# On the existence of bounded solution for nonlinear evolution equation of parabolic type

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# On the existence of bounded solution for nonlinear evolution equation of parabolic type<sup>(\*)</sup>

Mitsuhiro NAKAO (Received December 1, 1976)

#### §1. Introduction

Let V and W be two real separable reflexive Banach spaces and H be real Hilbert space with  $V \subset W \subset H$ . Let V be dense in W and in H and let the natural injections from V into W and from W into H be compact and continuous, respectively. In this paper we shall consider the following abstract evolution equation of parabolic type:

$$u'(t) + Au + Bu = f(t),$$
 (1.1)

where A is the Freshet derivative of a continuous convex functional  $F_A(u)$  on V and B is the one of a continuous functional  $F_B(u)$  on W. Precise conditions on them will be given in §2.

The aim of this paper is to give sufficient conditions under which a bounded solution on  $R^+=[0,\infty]$  for (1.1) with initial value  $u(0)=u_0 \in V$  or a bounded solution on  $R=(-\infty,\infty)$  for (1.1) without initial condition exists.

By our assumption A is a monotone operator from W into  $W^*$  (the dual space of W). If we assume moreover B is monotonic, related problems have been considered by several authors: L. Amerio G. Prouse [1], M Biroli [2, 3, 4], J. L. Lions [5] and others. However if B is not monotonic our problem seems to be unsolved and here we shall treat such case.

In a previous paper [6] the present author has investigated the boundedness, periodicity, and almost-periodicity of solutions of heat equations with nonlinear (possibly non monotonic) terms, but in case the principal parts are linear. There the linearity of the principal parts has played an essential role and the differential inequalities have been used effectively. As a matter of course that method is not applicable to our problem and we must employ another approach. Here some integral inequalities will be used

<sup>(\*)</sup> The result of this paper was announced at the spring conference of the mathematical society of Japan at Osaka University in April, 1975.

to obtain a bounded solution for (1.1). A typical example of our equation is:

$$\frac{\partial}{\partial t}u - \sum_{i=1}^{n} \frac{\partial}{\partial x_{i}} \left( \left| \frac{\partial}{\partial x_{i}} u \right|^{p-2} \frac{\partial}{\partial x_{i}} u \right) + \beta(x, u) = f(x, t)$$
(1.2)

with  $u|_{\partial g}=0$ , where  $\Omega$  is an open bounded domain in the *n*-dimensional Euclidean space  $R^n$  and  $\partial \Omega$  its boundary.

Regarding the initial-boundary problem for (1.2) Tsutsumi [8] gave existence and non-existence theorems concerning global non-negative solution in the case  $f \equiv 0$  and  $\beta(x, u) = \pm u^{1+\alpha}$ ,  $\alpha \ge 0$ . In [8] so-called the method of 'potential well' was used, and it is easy to see that the method is also available to our problem if we assume  $\int_{-\infty}^{\infty} ||f(t)||^2_{L^2(\mathfrak{g})} dt$  is sufficiently small. But this assumption is too restrictive and not so meaningful for the existence of bounded solution because all the periodic (in time) functions except trivial one do not satisfy this condition. Our method here will require instead of this that  $\sup_{t} \int_{t}^{t+1} ||f(s)||^2_{L^2(\mathfrak{g})} ds$  is small. This improvement seems to be important because it enables us to proceed to the research of periodic solutions for (1.1). In general under appropriate conditions bounded solutions of differential equations become periodic (almost-periodic) if the data are so. However we do not know at this time if our bounded solution for (1.1) is periodic or not, and this problem is open for future research.

#### §2. Preliminaries and results

Regarding A, B,  $F_A$ ,  $F_B$ , and f we shall assume that the following conditions are satisfied.

 $H_1$ .

A is the hemicontinuous Fréshet derivative of a continuous convex functional  $F_A(u)$  on V with the properties:

$$C_0 \|u\|_V^p \leq (Au, u) \leq C_1 \|u\|_V^p \qquad \text{for } u \in V$$

and

$$C_2 \|u\|_{\mathcal{V}}^p \leq F_A(u) \leq C_2 \|u\|_{\mathcal{V}}^p \qquad \text{for } u \in V,$$

where  $C_0$ ,  $C_1$ ,  $C_2$ ,  $C_3$  and  $p(\geq 2)$  are constants,  $\|\cdot\|_V$  denotes the norm in V, and (, ) denotes the relationship between  $V^*$  and V etc..  $H_2$ .

B is the continuous Fréshet derivative of a continuous functional  $F_B$  on

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W with the properties

 $||Bu_{W*}|| \leq k_0 ||u||_{a+1}^w \quad \text{for } u \in W$ 

and

$$|F_B(u)| \le k_1 ||u||_{a+2}^{w} \quad \text{for } u \in W$$

for some constants  $k_0$ ,  $k_1$  and  $\alpha(\geq 0)$ .  $H_3$ .

 $f \in S^2$  (R;H), that is,  $f \in L^2_{loc}(R;H)$  and  $M = \sup_{t \in R} \int_t^{t+1} ||f(s)||_H^2 ds < +\infty$ .

Now we state our definitions of bounded solutions.

DEFINITION 1. Let  $u_0 \in V$ . A function u (t) on V is said to be a bounded solution for (1.1) with initial value  $u_0$  if the following conditions are fulfilled:

(i) 
$$u \in L^{\infty}(R^+; V)$$
  $(R^+ \equiv [0, \infty))$   
(ii)  $Au \in L^{\infty}(R^+; V^*)$   
(iii)  $u' \in S^2(R^+; H)$   $\left(' \equiv \frac{d}{dt}\right)$ 

(iv) 
$$u'(t) + Au(t) + Bu(t) = f(t)$$
 for almost all  $t \in R^+$  (in  $V^*$ ) and

 $u(0) = u_0.$ 

For notation see e.g. Lions [5].

DEFINITION II. A function u(t) on V is said to be a bounded solution on R for (1.1) if the conditions (i)—(iv) above with  $R^+$  replaced by R are fulfilled except that  $u(0)=u_0$ .

To state our result some preparations are needed. We shall introduce certain functionals on V and some specific notations. Let us define functionals  $J_0(u)$ ,  $J_1(u)$ ,  $\tilde{J}_0(u)$  and  $\tilde{J}_1(u)$  on V as follows.

$$J_{0}(u) = (Au, u) + (Bu, u),$$
  

$$J_{1}(u) = F_{A}(u) + F_{B}(u),$$
  

$$\tilde{J}_{0}(u) = C_{0} ||u||\tilde{r} - k_{0}S^{\alpha+2} ||u||\tilde{r}^{+2},$$
  

$$\tilde{J}_{1}(u) = C_{2} ||u||\tilde{r} - k_{1}S^{\alpha+2} ||u||\tilde{r}^{+2}.$$

Here S denotes the imbedding constant from V into W, that is, the minimum constant satisfying

$$\|\boldsymbol{u}\|_{\boldsymbol{W}} \leq S \|\boldsymbol{u}\|_{\boldsymbol{V}}$$
 for  $\boldsymbol{u} \in \boldsymbol{V}$ .

By our assumptions it follows easily that

 $\tilde{J}_0(u) \leq J_0(u)$  and  $\tilde{J}_1(u) \leq J_1(u)$ .

For a moment let p < a+2. Associated with above functionals we determine  $D_0$  and  $D_1$  as follows:

$$D_{0} = \max_{x \ge 0} (C_{0}x^{p} - k_{0}S^{a+2}x^{a+2})$$
  

$$\equiv C_{0}\lambda_{0}^{p} - k_{0}S^{a+2}k_{0}^{a+2},$$
  

$$D_{1} = \max_{x \ge 0} (C_{2}x^{p} - k_{1}S^{a+2}x^{a+2})$$
  

$$\equiv C_{2}\lambda_{1}^{p} - k_{1}S^{a+2}\lambda_{1}^{a+2}.$$

where

$$\lambda_{0} = \left(\frac{PC_{0}}{k^{0}(\alpha+2)S^{\alpha+2}}\right)^{1/(\alpha+2-p)},$$
$$\lambda_{1} = \left(\frac{PC_{2}}{k_{1}(\alpha+2)S^{\alpha+2}}\right)^{1/(\alpha+2-p)}.$$

Put

$$\mathscr{W} = \{ u \in V | J_1(u) < D_1 \quad and \quad \|u\|_V < \lambda_1 \}.$$

The set  $\mathcal{W}$  is closely related to the so called 'potential well' (see Sattinger [7] and Tsutsumi [8]).

Now we are ready to state our Theorems.

THEOREM 1. Let us assume  $H_1 - H_3$  with R replaced by  $R^+$  and let  $p > \alpha + 2$ . Then the initial-value problem (1.1) with initial value  $u_0 \in V$  admits a bounded solution in the sense of Definition 1.

THEOREM 2. Let  $H_1 - H_3$  with R replaced by  $R^+$  be satisfied and let  $2 \le p < \alpha + 2$ . Then for any  $u_0 \in \mathcal{W}$  there exists a constant  $M_0 = M_0$   $(D_1 - J_1 (u_0))$  such that if  $M < M_0$  the problem (1.1) admits a bounded solution u with initial value  $u_0$  in the sense of Definition 1, u satisfying

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{v}} \leq K_1(M) \leq \lambda_1 \text{ for } t \in R^+,$$

where  $K_1(M)$  is a constant depending on M and  $J_1(u_0)$ .

THEOREM 3. Under the hypotheses  $H_1 - H_3$ ,

(i) if  $p > \alpha + 2$  the problem (1.1) admits a bounded solution u for any M in the sense of Definition 2,

and

(ii) if  $2 \le p < \alpha + 2$  the same result holds when  $M < \overline{M}_0 \equiv M_0$  ( $D_1$ ), and moreover u satisfies

$$\|u(t)\|_{\mathcal{V}} \leq K_2(M)$$
 for  $t \in \mathbb{R}$ .

where  $K_2(M)$  is a constant depending on M and tending to 0 (as  $M \rightarrow 0$ )

**REMARK.** The precise value of  $M_0$  in Theorem 2 (or 3) will be given in the proof in §3.

# § 3. Approximate solutions

In this section we shall construct approximate solutions. For this purpose we employ the Galerkin's method of approximation. Let  $\{w_j\}_{\substack{j=1,2,3,\cdots}}$  be a basis of V and consider the following system of ordinary differential equations:

$$(u'_{m}(t), w_{j}) + (Au_{m}(t), w_{j}) + (Bu_{m}(t), w_{j}) = (f(t), w_{j}), \qquad (3.1)$$

 $j=1, 2, 3, \ldots, m$ , with initial condition

$$u_m(0) = \sum_{j=1}^m \alpha_j(0) w_j, \qquad (3.2)$$

where

$$u_m(t) = \sum_{j=1}^m \alpha_j(t) w_j.$$

Here initial values  $u_m(0)$ ,  $m=1, 2, 3, \ldots$ , are chosen so that  $u_m(0) \rightarrow u_0$  strongly in V as  $m \rightarrow \infty$ . The standard theory of ordinary differential equations ensures that a solution  $u_m(t)$  of (3.1), (3.2) exists on an interval, say,  $[0, t_m]$ . To prove  $u_m(t)$  exists on  $[0, \infty)$  we must obtain a priori estimate of  $u_m(t)$ .

LEMMA 3.1. Let  $p < \alpha + 2$ . Then  $u_m(t)$  exists on  $R^+$  and the estimate

$$\|u_m(t)\|_{\mathcal{V}} \leq K_0(M, J_1(u_0)) \tag{3.3}$$

holds, where  $K_0(M, J_1(u_0))$  is a constant depending on M and  $J_1(u_0)$  but independent of m and t.

**PROOF.** In order to prove the lemma it suffices to show (3.3) for  $\forall t \in [0, t_m]$ . Multiplying (3.1) by  $\alpha_j'(t)$  and summing over j from 1 to m we obtain

$$\int_{0}^{t} (||u'_{m}(s)||_{H}^{2} + (Au_{m}(s), u'_{m}(s)) + (Bu_{m}(s), u'_{m}(s)))ds$$
  
= 
$$\int_{0}^{t} (f(s), u'_{m}(s))ds.$$
 (3.4)

Since  $(Au_m, u'_m) = \frac{d}{dt} F_A(u_m(t))$  and  $(Bu_m, u'_m) = \frac{d}{dt} F_B(u_m(t))$  it follows from (3.4) that

$$\int_{0}^{t} \|u'_{m}(s)\|_{H}^{2} ds + J_{1}(u_{m}(t)) = J_{1}(u_{m}(0)) + \int_{0}^{t} (f(s), u'_{m}(s)) ds \qquad (3.5)$$

and hence, by Young's inequality,

$$J_1(u_m(t)) \le J_1(u_m(0)) + \frac{1}{4}M^2 \quad \text{for} \quad t \le \min(t_m, 1).$$
 (3.6)

Therefore our assumptions  $H_1$  and  $H_2$  together with  $p > \alpha + 2$  yield

 $\|u_m(t)\|_V \le C_4(M, J_1(u_0))$  for  $0 \le t \le \min(t_m, 1)$ , (3.7)

where  $C_{i}(M, J_{1}(u_{0}))$  is a constant depending on M and  $J_{1}(u_{0})$  and other constants but independent of m and t. (In what follows we denote by  $C_{i}(Q)$ ,  $i = 4, 5, 6, \ldots$ , constants depending on Q)

By (3.7) we may assume  $t_m > 1$  and obtain

$$\|u_m(t)\|_V \le C_4(M, J_1(u_0)) \quad \text{for } t \in [0, 1].$$
(3.7)'

We shall show that there exists a constnat  $C_5(M, J_1(u_0))$  such that

$$J_1(u_m(t)) \le \max(\max_{t \in [0,1]} J_1(u_m(t)), C_5(M)), \quad t \le t_m,$$
(3.8)

which will give (3,3) by changing the notation. For (3,8) it suffices to prove

$$J_1(u_m(\bar{t}+1)) \le \max(J_1(u_m(\bar{t}), C_5(M)))$$
(3.9)

for  $\forall \bar{t} \leq t_m - 1$ . This is trivial if  $J_1(u_m(\bar{t}+1)) \leq J_1(u_m(\bar{t}))$ , and we assume  $J_1(u_m(\bar{t}+1)) \geq J_0(u_m(\bar{t}))$ . Then as in (3.5) we have

$$\int_{\overline{\iota}}^{\overline{\iota}+1} \|u'_{m}(s)\|_{H}^{2} ds \leq \int_{\overline{\iota}}^{\overline{\iota}+1} (f(s), u'_{m}(s)) ds$$
(3.10)

and

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$$\int_{\overline{\iota}}^{\overline{\iota}+1} \|u'_{m}(s)\|_{H}^{2} ds \leq \int_{\overline{\iota}}^{\overline{\iota}+1} \|f(s)\|_{H}^{2} ds \leq M^{2}.$$
(3.11)

On the other hand multiplication of (3.1) by  $\alpha_j(t)$ , summation over j and integration from  $\bar{t}$  to  $\bar{t}+1$  give

$$\begin{split} \int_{\overline{\tau}}^{\overline{\tau}+1} J_0(u_m(s)) ds &= -\int_{\overline{\tau}}^{\overline{\tau}+1} (u'_m(s), \ u_m(s)) ds + \int_{\overline{\tau}}^{\overline{\tau}+1} (f(s), u_m(s)) ds \\ &\leq \int_{\overline{\tau}}^{\overline{\tau}+1} ||u'_m(s)||_H ||u_m||_H ds + \int_{\overline{\tau}}^{\overline{\tau}+1} ||f(s)_H||u_m(s)_H ds \\ &\leq M^2 + \int_{\overline{\tau}}^{\overline{\tau}+1} ||u_m(s)||_H^2 ds \quad (by \ (3.11)), \end{split}$$

and .

$$\int_{\overline{\iota}}^{\overline{\iota}+1} \{ J_0(u_m(s)) - \|u_m(s)\|_{H}^2 \} ds \leq M^2.$$
(3.12)

Hence by (3.12) there exists a number  $t^* \in [t, t+1]$  such that

$$J_0(u_m(t^*)) - \|u_m(t^*)\|_H^2 \leq M^2.$$
(3.13)

The inequality (3.13) together with the assumption  $p > \alpha + 2$  implies

$$\|\boldsymbol{u}_{m}(t^{*})\|_{\boldsymbol{V}} \leq C_{6}(M). \tag{3.14}$$

Thus as in (3.5) we have

$$J_{1}(u_{m}(\bar{t}+1) = -\int_{\iota^{*}}^{\bar{\iota}+1} ||u'_{m}(s)||_{H}^{2} ds + J_{1}(u_{m}(t^{*})) + \int_{\iota^{*}}^{\bar{\iota}+1} (f(s), u'_{m}(s)) ds$$

$$\leq -\frac{1}{4} M^{2} + J_{1}(u_{m}(t^{*}))$$

$$\leq -\frac{1}{4} M^{2} + C_{3} ||u_{m}(t^{*})||_{F}^{p} + k_{1} S^{a+2} ||u_{m}(t^{*})||_{F}^{a+2}$$

$$\leq C_{7}(M) \text{ (by (3.14)).}$$

Since  $C_7(M)$  is independent of *m* and  $\bar{t}$ , the inequality (3.9) and consequently (3.8) are now proved with  $C_{\mathfrak{s}}(M) = C_7(M)$ . q.e.d.

In case  $p < \alpha + 2$  an argument somewhat complicated is needed.

LEMMA 3.2 Let  $2 \le p \le \alpha + 2$ . Then for  $u_0 \in \mathcal{W}$  there exist constants  $M_0 = M_0(D_1 - J_1(u_0))$  and  $m_0(\varepsilon)$  such that if  $M \le M_0$   $u_m(t)$  with  $m > m_0$  is defined on  $[0, +\infty)$  and the following estimate holds:

 $\|u_m(t)\|_V \leq K_1(M, \varepsilon) < \lambda_1$  for t and for  $m > m_0(\varepsilon)$ , (3.15) where  $\varepsilon$  is a small constant and  $K_1(M, \varepsilon)$  is a certain constant depending on M and  $\varepsilon$ .

**PROOF.** We shall show that there exists a constant  $M_0$  such that if  $M < M_0$  and  $m > m_0$  we have

$$(\tilde{J}_1(u_m(t)) \le) J_1(u_m(t)) < D_1, \quad t \le t_m.$$
 (3.16)

Then this inequality together with the initial condition  $||u(0)||_{\nu} = ||u_0||_{\nu} < \lambda_1$  will imply  $||u_m(t)||_{\nu} < \lambda_1, t \le t_m$ , and the former part of Lemma 3.2 will follow. Also (3.15) will be seen from the procedure of the proof of (3.16).

Let us begin with showing  $t_m>1$ . By (3.6) we know for  $M < M'_0 = 2\sqrt{D_1 - J_1(u_0)}$ 

$$J_1(u_m(t)) \le J_1(u_m(0)) + \frac{1}{4}M^2$$
(3.17)

$$\leq J_1(u_0) + \frac{1}{4}M^2 + \varepsilon < D_1$$
 ( $\varepsilon$ : small)

for  $0 \le t \le \min(1, t_m)$  and for  $m > m_0(\varepsilon)$ . Here we have used the fact that  $J_1(u_m(0))$  may be assumed to be as close to  $J_1(u_0)$  as one wants because  $J_1(u)$  is continuous with respect to u in the norm  $\|\cdot\|_V$ . Hereafter we assume  $M < M'_0$ . Then the inequality (3.17) implies  $t_m > 1$ . Let us assume (3.16) was false. Then there would exist a number  $t_1(>1)$  such that  $J_1(u_m(t_1)) = D_1$  and  $J_1(u_m(t)) < D_1$  for  $t < t_1$ . Thus by the same argument as in (3.5) we have

$$\int_{t_1-1}^{t_1} ||u'_m(s)||_{\dot{H}}^2 ds + J_1(u_m(t_1)) = J_1(u_m(t_1-1)) + \int_{t_1-1}^{t_1} (f(s), u'_m(s)) ds$$

and hence by the definition of  $t_1$ 

$$\int_{t_1-1}^{t_1} ||u'_m(s)||_{H}^2 ds \leq \int_{t_1-1}^{t_1} (f(s), u'_m(s)) ds,$$

and moreover

$$\int_{t_1-1}^{t_1} \|u'_m(s)\|_{H}^2 ds \leq \int_{t_1-1}^{t_1} (f(s), u'_m(s)) ds, M^2.$$

Also as in (3.12) we have

$$\int_{t_{1}-1}^{t_{1}} J_{0}(u_{m}(s)) ds \leq \int_{t_{1}-1}^{t_{1}} (\|u'_{m}(s)\|_{H} + \|f(s)\|_{H}) \|u_{m}(s)\|_{H} ds$$

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$$\leq 2M \max_{t \in [t_1^{-1}, t_1]} \|u_m(t)\|_H \leq 2MS_1 \lambda_1, \qquad (3.18)$$

where  $S_1$  is the imbedding constant from V into H. Therefore there exists a time  $t^* \in [t_1-1, t_1]$  such that

$$\tilde{J}_0(\boldsymbol{u}_m(t^*)) \leq J_0(\boldsymbol{u}_m(t^*)) \leq 2MS_1\lambda_1$$

and hence, if we choose  $M < M_0^* = D_0/2S_1\lambda_1$ ,

$$\|u_m(t^*)\|_V \leq r_0(M),$$
 (3.19)

where  $r_0(M)$  is the smaller root of the numerical equation

 $C_0 x^p - k_0 S^{\alpha+2} x^{\alpha+2} = 2M S_1 \lambda_1, x \ge 0.$ 

Note that  $r_0(M) \rightarrow 0$  as  $M \rightarrow 0$ . Now from (3.5) with t and 0 replaced by  $t_1$  and  $t^*$ , respectively, we have

$$\int_{t^{*}}^{t_{1}} ||u'_{m}(s)||_{\underline{H}}^{2} ds + J_{1}(u_{m}(t_{1})) = J_{1}(u_{m}(t^{*})) + \int_{t^{*}}^{t_{1}} (f, u'_{m}) ds$$

$$\leq C_{3} ||u_{m}(t^{*})||_{\underline{F}}^{2} + k_{1}S^{\alpha+2} ||u_{m}(t^{*})||_{\underline{F}}^{\alpha+2}$$

$$+ \frac{1}{4} \int_{t^{*}}^{t_{1}} ||f(s)||_{\underline{H}}^{2} ds + \int_{t^{*}}^{t_{1}} ||u'_{m}(s)||_{\underline{H}}^{2} ds,$$

and hence

$$D_{1} = J_{1}(u_{m}(t_{1})) \leq C_{3}r_{0}(M)^{p} + k_{1}S^{a+2}r_{0}(M)^{a+2} + \frac{1}{4}M^{2}$$
$$\equiv K_{2}(M).$$
(3.20)

This is a contradiction if we choose M < M''',  $M_0'''$  being the smallest constant such that

$$K_2(M_0''') = D_1,$$

which is possible since  $K_2(M)$  tends to 0 and  $\infty$  as  $M \rightarrow 0$  and  $\rightarrow \infty$ , respectively. Thus if we choose  $M_0$  as

$$M_0 = \min(M_0', M_0'', M_0'''),$$

the inequality (3.17) is valid for  $M < M_0$ . Also (3.15) follows evidently if we define

$$K_1(M,\varepsilon) \equiv \max(J_1(u_0) + \varepsilon + \frac{1}{4}M^2, K_2(M)).$$

q. e. d.

## §4. Proofs of theorems

Theorems 1 and 2 can be proved by the same way on the basis of Lemmas 3.1 and 3.2, respectively, and we shall carry out the proofs simultaneously.

Let  $\{u_m(t)\}\$  be approximate solutions constructed in §3.  $\{u_m(t)\}\$  exist and satisfy a priori estimates (3.3), (3.15) under our assumptions. From these estimates we can obtain easily

$$\sup_{\iota \in \mathbb{R}^+} \int_{\iota}^{\iota+1} \|u'_{m}(s)\|_{H}^{2} ds \leq C_{s}(M) < +\infty.$$
(4.1)

Then by standard compactness and monotonicity arguments (Lions [5]) we can extract a subsequence from  $\{u_m(t)\}$ , which will be denoted by the same symbol, such that

$$u_m(t) \rightarrow u(t)$$
 weakly star in  $L^{\infty}_{loc}(R^+; V)$ , (4.2)

$$u'_m(t) \rightarrow u'(t)$$
 weakly in  $L^2_{loc}(R^+;H)$ , (4.3)

$$u_m(t) \to u(t) \quad \text{strongly in } L^r{}_{loc}(R^+; W), r \ge 1, \tag{4.4}$$

$$u_m(t) \rightarrow u(t)$$
 a.e. in  $W$ , (4.5)

$$Au_m(t) \rightarrow Au(t)$$
 weakly star in  $L^{\infty}_{loc}(R^+, V^*)$ , (4.6)

$$Bu_m(t) \rightarrow Bu(t)$$
 weakly star in  $L^{\infty}_{loc}(R^+; W^*)$ , (4.7)

and the limit function u(t) satisfies

$$u'(t) + Au(t) + Bu(t) = f(t)$$
 a.e. in V\*

and

$$u(0) = u_0$$
.

Moreover the inequalities (3.3) (or (3.15) with  $K_1(M) = K_1(M, 0)$ ) and (4.1) remain valid for  $u_m = u$ . Thus the proofs of Theorems 1 and 2 are completed.

Next we shall proceed to the proof of Theorem 3. Consider the system of ordinary differential equations:

$$(u'_{m,r}(t), w_j) + (Au_{m,r}(t), w_j) + (Bu_{m,r}(t), w_j) = (f, w_j)$$
(4.8)

on  $[-r,\infty)$  with initial condition

$$u_{m,r}(-r) = 0,$$
 (4.9)

where

$$u_{m,r} = \sum_{j=1}^{m} \alpha_{j,r}(t) w_j.$$

Then we know by Lemmas 3.1 and 3.2 that if  $p > \alpha + 2$  we have

$$\|u_{m,r}(t)\|_{v} \leq K_{0}(M,0), \quad -r \leq t < \infty,$$
 (4.10)

and if  $2 \le p\alpha + 2$  we have for  $M < \overline{M}_0 \equiv M_0(D_1)$ 

$$||u_{m,r}(t)_{V} \leq K_{2}(M), \quad -r \leq t = \infty, \qquad (4.11)$$

where we recall

$$K_2(M) = C_3 r_0(M)^p + k_1 S^{\alpha+2} r_0(M)^{\alpha+2} + \frac{1}{4} M^2.$$

In both cases we have

$$\sup_{t \ge -r} \int_{t}^{t+1} ||u_{m,r}(s)||_{H}^{2} ds \le C_{g}(M) < +\infty.$$
(4.12)

From (4.10), (4.11) and (4.12) we conclude that after appropriate prolongations of  $u_{m,r}(t)$  and  $u'_{m,r}(t)$  for t < -r, we can extract subsequence of  $\{u_{m,r}(t)\}$  with respect to r (r=1, 2, 3, 4, ...) to obtain (4.2)—(4.7) with  $u_m(t)$ , u(t), m, and  $R^+$  replaced by  $u_{m,r}, u_m(t)$ , r and R, respectively, where  $u_m$ (t) is the limit function of  $u_{m,r}(t)$  as  $r \to \infty$ . Since the estimates (4.10), (4.11) and (4.12) remain valid for  $u_{m,r}(t)=u_m(t)$  on  $t \in R$ , we can extract once more again a subsequence from  $\{u_m(t)\}$  which satisfies the same convergency properties as in (4.2)—(4.7) with  $R^+$  replaced by R. Then the limit function u(t) becomes the required bounded solution on  $(-\infty, \infty)$ .

#### § 5. An example

As was mentioned in the introduction let us cosider the equation;

$$\frac{\partial}{\partial t}u - \sum_{i=1}^{n} \frac{\partial}{\partial x_{i}} \left( \left| \frac{\partial}{\partial x_{i}} u \right|^{p-2} \frac{\partial}{\partial x_{i}} u \right) + \beta(x, u) = f(x, t), \quad x \in \mathcal{Q}, \quad (5.1)$$

together with boundary condition

$$\boldsymbol{u}|_{\boldsymbol{\partial} \boldsymbol{\mathcal{G}}} = 0, \tag{5.2}$$

where  $\Omega$  is an open bounded domain in  $R_n$  and  $\partial \Omega$  is its boundary. We assume  $\sup_{t} \int_{t}^{t+1} ||f(\cdot, s)||^2 L^{2}(\Omega) ds \equiv M < +\infty$ , and  $\beta(x, u)$  is defined on  $\Omega \times R$  and measurable in x for each u and continuous in u for each x and satisfies

 $|\beta(x,u)| \le k_0 |u|^{\alpha+1} \text{ with } 0 \le \alpha < np/(n-p)-2 \text{ if } n > p > 2$ and  $0 < \alpha < +\infty$  if  $1 \le n \le p$ .

Put

$$V = W_0^{1,p}(\mathcal{Q}), \quad W = L^{\alpha+2}(\mathcal{Q}), \quad H = L^2(\mathcal{Q}),$$
$$Au = -\sum_{i=1}^n \frac{\partial}{\partial x_i} \left( \left| \frac{\partial}{\partial x_i} u \right|^{p-2} \frac{\partial}{\partial x_i} u \right) \text{ and } Bu = \beta(x, u).$$

Note that

$$||u||_{W_0^{1,p}(g)} = \left(\sum_{i=1}^n \int_g \left|\frac{\partial}{\partial x_i}u\right|^p dx\right)^{\frac{1}{p}}.$$

Then all the hypotheses in §2 are satisfied in this case with  $C_0 = C_1 = 1$ ,  $C_2 = C_3 = \frac{1}{p}$  and  $k_1 = \frac{1}{\alpha + 2}$ . S and  $S_1$  are defined as the Sobolev constants of the imbeddings from  $W_0^{1,p}$  into  $L^{*+2}(\mathcal{Q})$  and  $L^2(\mathcal{Q})$ , respectively. Thus our results are applicable directly to (5.1) - (5.2).

## FINAL REMARK.

In case  $p=\alpha+2$  the conclusions of Theorems 2 and 3 are valid under the assumption that S and S<sub>1</sub> are sufficiently small.

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