c) *If* $k>0$, $d(p, C(p)) \geqq \pi/\sqrt{k}$ *for all p of M, namely, any geodesics of length*$\leqq \pi/\sqrt{k}$ *are shortest. And if* $k \leqq 0$, $C(p)$ *is empty for all p of M, namely, any geodesics of M are shortest, and this is equivalent to "M is simply-connected".*

Especially when M is compact, "geodesic loop" of (b) *may be replaced by "closed geodesic". However in the case of* $k \leqq 0$, *all the conditions* (a), (b) *and the amended* (b) *are not statisfied since M is not simply-connected.*

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§ 2 Notations and known results

In the following, let M be a connected complete Riemannian manifold of dimension $m(\geqq 2)$ and M_p the tangent space of M at p. The sectional curvature of M with respect to 2-dimensional subspace σ of M_p generated by two linearly independent vectors v and w is denoted by $K\sigma$ or $K(v, w)$. All the geodesics are assumed to be parametrized by arc-length. The arc-length (resp. tangent vector) of any curve ϕ is denoted by $L(\phi)$ (resp. $\dot{\phi}$) and the distance between two points p and q by $d(p, q)$. A geodesic $c: [0, \ell] \to M$ $(\ell > 0)$ is called *a geodesic loop* if $c(0) = c(\ell)$, and *a closed geodesic if* $\dot{c}(0) = \dot{c}(\ell)$. Given distinct three points p_i and geodesics c_i: $[0, \ell_i] \to M (i=0, 1, 2)$ such that $c_i(\ell_i) = c_{i+1}(0) = p_{i+2} (mod\ 3)$ and $L(c_i) = d(c_i(0),\ c_i(\ell_i))$, the triple $\{c_0, c_1, c_2\}$ is said to form *a geodesic triangle* and denoted by Δ, (c_0, c_1, c_2) or (p_0, p_1, p_2). For each $i=0, 1, 2$, $\gamma_i = \pi - \angle(\dot{c}_{i+1}(\ell_{i+1}),\ \dot{c}_{i+2}(0))$ (mod 3) is called *the* (*interior*) *angle at* p_i, and p_i and c_i are called *the vertex* and *the side of* Δ respectively, where for v, w of M_p, $\angle(v, w)$ means the angle (of $[0, \pi]$) between v and w. When there are geodesic triangles on two Riemannian manifolds respectively, whose corresponding sides are of equal lengths, these two geodesic triangles are said to be *isometric* to each other.

For any $r>0$ and p of M we set $B_r(p) = \{v \in M_p \mid \|v\| < r\}$ and $U_r(p) = \{q \in M \mid d(p, q) < r\}$. For a geodesic c: $[0, \infty) \to M$ the point $c(t_0)$ such that $t_0 = \sup\{t \in \boldsymbol{R} \mid d(c(0), c(t)) = t\}$ is called *the cut point of* $c(0)$ *along* c, and the set of all cut points of p is called *the cut locus of p* and denoted by $C(p)$. Moreover $\inf\{d(p, C(p)) \mid p \in M\}$ is called *the injective radius of M* and denoted by $\iota(M)$, where we assume $d(p, C(p)) = \infty$ if $C(p)$ is empty. For a geodesic c, the point $c(t_0)$ is called *the conjugate point of* $c(0)$ *along*

c if the exponential map $\exp(c(0))$: $M_{c(0)} \to M$ is not of maximal rank at $t_0\dot{c}(0)$.

By $S^m(k)$ we mean the $m(\geqq 2)$-dimensional complete simply-conncted Riemannian manifold of constant curvature k, i.e., the Euclidean space $\boldsymbol{R}^m$ with natural Riemannian metric if $k=0$, the m-dimensional sphere of radius $1/\sqrt{k}$ in $\boldsymbol{R}^{m+1}$ with Riemannian metric induced from the natural one of $\boldsymbol{R}^{m+1}$ if $k>0$, and the m-dimensional open ball of radius $1/\sqrt{|k|}$ centered at the origin in $\boldsymbol{R}^m$ with Riemannian metric

$$ds^2 = \frac{4}{(1+k\sum_{i=1}^{m}(x^i)^2)^2}\sum_{i=1}^{m}(dx^i)^2 \quad \text{if } k<0.$$

It is well-known that in $S^m(k)$ the cosine formula holds for a geodesic triangle $\Delta=(c_0, c_1, c_2)$, that is to say,

$$\cos\sqrt{k}L(c_0) = \cos\sqrt{k}L(c_1)\cdot\cos\sqrt{k}L(c_2) + \sin\sqrt{k}L(c_1)\cdot\sin\sqrt{k}L(c_2)\cdot\cos\gamma_0$$

if $k>0$,

$$L(c_0)^2 = L(c_1)^2 + L(c_2)^2 - 2L(c_1)L(c_2)\cdot\cos\gamma_0 \quad \text{if } k=0,$$

$$\cosh\sqrt{|k|}L(c_0) = \cosh\sqrt{|k|}L(c_1)\cdot\cosh\sqrt{|k|}L(c_2) - \sinh\sqrt{|k|}L(c_1)\cdot\sinh\sqrt{|k|}L(c_2)\cdot\cos\gamma_0 \quad \text{if } k<0.$$

The following propositions are essential to prove our main theorem. As to the proofs of these propositions we refer to for example D. Gromoll, W. Klingenberg and W. Meyer [3].

PROPOSITION 1 *Let $c:[0, \ell]\to M$ be a geodesic and k be any positive constant.*

(a) *If $K\sigma\leqq k$ for any t of $[0, \ell]$ and σ such that $\dot{c}(t)$ belongs to σ, and if $\ell<\pi/\sqrt{k}$, then c has no conjugate point of $c(0)$.*

(b) *If $K\sigma\leqq 0$ for any t of $[0, \ell]$ and σ such that $\dot{c}(t)$ belongs to σ, then c has no conjugate point of $c(0)$.*

PROPOSITION 2 (a) *Let p and q be two points of M such that $d(p, q)=d(p, C(p))$ and q belongs to $C(p)$. If q is not a conjugate point of p along any minimal geodesic from p to q, there exists one and only one geodesic loop $c:[0, 2d(p, q)]\to M$ such that $c(0)=p$ and $c(d(p, q))=q$, except the directions.*

(b) *Suppose M is compact, then $\min_{p\in M} d(p, C(p))$ coincides with the mi-*

nimum of the set $\{\ell \mid$ *there exists a geodesic* $c\colon [0,\ell]\to M$ *such that* $c(\ell)$ *is a conjugate point of* $c(0)$ *along* $c\} \cup \{\ell/2 \mid$ *there exists a closed geodesic* $c\colon[0,\ell]\to M\}$.

PROPOSITION 3 *Let* $\tilde{M}(k)$ *be an m-dimensional complete Riemannian manifold of constant curvature* k *and* $\iota\colon M_p\to\tilde{M}(k)_{\tilde{p}}$ *be an isometric isomorphism for a point* p *(resp.* $\tilde{p}$*) of* M *(resp.* $\tilde{M}(k)$*). For a piecewise differentiable curve* $\phi\colon [a,b]\to M_p$ *we set* $\psi=\exp(p)\cdot\phi$ and $\tilde{\psi}=\exp(\tilde{p})\cdot\iota\cdot\phi$. *Now if* $K\sigma\leqq k$ *for all 2-dimensional tangent vector subspace* σ *of* M *and if one of the following* (a) *or* (b) *is satisfied;*

(a) $k\leqq 0$

(b) $k>0$ *and* $\|\phi(t)\|\leqq\pi/\sqrt{k}$ *for all* t *of* $[a,b]$,

then we have $L(\tilde{\psi})\leqq L(\psi)$.

PROPOSITION 4 *Let* p *be a point of* M. *Then the map* $\exp(p)\colon M_p \to M$ *is injective if and only if* M *is simply-connected and* $\exp(p)$ *is of maximal rank at every point of* M_p.

§ 3 Proof of the main theorem

To prove the main theorem we consider the following condition (b′);

(b′) There exists no geodesic loop c on M such that it can be divided into a geodesic triangle and that, if $k>0$, $L(c)<2\pi/\sqrt{k}$.

At first in the case of $k>0$ we prove that (a), (b′) and (c) are mutually equivalent.

Let's show that (a) implies (b′); Suppose that there is a geodesic loop c of length$<2\pi/\sqrt{k}$ which forms a geodesic triangle $\varDelta=(c_0, c_1, c_2)$. Then by (a) we can construct a geodesic triangle $\tilde{\varDelta}=(\tilde{c}_0, \tilde{c}_1, \tilde{c}_2)$ on $S^m(k)$ such that $L(c_i)=L(\tilde{c}_i)$ and $\gamma_i\leqq\tilde{\gamma}_i (i=0,1,2)$. However since $\gamma_i=\pi$ for at least two i, $\tilde{\varDelta}$ is a geodesic loop and hence a closed geodesic. This contradicts $L(\tilde{c}_0)+L(\tilde{c}_1)+L(\tilde{c}_2)<2\pi/\sqrt{k}$.

Next, (b′) implies (c); Suppose that $d(p_0, C(p_0))=\rho_0<\pi/\sqrt{k}$ for a fixed p_0 of M and that $d(p_0,q)=d(p_0,C(p_0))$ for a point q of $C(p_0)$. Then since proposition 1(a) means that there is no conjugate point of p_0 in $U_{\pi/\sqrt{k}}(p_0)$, there exists uniquely except orientations a geodesic loop $c\colon [0,2\rho_0]\to M$ such that $c(0)=p_0$ and $c(\rho_0)=q$ by proposition 2(a). Clearly $\varDelta=(c|_{[0,\rho_0-\varepsilon]}, c|_{[\rho_0-\varepsilon,\rho_0+\varepsilon]}, c|_{[\rho_0+\varepsilon,2\rho_0]})$ is a geodesic triangle for any small $\varepsilon>0$. This implies $2\rho_0=L(c)\geqq2\pi/\sqrt{k}$ by (b′) and contradicts our assumption.

(c) implies (a); We may assume $L(c_0)+L(c_1)+L(c_2)<2\pi/\sqrt{k}$. Hence Δ is contained in $U_{\pi/\sqrt{k}}(p_0)$, and by our assumption the map $\exp(p_0)|_{B_{\pi/\sqrt{k}}(p_0)}: B_{\pi/\sqrt{k}}(p_0)\to M$ is an into-diffeomorphism. Regarding $\tilde{M}(k)$ as $S^m(k)$ we apply proposition 3 and set by $\bar{\Delta}=(\bar{c}_0, \bar{c}_1, \bar{c}_2)$ the image of $\Delta=(c_0, c_1, c_2)$ under the map $\exp(\tilde{p}_0)\cdot\iota\cdot(\exp(p_0)|_{B_{\pi/\sqrt{k}}(p_0)})^{-1}$ and, on the one hand, constract a geodesic triangle $\tilde{\Delta}=(\tilde{c}_0, \tilde{c}_1, \tilde{c}_2)$ on $S^m(k)$ isometric to Δ. We have trivially $L(\tilde{c}_1)=L(\bar{c}_1)$ and $L(\tilde{c}_2)=L(\bar{c}_2)$, and $L(\bar{c}_0)\leqq L(c_0)=L(\tilde{c}_0)$ by proposition 3, hence $\gamma_0=\bar{\gamma}_0\leqq\tilde{\gamma}_0$ by applying the cosine formula on $S^m(k)$. The same arguments imply $\gamma_1\leqq\tilde{\gamma}_1$ and $\gamma_2\leqq\tilde{\gamma}_2$.

It is trivial that (b) implies (b') and (c) implies (b). Hence (a), (b'), (c) and (b) are equivalent.

In the case of M compact, we have only to prove that the amended (b) implies (c). In the process of proof of (c) from (b') we may assume $\rho_0=\min\{d(p, C(p))\mid p \text{ belongs to } M\}$ because M is compact. Then proposition 2(b) means that c is a closed geodesic, hence we may apply the amended (b).

Assume $k\leqq 0$. Since M has no conjugate point along any geodesics by proposition 1(b), $C(p)$ is empty, that is, $\exp(p): M_p\to M$ is an onto-diffeomorphism if and only if M is simply-connected for an arbitrarily fixed p of M by proposition 4. And clearly $S^m(k)$ has no non-trivial geodesic loop. The proof for the case of $k\leqq 0$ is similar as the above.

References

[1] A. D. ALEXANDROV: *Über eine Verallgemeinerung der Riemannschen Geometrie,* Schr. Forschungsinst. Math., **1** (1957), 33-84.

[2] J. CHEEGER and D. G. EBIN: *Comparison theorems in Riemannian Geometry,* North-Holland Math. Lib., **9** (1975).

[3] D. GROMOLL, W. KLINGENBERG and W. MEYER: *Riemannsche Geometrie im Grossen,* Springer Verlag, Lect. Notes in Math., **55** (1968).

[4] W. KLINGENBERG: *Contributions to Riemannian geometry in the large,* Ann. of Math., **69** (1959), 654-666.

[5] W. KLINGENBERG: *Über Riemannsche Mannigfaltigkeiten mit positiver Krümmung,* Comm. Math. Helv., **35** (1961), 47-54.

[6] V. A. TOPONOGOV: *Riemannian spaces having their curvature bounded below by a positive number,* Uspehi Math. Nauk, **14** (1959), 87-130; Amer. Math. Soc. Translations, **37** (1964), 291-336.

[7] Y. TSUKAMOTO: *On a theorem of A. D. Alexandrov,* Mem. of the Faculty of Sci. Kyushu Univ., Ser. A, **15** (1962), 83-89.

[8] Y. TSUKAMOTO and T. YAMAGUCHI: *On various comparison theorems,* Sūgaku, **21** (1969), 81-96, (in Japanese).