On a regularity of E. Hopf's weak solutions for the Navier-Stokes equations

Kato, Hisako
Department of Mathematics, College of General Education, Kyushu University

https://doi.org/10.15017/1448995

出版情報：九州大学教養部数学雑誌. 11 (1), pp.15-24, 1977-10. 九州大学教養部数学教室
バージョン：
権利関係：

Math. Rep.
XI-1, 1977.

On a regularity of E. Hopf's weak solutions for the Navier-Stokes equations

Hisako KATO

(Received December 15, 1976)

§1. Introduction and summary

Let us consider the initial value problem for the Navier-Stokes equations which is written in its classical form as

(1.1) $\frac{\partial u}{\partial t}-\nu\Delta u+(u\cdot\nabla)u=f-\nabla p,\quad x\in\Omega,\quad t>0,$

(1.2) $\mathrm{div}\,u=0,\quad x\in\Omega,\quad t>0,$

(1.3) $u|_{\partial\Omega}=0,\qquad t>0,$

(1.4) $u|_{t=0}=a,\qquad x\in\Omega,$

with the usual notations. Here $u(x,t)$ is the velocity field, $p(x,t)$ is the pressure, $a(x)$ is the initial velocity, $f(x,t)$ is the external force. In these equations u, p are unknown, and a, f are given.

For the problem, E. Hopf [3] (1951) succeeded in showing that there exists a global weak solution and that for an arbitrary domain in n-dimensional Euclidean space $R^n(n\geqq 2)$, but he left the investigation on the uniqueness and the smoothness of his solution for later works. Except the case $n=2$ no one has yet succeeded in proving or disproving the uniqueness theorem for his solution. To establish the uniqueness theorem together with an existence theorem, various researches on strong solutions have been made by many authors, especially by A. A. Kiselev and O. A. Ladyzhenskaia [5] (1957), P. E. Sobolevskii [9] (1959), S. Ito [4] (1961), H. Fujita and T. Kato [2] (1964) and others. Their strong solutions have been shown to be unique for the case $n=3$ in which we are interested, although none of them is global (in time) unless some smallness restriction on prescribed data is assumed.

In this paper, assuming that Ω is a bounded domain in R^3 with a sufficiently smooth boundary $\partial\Omega$ and that the external force f is absent, we will study, in some sense, the differentiability with respect to the time variable

t concerning the Hopf's weak solutions.

Now, we explane some notations and concepts.

DEFINITION 1.1. The space

$$\dot{J}(\Omega)=\left\{u(x)\mid u\in C_0^\infty(\Omega),\ \text{div}\ u=\sum_{i=1}^{3}\frac{\partial u^i}{\partial x^i}=0\right\},$$

and the associated Hilbert space

$\mathring{J}(\Omega)=$ the closure of $\dot{J}(\Omega)$ in $L_2(\Omega)$,

and also the Hilbert space

$J_2^1(\Omega)=$ the closure of $\dot{J}(\Omega)$ in $W_2^1(\Omega)$.

Here, for the vector-valued functions $u=u(x)$, $v=v(x)$ inner products are defined by

$$(u,v)_{L_2(\Omega)}=\int_\Omega\sum_{i=1}^{3}u^iv^i dx$$

$$(u,v)_{W_2^1(\Omega)}=\int_\Omega\left(\sum_{i=1}^{3}u^iv^i+\sum_{i=1}^{3}\sum_{k=1}^{3}\frac{\partial u^i}{\partial x^k}\frac{\partial v^i}{\partial x^k}\right)dx$$

$$=(u,v)_{L_2(\Omega)}+(\nabla u,\nabla v)_{L_2(\Omega)},$$

where u^i denotes the i-th component of the vector u.

DEFINITION 1.2. We denote by $W_2^{1\prime}(\hat{\Omega})=W_2^{1\prime}(\Omega\times(0,\mathrm{T}))$ the totality of vector-valued functions $u=u(x,t)$ which are measurable in $\hat{\Omega}=\Omega\times(0,\mathrm{T})$ and have the properties:

(i) $u\in L_2(\hat{\Omega})$

(ii) $\nabla u=\left(\frac{\partial u^i}{\partial x^k},\ i,k=1,2,3\right)\in L_2(\hat{\Omega})$.

The space $W_2^{1\prime}(\hat{\Omega})$ becomes a Hilbert space with the inner product

$$(u,v)_{W_2^{1\prime}(\hat{\Omega})}=\int_0^T\{(u,v)_{L_2(\Omega)}+(\nabla u,\nabla v)_{L_2(\Omega)}\}dt.$$

Therefore, concerning the derivative ∇u it holds the equality:

$$\int_0^T\int_\Omega u^i\frac{\partial\omega}{\partial x^k}\,dxdt=-\int_0^T\int_\Omega\frac{\partial u^i}{\partial x^k}\omega dxdt, \tag{1.5}$$

for any scalar test function $\omega=\omega(x,t)\in C_0^\infty(\hat{\Omega})\ (i,k=1,2,3)$.

DEFINITION 1.3. The space

$$\dot{J}(\Omega\times(0,\infty))=\left\{u(x,t)\mid u\in C^\infty(\bar{\Omega}\times[0,\infty)),\ \text{div}\ u=0\ \text{and}\ u\ \text{is of compact support in}\ \Omega\ \text{for any}\ t\right\}$$

and the associated Hilbert space

$$J_2^{1\prime}(\hat{\Omega})=J_2^{1\prime}(\Omega\times(0,T))=\text{the closure of } \mathring{J}(\Omega\times(0,\infty)) \text{ in } W_2^{1\prime}(\hat{\Omega}).$$

DEFINITION 1.4. Let P be the orthogonal projection from $L_2(\Omega)$ onto $\mathring{J}(\Omega)$. By A we denote the Friedrichs extension of the symmetric operator $-P\Delta$ in $\mathring{J}(\Omega)$ defined for every $u\in\mathring{J}(\Omega)$. A is a strictly positive self-adjoint operator in $\mathring{J}(\Omega)$ whose domain $D(A)$ is contained in $J_2^1(\Omega)$.
The relation $Au=w$ ($u\in D(A)$, $w\in\mathring{J}(\Omega)$) is true if and only if $u\in J_2^1(\Omega)$, $w\in\mathring{J}(\Omega)$, and

(1.6) $\qquad (\nabla u,\nabla v)_{L_2(\Omega)}=(w,v)_{L_2(\Omega)}$ for any $v\in J_2^1(\Omega)$.

Since the operator A is self-adjoint it admits a uniquely determined spectral resolution:

(1.7) $$A=\int_{-\infty}^{+\infty}\lambda dE(\lambda).$$

Moreover, since it is strictly positive:

(1.8) $\qquad (Au,u)_{L_2(\Omega)}=\|\nabla u\|^2_{L_2(\Omega)}\geqq\delta\|u\|^2_{L_2(\Omega)},\ u\in D(A),$

where $\delta=\inf\{(Au,u)_{L_2(\Omega)}\mid \|u\|_{L_2(\Omega)}\leqq 1,\ u\in D(A)\}$

(1.9) $\qquad =\inf\{\lambda\mid\lambda$ is a spectrum of $A\}$

$\qquad =$ the Minimum spectrum of A,

it holds the relations

(1.10) $$A=\int_{-\infty}^{+\infty}\lambda dE(\lambda)=\int_{\delta-\varepsilon}^{+\infty}\lambda dE(\lambda),\quad E(\delta-\varepsilon)=0,$$

(1.11) $$D(A)=\left\{u\in\mathring{J}(\Omega)\,\Big|\int_{\delta-\varepsilon}^{+\infty}\lambda^2 d\|E(\lambda)u\|^2_{L_2(\Omega)}<+\infty\right\}.$$

The inverse operator A^{-1} is defined in $\mathring{J}(\Omega)$ and it is bounded. The fractional power of the operator A is defined as follows,

(1.12) $$A^{\alpha}u=\int_{\delta-\varepsilon}^{+\infty}\lambda^{\alpha}dE(\lambda)u,\quad 0<\alpha<1,$$

(1.13) $$D(A^{\alpha})=\left\{u\in\mathring{J}(\Omega)\,\Big|\int_{\delta-\varepsilon}^{+\infty}\lambda^{2\alpha}d\|E(\lambda)u\|^2_{L_2(\Omega)}<+\infty\right\}$$

and also it holds

(1.14) $$A^{-1}u=\int_{\delta-\varepsilon}^{+\infty}\frac{1}{\lambda}dE(\lambda)u\quad\text{for any } u\in\mathring{J}(\Omega),$$

(1.15) $$A^{-\alpha}u=\int_{\delta-\varepsilon}^{+\infty}\frac{1}{\lambda^{\alpha}}dE(\lambda)u\quad\text{for any } u\in\mathring{J}(\Omega).$$

With respect to the relation between the domain $D(A^{\frac{1}{2}})$ and the space $J_2^1(\Omega)$ which are introduced in the above, the following lemma is well known.

LEMMA 1.1. *It holds*

$$D(A^{\frac{1}{2}})=J_2^1(\Omega) \text{ and } \|A^{\frac{1}{2}}u\|_{L_2(\Omega)}=\|\nabla u\|_{L_2(\Omega)}.$$

Secondly, we state the properties of the Hopf's weak solution which have been proved ([3][7]).

LEMMA 1.2. *For an initial value $a(x)\in \dot{J}(\Omega)$ there exists a global weak solution $u(x,t)$ of the Navier-Stokes equations, i.e. the vector-valued function $u(x,t)$ is measurable in $\Omega\times(0,\infty)$, and for any finite $T>0$ satisfies the relations*

(i) $u\in J_2^{1\prime}(\Omega\times(0,T))$,

(ii) $u(x,t)\in\dot{J}(\Omega)$ *for any* $t\in[0,T]$,

(1.16) (iii) $$\int_0^t\{(u,\frac{\partial\varphi}{\partial t})_{L_2(\Omega)}+\nu(u,\Delta\varphi)_{L_2(\Omega)}+(u,u\cdot\nabla\varphi)_{L_2(\Omega)}\}dt$$
$$=(u(x,t),\varphi(x,t))_{L_2(\Omega)}-(a(x),\varphi(x,0))_{L_2(\Omega)}$$
for any $\varphi(x,t)\in\dot{J}(\Omega\times(0,\infty))$ *and for any* $t\in[0,T]$,

where by the notation $(u,u\cdot\nabla\varphi)_{L_2(\Omega)}$ we mean the integral $\int_\Omega\sum_{i=1}^3\sum_{k=1}^3 u^iu^k\frac{\partial\varphi^i}{\partial x^k}dx$, more generally, by the notation $(f,g\cdot\nabla h)_{L_2(\Omega)}$ we will denote the integral $\int_\Omega\sum_{i=1}^3\sum_{k=1}^3 f^ig^k\frac{\partial h^i}{\partial x^k}dx$ in this paper, and moreover

(iv) $u(x,t)\in\dot{J}_2^1(\Omega)$ *a.e.* $t\in[0,\infty)$,

(1.17) (v) $$\|u(x,t)\|^2_{L_2(\Omega)}+2\nu\int_0^t\|\nabla u(x,t)\|^2_{L_2(\Omega)}dt\leqq\|a(x)\|^2_{L_2(\Omega)}$$
for any $t\in[0,\infty)$,

(vi) $\lim_{t\to t_0}(u(x,t),\ \varphi(x))_{L_2(\Omega)}=(u(x,t_0),\varphi(x))_{L_2(\Omega)}$

for any $t_0\in[0,\infty)$ *and for any* $\varphi(x)\in\dot{J}(\Omega)$,

(vii) $\lim_{t\downarrow 0}\|u(x,t)-a(x)\|_{L_2(\Omega)}=0$.

In this situation we will show the following theorem.

THEOREM. *Let $u(x,t)$ be the weak solution in the above lemma. Then we have*

(i) *The function $(A^{-\frac{1}{2}}u(t),\ \varphi)_{L_2(\Omega)}$ is absolutely continuous with respect to $t\in[0,T]$ for any $\varphi\in\dot{J}(\Omega)$.*

(ii) *There exists the derivative* $\frac{d}{dt}(A^{-\frac{1}{2}}u(t), \varphi)_{L_2(\Omega)}$ *a.e. t.*

(iii) *The vector-valued function* $A^{-\frac{1}{2}}u(x,t)$ *is measurable and locally integrable in* $\Omega\times(0,T)$ *and there exists the derivative* $\frac{\partial}{\partial t}A^{-\frac{1}{2}}u(x,t)$ *i.e. there exists the measurable and locally integrable vector-valued function* $g(x,t)$ *such that*

$$\int_0^T\int_\Omega A^{-\frac{1}{2}}u(x,t)\frac{\partial\omega(x,t)}{\partial t}\,dxdt=-\int_0^T\int_\Omega g(x,t)\omega(x,t)dxdt \tag{1.18}$$

for any test vector-valued function $\omega(x,t)\in C_0^\infty(\Omega\times(0,T))$.

(iv) *It holds the relation*

$$\frac{\partial}{\partial t}A^{-\frac{1}{2}}u(x,t)=-\nu A^{\frac{1}{2}}u(x,t)-A^{-\frac{1}{2}}Pu\cdot\nabla u(x,t),\quad a.e.\ (x,t)\in\Omega\times(0,T).$$

§ 2 Proof of Theorem

We will give the proof of the theorem step by step.

[1°] We note that there exists an absolute constant C_1 such that the inequality ([2] [9])

$$\|A^{-\frac{1}{4}}P(v\cdot\nabla)w\|_{L_2(\Omega)}\leqq C_1\|A^{\frac{1}{2}}v\|_{L_2(\Omega)}\|A^{\frac{1}{2}}w\|_{L_2(\Omega)} \tag{2.1}$$

holds for any v, $w\in\dot{J}(\Omega)$ and therefore we may define the operator $Hw=A^{-\frac{1}{4}}P(w\cdot\nabla)w$ for every $w\in D(A^{\frac{1}{2}})$ as follows. For any $w\in D(A^{\frac{1}{2}})=J_2^1(\Omega)$ there exists a sequence $w_n\in\dot{J}(\Omega)\,(n=1,2,3,\dots)$ tending to w in $J_2^1(\Omega)$ and it holds

$$\begin{aligned}&\|A^{-\frac{1}{4}}P(w_n\cdot\nabla)w_n-A^{-\frac{1}{4}}P(w_m\cdot\nabla)w_m\|_{L_2(\Omega)}\\ &\leqq\|A^{-\frac{1}{4}}Pw_n\cdot(\nabla w_n-\nabla w_m)\|_{L_2(\Omega)}+\|A^{-\frac{1}{4}}P(w_n-w_m)\cdot\nabla w_m\|_{L_2(\Omega)}\\ &\leqq C_1\|A^{\frac{1}{2}}w_n-A^{\frac{1}{2}}w_m\|_{L_2(\Omega)}\{\|A^{\frac{1}{2}}w_n\|_{L_2(\Omega)}+\|A^{\frac{1}{2}}w_m\|_{L_2(\Omega)}\}.\end{aligned} \tag{2.2}$$

By the inequalities (2.2) we have that the sequence $A^{-\frac{1}{4}}P(w_n\cdot\nabla)w_n$ tends to an element $v\in\dot{J}(\Omega)$ in $\dot{J}(\Omega)$, and then we define Hw by v, i.e.

$$Hw=\lim_{n\to\infty}A^{-\frac{1}{4}}P(w_n\cdot\nabla)w_n. \tag{2.3}$$

We remark that Hw is determined uniquely for w. In fact if there exists another sequence $\tilde{w}_n\in\dot{J}(\Omega)$ tending to w in $J_2^1(\Omega)$ the sequence $A^{-\frac{1}{4}}P(\tilde{w}_n\cdot\nabla)\tilde{w}_n$ tends to an element $\tilde{v}\in\dot{J}(\Omega)$. Since we have the inequalities

$$\|\tilde{v}-v\|_{L_2(\Omega)}\leqq\|\tilde{v}-A^{-\frac{1}{4}}P(\tilde{w}_n\cdot\nabla)\tilde{w}_n\|_{L_2(\Omega)}+\|A^{-\frac{1}{4}}P(w_n\cdot\nabla)w_n-v\|_{L_2(\Omega)}$$
$$+\|A^{-\frac{1}{4}}P(\tilde{w}_n\cdot\nabla)\tilde{w}_n-A^{-\frac{1}{4}}P(w_n\cdot\nabla)w_n\|_{L_2(\Omega)},$$

and

$$\|A^{-\frac{1}{4}}P(\tilde{w}_n\cdot\nabla)\tilde{w}_n-A^{-\frac{1}{4}}P(w_n\cdot\nabla)w_n\|_{L_2(\Omega)}$$
$$\leqq C_1\|A^{\frac{1}{2}}\tilde{w}_n-A^{\frac{1}{2}}w_n\|_{L_2(\Omega)}\{\|A^{\frac{1}{2}}\tilde{w}_n\|_{L_2(\Omega)}+\|A^{\frac{1}{2}}w_n\|_{L_2(\Omega)}\},$$

it gives the equality $\tilde{v}=v$.

[2°] We will show the identity

$$(w\cdot\nabla w,\ A^{-\frac{1}{2}}\psi)_{L_2(\Omega)}=(A^{-\frac{1}{4}}Hw,\ \psi)_{L_2(\Omega)} \tag{2.4}$$

for any $w\in D(A^{\frac{1}{2}})=J_2^1(\Omega)$ and for any $\psi\in\mathring{J}(\Omega)$.

For the $w_n\in\dot{J}(\Omega)$ in [1°] which tends to $w\in J_2^1(\Omega)$, it holds

$$(w_n\cdot\nabla w_n,\ A^{-\frac{1}{2}}\psi)_{L_2(\Omega)}=(A^{-\frac{1}{4}}P(w_n\cdot\nabla)w_n,\ A^{-\frac{1}{4}}\psi)_{L_2(\Omega)},$$

and

$$|(w_n\cdot\nabla w_n,\ A^{-\frac{1}{2}}\psi)_{L_2(\Omega)}-(w\cdot\nabla w,\ A^{-\frac{1}{2}}\psi)_{L_2(\Omega)}|$$
$$=|(w_n\cdot(\nabla w_n-\nabla w)+(w_n-w)\cdot\nabla w,\ A^{-\frac{1}{2}}\psi)_{L_2(\Omega)}|$$
$$\leqq C_2\|\nabla w_n-\nabla w\|_{L_2(\Omega)}\{\|\nabla w_n\|_{L_2(\Omega)}+\|\nabla w\|_{L_2(\Omega)}\}\|\nabla A^{-\frac{1}{2}}\psi\|_{L_2(\Omega)}.$$

Therefore, by letting $n\to\infty$ it gives (2.4) i.e.

$$(w\cdot\nabla w,\ A^{-\frac{1}{2}}\psi)_{L_2(\Omega)}=(Hw,\ A^{-\frac{1}{4}}\psi)_{L_2(\Omega)}=(A^{-\frac{1}{4}}Hw,\ \psi)_{L_2(\Omega)}.$$

[3°] For the weak solution $u(x,t)$ in the theorem, we will show the identity

$$(u(t),\ \varphi)_{L_2(\Omega)}-(a,\varphi)_{L_2(\Omega)} =-\int_0^t\{\nu(\nabla u,\nabla\varphi)_{L_2(\Omega)}+(u\cdot\nabla u,\varphi)_{L_2(\Omega)}\}dt \tag{2.5}$$

for any $\varphi\in J_2^1(\Omega)$ and for any $t\geqq0$.

In the weak equation (1.16) in the lemma 1.2 since we may set $\varphi(x,t)=\psi(x)\in\dot{J}(\Omega)$ especially, we get

$$\int_0^t\{\nu(u,\Delta\psi)_{L_2(\Omega)}+(u,u\cdot\nabla\psi)_{L_2(\Omega)}\}dt =(u(t),\psi)_{L_2(\Omega)}-(a,\psi)_{L_2(\Omega)} \tag{2.6}$$

for any $\psi\in\dot{J}(\Omega)$ and for any $t\geqq0$.

For any $\varphi\in J_2^1(\Omega)$ there exists a sequence $\psi_n\in\dot{J}(\Omega)$ tending to φ in $J_2^1(\Omega)$ and the identity (2.6) holds for ψ_n ($n=1,2,\dots$) also, and following relations hold

$$\int_0^t(u,\Delta\psi_n)_{L_2(\Omega)}dt=-\int_0^t(\nabla u,\nabla\psi_n)_{L_2(\Omega)}dt\to-\int_0^t(\nabla u,\nabla\varphi)_{L_2(\Omega)}dt\quad(n\to\infty),$$
$$(u(t),\psi_n)_{L_2(\Omega)}\to(u(t),\varphi)_{L_2(\Omega)},\ (a,\psi_n)_{L_2(\Omega)}\to(a,\varphi)_{L_2(\Omega)}\quad(n\to\infty),$$
$$\left|\int_0^t(u,u\cdot\nabla\psi_n)_{L_2(\Omega)}dt-\int_0^t(u,u\cdot\nabla\varphi)_{L_2(\Omega)}dt\right|$$

$$=|\int_0^t (u, u\cdot[\nabla\psi_n-\nabla\varphi])_{L_2(\Omega)}dt|\leqq C_2\|\nabla\psi_n-\nabla\varphi\|_{L_2(\Omega)}\int_0^t\|\nabla u\|^2_{L_2(\Omega)}dt$$

$$\leqq\frac{1}{2\nu}C_2\|a\|^2_{L_2(\Omega)}\|\nabla\psi_n-\nabla\varphi\|_{L_2(\Omega)}\to 0 \quad (n\to\infty).$$

It gives the identity (2.5)

[4°] Let us show the identity

$$(2.7) \qquad (A^{-\frac{1}{2}}u(t),\ \varphi)_{L_2(\Omega)}-(A^{-\frac{1}{2}}a,\varphi)_{L_2(\Omega)}$$

$$=-\int_0^t\{\nu(A^{\frac{1}{2}}u,\varphi)_{L_2(\Omega)}+(A^{-\frac{1}{4}}Hu,\varphi)_{L_2(\Omega)}\}dt$$

for any $\varphi\in\mathring{J}(\Omega)$ and for any $t\geqq 0$, where the operator H is defined in [1°], [2°]. By this fact we obtain that the function $(A^{-\frac{1}{2}}u(t),\ \varphi)_{L_2(\Omega)}$ is absolutely continuous and thereby that there exists the derivative $\frac{d}{dt}(A^{-\frac{1}{2}}u(t),\ \varphi)_{L_2(\Omega)}$.

In fact, for any $\varphi\in\mathring{J}(\Omega)$ we set $A^{-\frac{1}{2}}\varphi=\psi$ and then $\psi\in D(A^{\frac{1}{2}})=J_2^1(\Omega)$. Consequently, from the identity (2.5) we have

$$(u(t),\ A^{-\frac{1}{2}}\varphi)_{L_2(\Omega)}-(a, A^{-\frac{1}{2}}\varphi)_{L_2(\Omega)}$$

$$=-\int_0^t\{\nu(A^{\frac{1}{2}}u,\varphi)_{L_2(\Omega)}+(A^{-\frac{1}{4}}Hu,\varphi)_{L_2(\Omega)}\}dt.$$

Here we used the relations (2.4) in [2°], and

$u(x,t)\ \in J_2^1(\Omega)$ a.e. t, and

$$(A^{\frac{1}{2}}u,\varphi)_{L_2(\Omega)}=(A^{\frac{1}{2}}u,A^{\frac{1}{2}}A^{-\frac{1}{2}}\varphi)_{L_2(\Omega)}=(\nabla u,\nabla(A^{-\frac{1}{2}}\varphi))_{L_2(\Omega)}.$$

This means (2.7).

[5°] We will note that $A^{-\frac{1}{2}}u(x,t)$ is measurable and locally integrable in $\Omega\times$ (0, T). For, since the function

$$(A^{-\frac{1}{2}}u(t),\varphi)_{L_2(\Omega)}=(A^{-\frac{1}{2}}u(t),\varphi_1)_{L_2(\Omega)}+(A^{-\frac{1}{2}}u(t),\varphi_2)_{L_2(\Omega)}$$
$$=(A^{-\frac{1}{2}}u(t),\varphi_1)_{L_2(\Omega)}$$

is continuous function with respect to t for any $\varphi\in L_2(\Omega)$, which is represented as $\varphi=\varphi_1+\varphi_2$, $\varphi_1\in\mathring{J}(\Omega)$, $\varphi_2\in\mathring{J}(\Omega)^{\perp}$, $(A^{-\frac{1}{2}}u(t),\varphi)_{L_2(\Omega)}$ is measurable. $\mathring{J}(\Omega)^{\perp}$ is an orthogonal complement of $\mathring{J}(\Omega)$ in $L_2(\Omega)$. Moreover

$$\int_0^T\|A^{-\frac{1}{2}}u(t)\|^2_{L_2(\Omega)}dt\leqq C\int_0^T\|u(t)\|^2_{L_2(\Omega)}\leqq CT\|a\|^2_{L_2(\Omega)}.$$

Hence $A^{-\frac{1}{2}}u(t)$ belongs to the space $L_2((0,T);\ L_2(\Omega))$. Since $L_2((0,T);\ L_2(\Omega))=L_2(\Omega\times(0,T))$ ([1] [8]), we have $A^{-\frac{1}{2}}u(x,t)\in L_2(\Omega\times(0,T))$.

[6°] We note that $A^{\frac{1}{2}}u(x,t)$, $A^{-\frac{1}{4}}Hu(x,t)$ are measurable and locally integrable in $\Omega\times(0,T)$. In fact, for any $\varphi\in\mathring{J}(\Omega)$ the function

$$(A^{\frac{1}{2}}u(t),\ \varphi)_{L_2(\Omega)}=(\nabla u,\nabla(A^{-\frac{1}{2}}\varphi))_{L_2(\Omega)}$$

is measurable with respect to t and

$$\int_0^T \|A^{\frac{1}{2}}u(t)\|^2_{L_2(\Omega)} = \int_0^T \|\nabla u\|^2_{L_2(\Omega)} dt \leqq \frac{1}{2\nu}\|a\|^2_{L_2(\Omega)}.$$

Hence $A^{\frac{1}{2}}u \in L_2((0, T);\ L_2(\Omega)) = L_2(\Omega\times(0, T))$. For any $\varphi \in L_\infty(\Omega) \subset L_2(\Omega)$ the function

$$\int_\Omega A^{-\frac{1}{4}}Hu(x, t)\varphi(x)dx = (A^{-\frac{1}{4}}Hu, \varphi)_{L_2(\Omega)} = (A^{-\frac{1}{4}}Hu, P\varphi)_{L_2(\Omega)}$$
$$= (u\cdot\nabla u, A^{-\frac{1}{2}}P\varphi)_{L_2(\Omega)}$$

is measurable with respect to t and

$$\int_0^T \|A^{-\frac{1}{4}}Hu\|_{L_1(\Omega)} dt \leqq C\int_0^T \|A^{-\frac{1}{4}}Hu\|_{L_2(\Omega)} dt \leqq C'\int_0^T \|Hu\|_{L_2(\Omega)} dt$$
$$\leqq C'C_1\int_0^T \|A^{\frac{1}{2}}u\|^2_{L_2(\Omega)} dt \leqq \frac{C'C_1}{2\nu}\|a\|^2_{L_2(\Omega)}.$$

Hence $A^{-\frac{1}{4}}Hu \in L_1((0, T);\ L_1(\Omega)) = L_1(\Omega\times(0, T))$.

[7°] Let us show that it holds

$$(2.8)\quad \int_0^T\int_\Omega A^{-\frac{1}{2}}u(x, t)\frac{\partial\omega(x, t)}{\partial t}dxdt = \int_0^T\int_\Omega \{\nu A^{\frac{1}{2}}u(x, t) + A^{-\frac{1}{4}}Hu(x, t)\}\omega(x, t)\, dxdt$$

for any $\omega(x, t) \in C_0^\infty(\Omega\times(0, T))$.

In fact, from the identity (2.7) in [4°] it gives

$$(2.9)\quad \frac{d}{dt}(A^{-\frac{1}{2}}u(t), \varphi)_{L_2(\Omega)} = -\nu(A^{\frac{1}{2}}u(t), \varphi)_{L_2(\Omega)} - (A^{-\frac{1}{4}}Hu(t), \varphi)_{L_2(\Omega)} \text{ a. e. } t,$$

for any $\varphi \in \dot{J}(\Omega)$. For any $\varphi(x) \in C_0^\infty(\Omega)$ it holds

$$\frac{d}{dt}(A^{-\frac{1}{2}}u(t), \varphi)_{L_2(\Omega)} = \frac{d}{dt}(A^{-\frac{1}{2}}u(t), P\varphi)_{L_2(\Omega)}$$
$$= -\nu(A^{\frac{1}{2}}u(t), P\varphi)_{L_2(\Omega)} - (A^{-\frac{1}{4}}Hu(t), P\varphi)_{L_2(\Omega)}$$
$$= -\nu(A^{\frac{1}{2}}u(t), \varphi)_{L_2(\Omega)} - (A^{-\frac{1}{4}}Hu(t), \varphi)_{L_2(\Omega)}.$$

Hence we have

$$(2.10)\quad \int_0^T (A^{-\frac{1}{2}}u(t), \varphi)_{L_2(\Omega)}\psi'(t)dt = \int_0^T \{\nu(A^{\frac{1}{2}}u(t), \varphi)_{L_2(\Omega)}$$
$$+ (A^{-\frac{1}{4}}Hu(t), \varphi)_{L_2(\Omega)}\}\psi(t)dt$$

for any $\varphi \in C_0^\infty(\Omega)$ and for any $\psi \in C_0^\infty(0, T)$. Equivalently,

$$(2.11)\quad \int_0^T\int_\Omega A^{-\frac{1}{2}}u(x, t)\varphi(x)\psi'(t)\, dxdt = \int_0^T\int_\Omega \{\nu A^{\frac{1}{2}}u(x, t)$$
$$+ A^{-\frac{1}{4}}Hu(x, t)\}\varphi(x)\psi(t)dxdt.$$

The Hilbert space $\overset{\circ}{W}{}_2^1(\Omega)$ which is the closure of $C_0^\infty(\Omega)$ being separable, there exists a complete orthonormal system $\{\varphi_l(x),\ l=1, 2, 3, , , \}$ in $\overset{\circ}{W}{}_2^1(\Omega)$ consisting of the elements belonging to the space $C_0^\infty(\Omega)$. For any $\omega(x, t) \in$

$C_0^\infty(\Omega\times(0, T))$, let

(2.12) $\omega(x,t)=\sum_{l=1}^{\infty}\psi_l(t)\varphi_l(x)$, where $\psi_l(t)=(\omega(x,t),\varphi_l(x))_{W_2^2(\Omega)}$

be the Fourier expansion of $\omega(x,t)$ and $\omega_m(x,t)=\sum_{l=1}^{m}\psi_l(t)\varphi_l(x)$ be the corresponding partial sum. According to the theory of Fourier series it holds for fixed $t\in[0,T]$

(2.13) $\omega_m(x,t)=\sum_{l=1}^{m}\psi_l(t)\varphi_l(x)\to\omega(x,t)$ in $W_2^2(\Omega)$,

(2.14) $\dfrac{\partial\omega_m(x,t)}{\partial t}=\sum_{l=1}^{m}\psi'_l(t)\varphi_l(x)\to\dfrac{\partial\omega(x,t)}{\partial t}$ in $W_2^2(\Omega)$,

as it holds $\psi'_l(t)=\left(\dfrac{\partial\omega(x,t)}{\partial t},\ \varphi_l(x)\right)_{W_2^2(\Omega)}$.

More precisely, we can show that the convergence in (2.13), (2.14) is uniform in $t\in[0,T]$. Because of the uniform continuity of $\omega(x,t)$ with respect to $t\in[0,T]$ as an element of $W_2^2(\Omega)$, for any $\varepsilon>0$ there exists a partition $0=t_0<t_1<\cdots<t_{N-1}<t_N=T$ of the interval $[0,T]$ such that it holds

$$\|\omega(x,t')-\omega(x,t'')\|_{W_2^2(\Omega)}<\frac{\varepsilon}{3} \text{ for any } t',t''\in[t_{k-1},t_k].$$

Using Minkowski's inequality we have

$$\|\omega_m(x,t)-\omega(x,t)\|_{W_2^2(\Omega)}\leqq\|\omega_m(x,t)-\omega_m(x,t_k)\|_{W_2^2(\Omega)}$$
$$+\|\omega_m(x,t_k)-\omega(x,t_k)\|_{W_2^2(\Omega)}+\|\omega(x,t_k)-\omega(x,t)\|_{W_2^2(\Omega)}.$$

Let $t\in[0,T]$ belong to the subinterval $[t_{k-1},\ t_k]$. Then we have

$$\|\omega(x,t_k)-\omega(x,t)\|_{W_2^2(\Omega)}<\frac{\varepsilon}{3},$$

and

$$\|\omega_m(x,t)-\omega_m(x,t_k)\|_{W_2^2(\Omega)}\leqq\|\omega(x,t)-\omega(x,t_k)\|_{W_2^2(\Omega)}<\frac{\varepsilon}{3}.$$

As for the quantities $\|\omega_m(x,t_k)-\omega(x,t_k)\|_{W_2^2(\Omega)}$ $(k=1,2,\dots N)$ we can choose a sufficiently large index $m_0(\varepsilon)$ such that

$$\|\omega_m(x,t_k)-\omega(x,t_k)\|_{W_2^2(\Omega)}<\frac{\varepsilon}{3}\quad(k=1,2,3,\dots,N) \text{ for } m\geqq m_0(\varepsilon).$$

Thus, we have a uniform approximation in $[0,\ T]$ such that

$$\|\omega_m(x,t)-\omega(x,t)\|_{W_2^2(\Omega)}<\varepsilon \quad\text{for } m\geqq m_0(\varepsilon).$$

Similarly we have uniformly in $[0,T]$

$$\left\|\frac{\partial\omega_m(x,t)}{\partial t}-\frac{\partial\omega(x,t)}{\partial t}\right\|_{W_2^2(\Omega)}<\varepsilon \quad\text{for } m\geqq m_1(\varepsilon).$$

Finally, applying the Sobolev's lemma we have in $[0,T]$

(2.15) $\operatorname*{Max}_{\bar{\Omega}}|\omega_m(x,t)-\omega(x,t)|\leqq C_3\|\omega_m(x,t)-\omega(x,t)\|_{W_2^2(\Omega)}$,

(2.16) $\operatorname*{Max}_{\bar{\Omega}}\left|\frac{\partial\omega_m(x,t)}{\partial t}-\frac{\partial\omega(x,t)}{\partial t}\right|\leqq C_3\left\|\frac{\partial\omega_m(x,t)}{\partial t}-\frac{\partial\omega(x,t)}{\partial t}\right\|_{W_2^2(\Omega)}$.

The uniform convergence follows from these estimates. For $\omega_m(x,t)=\sum_{l=1}^{m}\psi_l(t)\varphi_l(x)$, where $\psi_l(t)\in C_0^\infty(0,T)$, $\varphi_l(x)\in C_0^\infty(\Omega)$, it holds the relation (2.11). Hence it holds also for any $\omega(x,t)\in C_0^\infty$ $(\Omega\times(0,T))$. This means that the weak solution $u(x,t)$ satisfies the relation (2.8) and moreover

$$\frac{\partial}{\partial t}A^{-\frac{1}{2}}u(x,t)=-\nu A^{\frac{1}{2}}u(x,t)-A^{-\frac{1}{4}}Hu(x,t)\qquad \text{a.e. } (x,t)\in\ \Omega\times(0,T).$$

The proof of the theorem is complete.

References

[1] D. C. J. BURGESS: *Abstract moment problems with applications to the l^p and L^p spaces,* Proc. London Math. Soc. (3), 4(1954), 107-128.

[2] H. FUJITA and T. KATO: *On the Navier-Stokes initial value problem. I,* Arch. Rational Mech. and Anal., 16(1964), 269-315.

[3] E. HOPF: *Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen,* Math. Nachr., 4(1951), 213-231.

[4] S. ITO: *The existence and the uniqueness of regular solution of nonstationary Navier-Stokes equation,* J. Fac. Sci., Univ. Tokyo, Sec. I, 9(1961), 103-140.

[5] A. A. KISELEV and O. A. LADYZHENSKAIA: *On existence and uniqueness of the solution of the non-stationary problem for any incompressible viscous fluid,* Izv. Akad. Nauk SSSR, 21(1957), 655-680.

[6] O. A. LADYZHENSKAIA: Mathematical problems for dynamics of viscous incompressible fluids, Moscow, 1961.
The mathematical theory of viscous incompressible flow, Eng. tr., Gordon and Breach, 1963.

[7] W. SIBAGAKI and H. RIKIMARU: *On the E. Hopf's weak solution of the initial value problem for the Navier-Stokes equations,* Mem. Fac. Sci., Kyushu Univ. Ser. A, 21(1967), 194-240.

[8] H. RIKIMARU: *On the (x,t)-measurability of the function $P\Delta u(x,t)$ related to the Navier-Stokes initial value problem,* Mem. Fac. Sci., Kyushu Univ. Ser. A, 22(1968), 46-55.

[9] P. E. SOBOLEVSKII: *On non-stationary equations of hydrodynamics for viscous fluid,* Dokl. Akad. Nauk SSSR, 128 (1959), 45-48.