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On the existence of bounded solution for nonlinear evolution equation of parabolic type(*)

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§ 1. Introduction

Let V and W be two real separable reflexive Banach spaces and H be real Hilbert space with $V\subset W\subset H$. Let V be dense in W and in H and let the natural injections from V into W and from W into H be compact and continuous, respectively. In this paper we shall consider the following abstract evolution equation of parabolic type:

$$u'(t)+Au+Bu=f(t), \tag{1.1}$$

where A is the Freshet derivative of a continous convex functional $F_A(u)$ on V and B is the one of a continuous functional $F_B(u)$ on W. Precise conditions on them will be given in § 2.

The aim of this paper is to give sufficient conditions under which a bounded solution on $R^+=[0,\infty]$ for (1.1) with initial value $u(0)=u_0\in V$ or a bounded solution on $R=(-\infty,\infty)$ for (1.1) without initial condition exists.

By our assumption A is a monotone operator from W into W^* (the dual space of W). If we assume moreover B is monotonic, related problems have been considered by several authors: L. Amerio G. Prouse [1], M Biroli [2, 3, 4], J. L. Lions [5] and others. However if B is not monotonic our problem seems to be unsolved and here we shall treat such case.

In a previous paper [6] the present author has investigated the boundedness, periodicity, and almost-periodicity of solutions of heat equations with nonlinear (possibly non monotonic) terms, but in case the principal parts are linear. There the linearity of the principal parts has played an essential role and the differential inequalities have been used effectively. As a matter of course that method is not applicable to our problem and we must employ another approach. Here some integral inequalities will be used

(*) The result of this paper was announced at the spring conference of the mathematical society of Japan at Osaka University in April, 1975.

to obtain a bounded solution for (1. 1). A typical example of our equation is:

$$\frac{\partial}{\partial t}u-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial}{\partial x_i}u\right|^{p-2}\frac{\partial}{\partial x_i}u\right)+\beta(x,u)=f(x,t) \tag{1.2}$$

with $u|_{\partial\Omega}=0$, where Ω is an open bounded domain in the n-dimensional Euclidean space R^n and $\partial\Omega$ its boundary.

Regarding the initial-boundary problem for (1. 2) Tsutsumi [8] gave existence and non-existence theorems concerning global non-negative solution in the case $f\equiv 0$ and $\beta(x,u)=\pm u^{1+\alpha}$, $\alpha\geq 0$. In [8] so-called the method of 'potential well' was used, and it is easy to see that the method is also available to our problem if we assume $\int_{-\infty}^{\infty}\|f(t)\|^2_{L^2(\Omega)}\,dt$ is sufficiently small. But this assumption is too restrictive and not so meaningful for the existence of bounded solution because all the periodic (in time) functions except trivial one do not satisfy this condition. Our method here will require instead of this that $\sup_t\int_t^{t+1}\|f(s)\|^2_{L^2(\Omega)}\,ds$ is small. This improvement seems to be important because it enables us to proceed to the research of periodic solution for (1. 1). In general under appropriate conditions bounded solutions of differential equations become periodic (almost-periodic) if the data are so. However we do not know at this time if our bounded solution for (1. 1) is periodic or not, and this problem is open for future research.

§ 2. Preliminaries and results

Regarding A, B, F_A, F_B, and f we shall assume that the following conditions are satisfied.

H_1.

A is the hemicontinuous Fréshet derivative of a continuous convex functional $F_A(u)$ on V with the properties:

$$C_0\|u\|_V^p\leq(Au,u)\leq C_1\|u\|_V^p \qquad \text{for } u\in V$$

and

$$C_2\|u\|_V^p\leq F_A(u)\leq C_2\|u\|_V^p \qquad \text{for } u\in V,$$

where C_0, C_1, C_2, C_3 and $p(\geq 2)$ are constants, $\|\cdot\|_V$ denotes the norm in V, and (,) denotes the relationship between V^* and V etc..

H_2.

B is the continuous Fréshet derivative of a continuous functional F_B on

W with the properties

$$\|Bu_{W*}\| \leq k_0 \|u\|^{W}_{\alpha+1} \qquad \text{for } u \in W$$

and

$$|F_B(u)| \leq k_1 \|u\|^{W}_{\alpha+2} \qquad \text{for } u \in W$$

for some constants k_0, k_1 and $\alpha(\geq 0)$.

H_3.

$f \in S^2\ (R; H)$, that is, $f \in L^2_{loc}(R; H)$ and $M \equiv \sup_{t \in R} \int_t^{t+1} \|f(s)\|_H^2 ds < +\infty$.

Now we state our definitions of bounded solutions.

DEFINITION 1. Let $u_0 \in V$. A function u (t) on V is said to be a bounded solution for (1.1) with initial value u_0 if the following conditions are fulfilled:

(i) $u \in L^\infty(R^+; V)$ $\qquad (R^+ \equiv [0, \infty))$

(ii) $Au \in L^\infty(R^+; V^*)$

(iii) $u' \in S^2(R^+; H)$ $\qquad \left(' \equiv \frac{d}{dt}\right)$

(iv) $u'(t) + Au(t) + Bu(t) = f(t)$ for almost all $t \in R^+$ (in V^*) and

$u(0) = u_0$.

For notation see e. g. Lions [5].

DEFINITION II. A function $u(t)$ on V is said to be a bounded solution on R for (1.1) if the conditions (i)—(iv) above with R^+ replaced by R are fulfilled except that $u(0) = u_0$.

To state our result some preparations are needed. We shall introduce certain functionals on V and some specific notations. Let us define functionals $J_0(u)$, $J_1(u)$, $\tilde{J}_0(u)$ and $\tilde{J}_1(u)$ on V as follows.

$$J_0(u) = (Au, u) + (Bu, u),$$
$$J_1(u) = F_A(u) + F_B(u),$$
$$\tilde{J}_0(u) = C_0\|u\|_V^p - k_0 S^{\alpha+2} \|u\|_V^{\alpha+2},$$
$$\tilde{J}_1(u) = C_2\|u\|_V^p - k_1 S^{\alpha+2} \|u\|_V^{\alpha+2}.$$

Here S denotes the imbedding constant from V into W, that is, the minimum constant satisfying

$$\|u\|_W \leq S \|u\|_V \text{ for } u \in V.$$

By our assumptions it follows easily that

$$\tilde{J}_0(u) \leq J_0(u) \quad \text{and} \quad \tilde{J}_1(u) \leq J_1(\mathrm{u}).$$

For a moment let $p<\alpha+2$. Associated with above functionals we determine D_0 and D_1 as follows:

$$\begin{aligned} D_0 &= \max_{x\geq 0} (C_0 x^p - k_0 S^{\alpha+2} x^{\alpha+2}) \\ &\equiv C_0 \lambda_0^p - k_0 S^{\alpha+2} k_0^{\alpha+2}, \\ D_1 &= \max_{x\geq 0} (C_2 x^p - k_1 S^{\alpha+2} x^{\alpha+2}) \\ &\equiv C_2 \lambda_1^p - k_1 S^{\alpha+2} \lambda_1{}^{\alpha+2}, \end{aligned}$$

where

$$\lambda_0 = \left(\frac{PC_0}{k^0(\alpha+2)S^{\alpha+2}}\right)^{1/(\alpha+2-p)},$$

$$\lambda_1 = \left(\frac{PC_2}{k_1(\alpha+2)S^{\alpha+2}}\right)^{1/(\alpha+2-p)}.$$

Put

$$\mathscr{W} \equiv \{u \in V \mid J_1(u) < D_1 \quad \textit{and} \quad \|u\|_V < \lambda_1\}.$$

The set $\mathscr{W}$ is closely related to the so called 'potential well' (see Sattinger [7] and Tsutsumi [8]).

Now we are ready to state our Theorems.

THEOREM 1. *Let us assume H_1—H_3 with R replaced by R^+ and let $p>\alpha+2$. Then the initial-value problem* (1.1) *with initial value $u_0 \in V$ admits a bounded solution in the sense of Definition* 1.

THEOREM 2. *Let H_1—H_3 with R replaced by R^+ be satisfied and let $2 \leq p < \alpha+2$. Then for any $u_0 \in \mathscr{W}$ there exists a constant $M_0 \equiv M_0\,(D_1 - J_1(u_0))$ such that if $M<M_0$ the problem* (1.1) *admits a bounded solution u with initial value u_0 in the sense of Definition* 1, *u satisfying*

$$\|u(t)\|_V \leq K_1(M) \leq \lambda_1 \quad \textit{for } t \in R^+,$$

where $K_1(M)$ is a constant depending on M and $J_1(u_0)$.

THEOREM 3. *Under the hypotheses H_1—H_3,*
(i) *if $p>\alpha+2$ the problem* (1.1) *admits a bounded solution u for any M in the sense of Definition* 2,

and

(ii) *if* $2 \leq p < \alpha + 2$ *the same result holds when* $M < \bar{M}_0 \equiv M_0(D_1)$, *and moreover u satisfies*

$$\|u(t)\|_V \leq K_2(M) \quad \text{for } t \in R.$$

where $K_2(M)$ *is a constant depending on M and tending to* 0 (*as* $M \to 0$)

REMARK. The precise value of M_0 in Theorem 2 (or 3) will be given in the proof in § 3.

§ 3. Approximate solutions

In this section we shall construct approximate solutions. For this purpose we employ the Galerkin's method of approximation. Let $\{w_j\}_{j=1,2,3,\cdots}$ be a basis of V and consider the following system of ordinary differential equations:

$$(u'_m(t), w_j) + (Au_m(t), w_j) + (Bu_m(t), w_j) = (f(t), w_j), \tag{3.1}$$

$j = 1, 2, 3, \ldots, m$, with initial condition

$$u_m(0) = \sum_{j=1}^{m} \alpha_j(0) w_j, \tag{3.2}$$

where;

$$u_m(t) = \sum_{j=1}^{m} \alpha_j(t) w_j.$$

Here initial values $u_m(0)$, $m = 1, 2, 3, \ldots$, are chosen so that $u_m(0) \to u_0$ strongly in V as $m \to \infty$. The standard theory of ordinary differential equations ensures that a solution $u_m(t)$ of (3.1), (3.2) exists on an interval, say, $[0, t_m]$. To prove $u_m(t)$ exists on $[0, \infty)$ we must obtain a priori estimate of $u_m(t)$.

LEMMA 3.1. *Let* $p < \alpha + 2$. *Then* $u_m(t)$ *exists on* R^+ *and the estimate*

$$\|u_m(t)\|_V \leq K_0(M, J_1(u_0)) \tag{3.3}$$

holds, where $K_0(M, J_1(u_0))$ *is a constant depending on M and* $J_1(u_0)$ *but independent of m and t.*

PROOF. In order to prove the lemma it suffices to show (3.3) for $\forall t\in[0, t_m]$. Multiplying (3.1) by $\alpha_j'(t)$ and summing over j from 1 to m we obtain

$$\int_0^t (\|u'_m(s)\|_H^2+(Au_m(s), u'_m(s))+(Bu_m(s), u'_m(s)))ds$$
$$=\int_0^t (\mathrm{f}(s), u'_m(s))ds. \tag{3.4}$$

Since $(Au_m, u'_m)=\frac{d}{dt}F_A(u_m(t))$ and $(Bu_m, u'_m)=\frac{d}{dt}F_B(u_m(t))$ it follows from (3.4) that

$$\int_0^t \|u'_m(s)\|_H^2 ds+J_1(u_m(t))=J_1(u_m(0))+\int_0^t (f(s), u'_m(s))ds \tag{3.5}$$

and hence, by Young's inequality,

$$J_1(u_m(t))\leq J_1(u_m(0))+\frac{1}{4}M^2 \quad \text{for} \quad t\leq min(t_m, 1). \tag{3.6}$$

Therefore our assumptions H_1 and H_2 together with $p>\alpha+2$ yield

$$\|u_m(t)\|_V\leq C_4(M, J_1(u_0)) \qquad \text{for } 0\leq t\leq \min(t_m, 1), \tag{3.7}$$

where $C_4(M, J_1(u_0))$ is a constant depending on M and $J_1(u_0)$ and other constants but independent of m and t. (In what follows we denote by $C_i(Q)$, $i=4, 5, 6, \ldots$, constants depending on Q)

By (3.7) we may assume $t_m>1$ and obtain

$$\|u_m(t)\|_V\leq C_4(M, J_1(u_0)) \qquad \text{for } t\in[0, 1]. \tag{3.7$'$}$$

We shall show that there exists a constnat $C_5(M, J_1(u_0))$ such that

$$J_1(u_m(t))\leq \max\ (\max_{t\in[0,1]} J_1(u_m(t)), C_5(M)), \quad t\leq t_m, \tag{3.8}$$

which will give (3.3) by changing the notation. For (3.8) it suffices to prove

$$J_1(u_m(\bar{t}+1))\leq \max(J_1(u_m(\bar{t}), C_5(M)) \tag{3.9}$$

for $\forall \bar{t}\leq t_m-1$. This is trivial if $J_1(u_m(\bar{t}+1))\leq J_1(u_m(\bar{t}))$, and we assume $J_1(u_m(\bar{t}+1))\geq J_0(u_m(\bar{t}))$. Then as in (3.5) we have

$$\int_{\bar{t}}^{\bar{t}+1} \|u'_m(s)\|_H^2 ds\leq \int_{\bar{t}}^{\bar{t}+1} (f(s), u'_m(s))ds \tag{3.10}$$

and

$$\int_{\bar{t}}^{\bar{t}+1}\|u'_m(s)\|_H^2 ds \le \int_{\bar{t}}^{\bar{t}+1}\|f(s)\|_H^2 ds \le M^2. \tag{3.11}$$

On the other hand multiplication of (3.1) by $\alpha_j(t)$, summation over j and integration from $\bar{t}$ to $\bar{t}+1$ give

$$\begin{aligned}\int_{\bar{t}}^{\bar{t}+1} J_0(u_m(s))ds &= -\int_{\bar{t}}^{\bar{t}+1}(u'_m(s),\ u_m(s))ds + \int_{\bar{t}}^{\bar{t}+1}(f(s),u_m(s))ds \\ &\le \int_{\bar{t}}^{\bar{t}+1}\|u'_m(s)\|_H\|u_m\|_H ds + \int_{\bar{t}}^{\bar{t}+1}\|f(s)_H\|u_m(s)_H ds \\ &\le M^2 + \int_{\bar{t}}^{\bar{t}+1}\|u_m(s)\|_H^2 ds \quad \text{(by (3.11))},\end{aligned}$$

and.

$$\int_{\bar{t}}^{\bar{t}+1}\{J_0(u_m(s)) - \|u_m(s)\|_H^2\} ds \ \le M^2. \tag{3.12}$$

Hence by (3.12) there exists a number $t^* \in [t, t+1]$ such that

$$J_0(u_m(t^*)) - \|u_m(t^*)\|_H^2 \le M^2. \tag{3.13}$$

The inequality (3.13) together with the assumption $p>\alpha+2$ implies

$$\|u_m(t^*)\|_V \le C_6(M). \tag{3.14}$$

Thus as in (3.5) we have

$$\begin{aligned}J_1(u_m(\bar{t}+1)) &= -\int_{t^*}^{\bar{t}+1}\|u'_m(s)\|_H^2 ds + J_1(u_m(t^*)) + \int_{t^*}^{\bar{t}+1}(f(s), u'_m(s))ds \\ &\le \frac{1}{4}M^2 + J_1(u_m(t^*)) \\ &\le \frac{1}{4}M^2 + C_3\|u_m(t^*)\|_V^p + k_1 S^{\alpha+2}\|u_m(t^*)\|_V^{\alpha+2} \\ &\le C_7(M) \ \text{(by (3.14))}.\end{aligned}$$

Since $C_7(M)$ is independent of m and $\bar{t}$, the inequality (3.9) and consequently (3.8) are now proved with $C_5(M) \equiv C_7(M)$. q. e. d.

In case $p<\alpha+2$ an argument somewhat complicated is needed.

LEMMA 3.2 *Let $2 \le p < \alpha+2$. Then for $u_0 \in \mathscr{W}$ there exist constants $M_0 \equiv M_0(D_1 - J_1(u_0))$ and $m_0(\varepsilon)$ such that if $M<M_0$ $u_m(t)$ with $m>m_0$ is defined on $[0, +\infty)$ and the following estimate holds:*

$$\|u_m(t)\|_V \leq K_1(M, \varepsilon) < \lambda_1 \text{ for } t \text{ and for } m > m_0(\varepsilon), \tag{3.15}$$

where ε is a small constant and $K_1(M, \varepsilon)$ is a certain constant depending on M and ε.

PROOF. We shall show that there exists a constant M_0 such that if $M < M_0$ and $m > m_0$ we have

$$(\tilde{J}_1(u_m(t)) \leq) J_1(u_m(t)) < D_1, \quad t \leq t_m. \tag{3.16}$$

Then this inequality together with the initial condition $\|u(0)\|_V = \|u_0\|_V < \lambda_1$ will imply $\|u_m(t)\|_V < \lambda_1, t \leq t_m$, and the former part of Lemma 3.2 will follow. Also (3.15) will be seen from the procedure of the proof of (3.16).

Let us begin with showing $t_m > 1$. By (3.6) we know for $M < M'_0 \equiv 2\sqrt{D_1 - J_1(u_0)}$

$$J_1(u_m(t)) \leq J_1(u_m(0)) + \frac{1}{4}M^2 \tag{3.17}$$

$$\leq J_1(u_0) + \frac{1}{4}M^2 + \varepsilon < D_1 \quad (\varepsilon\text{: small})$$

for $0 \leq t \leq \min(1, t_m)$ and for $m > m_0(\varepsilon)$. Here we have used the fact that $J_1(u_m(0))$ may be assumed to be as close to $J_1(u_0)$ as one wants because $J_1(u)$ is continuous with respect to u in the norm $\|\cdot\|_V$. Hereafter we assume $M < M'_0$. Then the inequality (3.17) implies $t_m > 1$. Let us assume (3.16) was false. Then there would exist a number $t_1 (> 1)$ such that $J_1(u_m(t_1)) = D_1$ and $J_1(u_m(t)) < D_1$ for $t < t_1$. Thus by the same argument as in (3.5) we have

$$\int_{t_1-1}^{t_1} \|u'_m(s)\|_H^2 ds + J_1(u_m(t_1)) = J_1(u_m(t_1 - 1)) + \int_{t_1-1}^{t_1} (f(s), u'_m(s)) ds$$

and hence by the definition of t_1

$$\int_{t_1-1}^{t_1} \|u'_m(s)\|_H^2 ds \leq \int_{t_1-1}^{t_1} (f(s), u'_m(s)) ds,$$

and moreover

$$\int_{t_1-1}^{t_1} \|u'_m(s)\|_H^2 ds \leq \int_{t_1-1}^{t_1} (f(s), u'_m(s)) ds, M^2.$$

Also as in (3.12) we have

$$\int_{t_1-1}^{t_1} J_0(u_m(s)) ds \leq \int_{t_1-1}^{t_1} (\|u'_m(s)\|_H + \|f(s)\|_H) \|u_m(s)\|_H ds$$

$$\leq 2M \max_{t\in[t_1-1,t_1]} \|u_m(t)\|_H \leq 2MS_1\lambda_1, \tag{3.18}$$

where S_1 is the imbedding constant from V into H. Therefore there exists a time $t^*\in[t_1-1, t_1]$ such that

$$\tilde{J}_0(u_m(t^*))\leq J_0(u_m(t^*))\leq 2MS_1\lambda_1$$

and hence, if we choose $M<M_0'\equiv D_0/2S_1\lambda_1$,

$$\|u_m(t^*)\|_V\leq r_0(M), \tag{3.19}$$

where $r_0(M)$ is the smaller root of the numerical equation

$$C_0x^p-k_0S^{\alpha+2}x^{\alpha+2}=2MS_1\lambda_1,\ x\geq 0.$$

Note that $r_0(M)\to 0$ as $M\to 0$. Now from (3.5) with t and 0 replaced by t_1 and t^*, respectively, we have

$$\int_{t^*}^{t_1}\|u'_m(s)\|_H^2ds+J_1(u_m(t_1))=J_1(u_m(t^*))+\int_{t^*}^{t_1}(f, u'_m)ds$$
$$\leq C_3\|u_m(t^*)\|_V^p+k_1S^{\alpha+2}\|u_m(t^*)\|_V^{\alpha+2}$$
$$+\frac{1}{4}\int_{t^*}^{t_1}\|f(s)\|_H^2ds+\int_{t^*}^{t_1}\|u'_m(s)\|_H^2ds,$$

and hence

$$D_1=J_1(u_m(t_1))\leq C_3r_0(M)^p+k_1S^{\alpha+2}r_0(M)^{\alpha+2}+\frac{1}{4}M^2$$
$$\equiv K_2(M). \tag{3.20}$$

This is a contradiction if we choose $M<M'''$, M_0''' being the smallest constant such that

$$K_2(M_0''')=D_1,$$

which is possible since $K_2(M)$ tends to 0 and ∞ as $M\to 0$ and $\to\infty$, respectively. Thus if we choose M_0 as

$$M_0\equiv\min(M_0', M_0'', \mathrm{M}_0'''),$$

the inequality (3.17) is valid for $M<M_0$. Also (3.15) follows evidently if we define

$$K_1(M, \varepsilon)\equiv\max(J_1(u_0)+\varepsilon+\frac{1}{4}M^2, K_2(M)).$$

q. e. d.

§ 4. Proofs of theorems

Theorems 1 and 2 can be proved by the same way on the basis of Lemmas 3.1 and 3.2, respectively, and we shall carry out the proofs simultaneously.

Let $\{u_m(t)\}$ be approximate solutions constructed in § 3. $\{u_m(t)\}$ exist and satisfy a priori estimates (3.3), (3.15) under our assumptions. From these estimates we can obtain easily

$$\sup_{t\in R^+}\int_t^{t+1}\|u'_m(s)\|_H^2 ds \leq C_8(M) < +\infty. \tag{4.1}$$

Then by standard compactness and monotonicity arguments (Lions [5]) we can extract a subsequence from $\{u_m(t)\}$, which will be denoted by the same symbol, such that

$$u_m(t) \to u(t) \quad \text{weakly star in } L^\infty{}_{loc}(R^+; V), \tag{4.2}$$

$$u'_m(t) \to u'(t) \quad \text{weakly in } L^2{}_{loc}(R^+; H), \tag{4.3}$$

$$u_m(t) \to u(t) \quad \text{strongly in } L^r{}_{loc}(R^+; W), r \geq 1, \tag{4.4}$$

$$u_m(t) \to u(t) \quad \text{a.e. in } W, \tag{4.5}$$

$$Au_m(t) \to Au(t) \quad \text{weakly star in } L^\infty{}_{loc}(R^+, V^*), \tag{4.6}$$

$$Bu_m(t) \to Bu(t) \quad \text{weakly star in } L^\infty{}_{loc}(R^+; W^*), \tag{4.7}$$

and the limit function $u(t)$ satisfies

$$u'(t) + Au(t) + Bu(t) = f(t) \quad \text{a.e. in } V^*$$

and

$$u(0) = u_0.$$

Moreover the inequalites (3.3) (or (3.15) with $K_1(M) \equiv K_1(M, 0)$) and (4.1) remain valid for $u_m = u$. Thus the proofs of Theorems 1 and 2 are completed.

Next we shall proceed to the proof of Theorem 3. Consider the system of ordinary differential equations:

$$(u'_{m,r}(t), w_j) + (Au_{m,r}(t), w_j) + (Bu_{m,r}(t), w_j) = (f, w_j) \tag{4.8}$$

on $[-r, \infty)$ with initial condition

$$u_{m,r}(-r) = 0, \tag{4.9}$$

where

$$u_{m,r}=\sum_{j=1}^{m} a_{j,r}(t)w_j.$$

Then we know by Lemmas 3.1 and 3.2 that if $p>\alpha+2$ we have

$$\|u_{m,r}(t)\|_V\leq K_0(M,0),\quad -r\leq t<\infty, \tag{4.10}$$

and if $2\leq p\alpha+2$ we have for $M<\bar{M}_0\equiv M_0(D_1)$

$$\|u_{m,r}(t)_V\leq K_2(M),\quad -r\leq \mathrm{t}=\infty, \tag{4.11}$$

where we recall

$$K_2(M)=C_3r_0(M)^p+k_1S^{\alpha+2}r_0(M)^{\alpha+2}+\frac{1}{4}M^2.$$

In both cases we have

$$\sup_{t\geq -r}\int_t^{t+1}\|u_{m,r}(s)\|_H^2 ds\leq C_9(M)<+\infty. \tag{4.12}$$

From (4.10), (4.11) and (4.12) we conclude that after appropriate prolongations of $u_{m,r}(t)$ and $u'_{m,r}(t)$ for $t<-r$, we can extract subsequence of $\{u_{m,r}(t)\}$ with respect to r $(r=1,2,3,4,\dots)$ to obtain (4.2)—(4.7) with $u_m(t)$, $\mathrm{u}(t)$, m, and R^+ replaced by $u_{m,r}, u_m(t)$, r and R, respectively, where $u_m(t)$ is the limit function of $u_{m,r}(t)$ as $r\to\infty$. Since the estimates (4.10), (4.11) and (4.12) remain valid for $u_{m,r}(t)=u_m(t)$ on $t\in R$, we can extract once more again a subsequence from $\{u_m(t)\}$ which satisfies the same convergency properties as in (4.2)—(4.7) with R^+ replaced by R. Then the limit function $u(t)$ becomes the required bounded solution on $(-\infty,\infty)$.

§ 5. An example

As was mentioned in the introduction let us cosider the equation;

$$\frac{\partial}{\partial t}u-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial}{\partial x_i}u\right|^{p-2}\frac{\partial}{\partial x_i}\mathrm{u}\right)+\beta(x,u)=f(x,t),\quad x\in\Omega, \tag{5.1}$$

together with boundary condition

$$u|_{\partial\Omega}=0, \tag{5.2}$$

where Ω is an open bounded domain in R_n and $\partial\Omega$ is its boundary. We assume $\sup_t\int_t^{t+1}\|f(\cdot,s)\|^2_{L^2(\Omega)}ds\equiv M<+\infty$, and $\beta(x,u)$ is defined on $\Omega\times R$ and measurable in x for each u and continuous in u for each x and satisfies

$|\beta(x,u)|\leq k_0|u|^{\alpha+1}$ with $0\leq\alpha<np/(n-p)-2$ if $n>p>2$ and $0<\alpha<+\infty$ if $1\leq n\leq p$.

Put

$$V=W_0^{1,p}(\Omega),\quad W=L^{\alpha+2}(\Omega),\quad H=L^2(\Omega),$$

$$Au=-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial}{\partial x_i}u\Big|^{p-2}\frac{\partial}{\partial x_i}u\Big) \text{ and } Bu=\beta(x,u).$$

Note that

$$\|u\|_{W_0^{1,p}(\Omega)}=\Big(\sum_{i=1}^{n}\int_\Omega\Big|\frac{\partial}{\partial x_i}u\Big|^p dx\Big)^{\frac{1}{p}}.$$

Then all the hypotheses in § 2 are satisfied in this case with $C_0=C_1=1$, $C_2=C_3=\frac{1}{p}$ and $k_1=\frac{1}{\alpha+2}$. S and S_1 are defined as the Sobolev constants of the imbeddings from $W_0^{1,p}$ into $L^{\alpha+2}(\Omega)$ and $L^2(\Omega)$, respectively. Thus our results are applicable directly to (5.1)—(5.2).

FINAL REMARK.

In case $p=\alpha+2$ the conclusions of Theorems 2 and 3 are valid under the assumption that S and S_1 are sufficiently small.

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