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Existence and decay of solutions for a semilinear parabolic equation

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1. Introduction

In this paper we are concerned with the existence and an asymptotic property of solutions of a semilinear parabolic equation:

$$(1.1)\quad \begin{cases} \dfrac{\partial u}{\partial t}-\Delta u+\beta(u)+f(x,t)=0 & \text{in } R^n\times(0,\infty) \\ u(x,0)=u_0(x), & x\in R^n, \end{cases}$$

where Δ is the Laplacian and $\beta(u)$ is a function satisfying $\frac{d}{du}\beta(u)>0$ at $u=0$.

Recently Chafee [1] has proved the existence of solution of (1.1) and a stability of the trivial solution in the case $f(x,t)\equiv 0$ and $n=1$. He used the method of the Liapunov functional and the theory of dynamical system. His method, however, does not apply in its original form to the case $f\not\equiv 0$.

Here, using the methods of our previous paper [4], [6] (see also [5]), we shall prove the existence as well as a decay property of solutions for the problem (1.1) with $f\not\equiv 0$ and $n\geqq 1$. In [6] a semilinear parabolic equation in a bounded domain has been considered and in [4] a decay or stability property of solutions of a semlinear wave equation has been obtained.

We treat the equation (1.1) in the same framework as [1] and our result can be regarded as an extension of the one in [1].

When $\frac{d}{du}\beta(u)\leqq 0$, a bifurcation of solution may occur. This problem has been investigated by Chafee and Infante [2].

2. Preliminaries

Following Chafee [1], we introduce some fuction spaces.

Let Z_0 denote the set of all integer $l\geq 0$. For any $l\in Z_0$, we let $X^l_\infty(R^n)$

be the space of all l-times continuously differentiable function ϕ: $R^n \to R$ such that all derivatives of ϕ up to order l are uniformly continuous in R^n and tend to 0 with order $o(|x|^{1-n})$ as $|x| \to \infty$.(*)

We define a norm $\|\ \|_\infty^{(l)}$ on X_∞^l by setting

$$(2.1) \quad \|\phi\|_\infty^{(l)} = \sum_{j=0}^{l} \sup_{x \in R^n} |D_x^j \phi(x)| \quad (\phi \in X_\infty^l(R^n))$$

For any $p \in [1, \infty)$, we let $X_p^l(R^n)$ be the space of $\phi \in X_\infty^l(R^n)$ such that all derivatives $D_x^j \phi(x)$ $(j=0, 1, \cdots, l)$ are p-th power Lebesgue integrable in R^n. We define a norm $\|\ \|_p^{(l)}$ on $X_p^l(R^n)$ by setting

$$(2.2) \quad \|\phi\|_p^{(l)} = \|\phi\|_\infty^{(l)} + \sum_{j=0}^{l} \left[\int_{R^n} |D_x^j \phi(x)|^p dx \right]^{1/p} (\phi \in X_p^l(R^n)).$$

For brevity, we will write $X_p^l(X_\infty^l)$ instead of $X_p^l(R^n)$ $(X_\infty^l(R^n))$.

DEFINITION 2.1.

A function $u(x, t)$ defined in $R^n \times (0, s)$ $(0 < s \leqq \infty)$ is called a solution of the Cauchy problem (1.1) in $R^n \times (0, s)$ if:

(i) $u(\cdot, t) \in X_2^0$ for every $t \in [0, s)$,

(ii) the map $t \to u(\cdot, t)$ from $[0, s)$ into X_2^0 is continuous on $[0, s)$,

(iii) the partial derivative $u_{x_i x_i}$ $(i=1, 2, \cdots, n)$ and u_t exist and continuous on $R^n \times (0, s)$,

(iv) u satisfies (1.1) in $R^n \times (0, s)$.

Now we state our hypotheses.

(H_1) $\beta(s)$ is defined, real valued and C^2-smooth on R, and satisfies $\beta(0)=0$.

(H_2) There exists a positive number γ_0 such that

$$u\beta(u) \geqq K_0 u^2 \qquad (|u| < \gamma_0)$$

where K_0 is some positive constant.

(H_3) $f(\cdot, t) \in X_2^1$ $(0 \leqq {}^\forall t < \infty)$ and $f(x, t)$ satisfies

$$\sup_{R^n \times [0, \infty)} |f(x, t)| < K_0 \gamma_0$$

where K_0 and γ_0 are constants appeared in (H_2).

(H_4) There exist positive constants θ and K_1 such that

$$\|f(\cdot, t)\|_{L^2} \leqq K_1 \exp(-\theta t).$$

(H_5) $u_0(x) \in X_2^1$ and $\|u_0\|_\infty^{(0)} < \gamma_0$ with γ_0 in (H_2).

By a standard argument (see [3]), we have

(*) When $n=1$, this property may be replaced by the boundedness condition.

LEMMA 2.1. *Let u be a function defined on $R^n\times[0,s)$, $0<s\leqq\infty$, such that*

(i) $u(\cdot,t)\in X_2^0$ *for every* $t\in[0,s)$,

(ii) *the map* $t\to u(\cdot,t)$ *from* $[0,s)$ *into* X_2^0 *is continuous on* $[0,s)$; *then* u *is a solution of* (1.1) *if and only if*

$$(2.3)\qquad u(\cdot,t)=T(t)u_0-\int_0^t T(t-\tau)\,[\beta(u(\cdot,\tau)+f(\cdot,\tau)]\,d\tau \qquad (0\leqq t<s),$$

where we set

$$(2.4)\qquad [T(t)\phi](x)=\frac{1}{(\sqrt{\pi})^n}\int_{R^n}\exp(-|\eta|^2)\varphi(x+2\eta\sqrt{t})d\eta$$
$$(\varphi\in X_\infty^1,\ x\in R^n,\ t\in[0,\infty)).$$

On the basis of Eq. (2.3), we can prove the following lemma.

LEMMA 2.2. *For any* $u_0\in X_2^0$, *Eq.* (1.1) *has a unique noncontinuable solution* $u(x,t;\ u_0)$, *defined on* $R^n\times[0,s(u_0))$, *where* $0<s(u_0)\leqq\infty$. *Furthermore, if there exists an* $r\in(0,\infty)$ *such that*

$$\|u(\cdot,t;\ u_0)\|_2^{(0)}<r\quad(0\leqq t<s(u_0)),$$

then we may take $s(u_0)=\infty$.

Now we shall state some regularity properties of the local solution $u(x,t;\ u_0)$ due to Chafee [1].

LEMMA 2.3. *For any* $u_0\in X_2^0$ *and any* $t\in(0,s(u_0))$, *we have;*

(i) $u(\cdot,t;\ u_0)\in X_2^3$ *and the map* $t\to u(\cdot,t;\ u_0)$ *from* $(0,s(u_0))$ *into* X_2^j *is continuous* $(j=0,1,2,3)$,

(ii) $u_t(\cdot,t;\ u_0)\in X_2^0$ *and the map* $t\to u_t(\cdot,t;\ u_0)$ *from* $(0,s(u_0))$ *into* X_2^0 *is continuous,*

(iii) $u_{x_it}(\cdot,t;\ u_0)\in X_2^0$ $(i=1,2,\cdots,n)$ *and the map* $t\to u_{x_it}(\cdot,t;\ u_0)$ *from* $(0,s(u_0))$ *into* X_2^0 *is continuous* $(i=1,2,\cdots,n)$.

The following lemma is easily proved and is useful in analyzing the behavior of solutions. (See Nakao [3]).

LEMMA 2.4. *Let* $\varphi(t)$ *be a bounded nonnegation function defined on* $[0,\infty)$, *satisfying*

$$\operatorname*{Max}_{s\in[t,t+1]}\varphi(s)\leqq\widetilde{K}_0[\varphi(t)-\varphi(t+1)]+g(t)$$

for some positive constant $\widetilde{K}_0$ *and a nonnegative function* $g(t)$ *with* $0\leqq g(t)\leqq\widetilde{K}_1$ $\exp(-\lambda_0 t)$ ($\widetilde{K}_1$, λ_0 *are positive constants*).

Then there exist constants C_2 *and* λ_1 *which depend on* $\widetilde{K}_0$, $\widetilde{K}_1$ *and* λ_0 *such that*

$0 \leqq \varphi(t) \leqq C_2 \exp(-\lambda_1 t) \quad (t \geqq 0).$

3. Theorem

Now we are ready to state and prove our theorem.

THEOREM 3.1. *Under the hypotheses* $(H_1)-(H_5)$, *the problem* (1.1) *has a bounded solution* $u(x, t)$ *with the following estimates*;

(3.1) $$\sup_{x \in R^n} |u(x, t)| \leqq \gamma_0 \qquad (0 \leqq t < \infty);$$

(3.2) $$\int_{R^n} \{|\nabla_x u(x, t)|^2 + |u(x, t)|^2\} dx \leqq C_1 \exp(-\theta_1 t) \qquad (0 < t < \infty),$$

and

(3.3) $$\int_t^{t+1} \int_{R^n} |u_t(x, s)|^2 \, dx \, ds \leqq C_2 \exp(-\theta_1 t) \qquad (0 < t < \infty),$$

where C_1, C_2 *and* θ_1 *are positive constants, depending only on* u_0, f *and* γ_0.

PROOF. For the proof of theorem, it sufficies by Lemma 2.2 to verify the follwing estimates;

(3.4) $$\sup_{x \in R^n} |u(x, t)| < \gamma_0 \qquad (t \in [0, s(u_0))),$$

(3.5) $$\int_{R^n} \{|\nabla_x u(x, t)|^2 + |u(x, t)|^2\} dx \leqq C_1 \exp(-\theta_1 t) \qquad (0 \leqq t < s(u_0)),$$

where C_1 and θ_1 are positive constants, depending only on u_0, f, and γ_0.

First, suppose that (3.4) is false.

Then there would exist the smallest time $\bar{t} \in [0, s(u_0))$ such that

(3.6) $$\sup_{x \in R^n} |u(x, \bar{t})| = \gamma_0 \quad \text{and} \quad \sup_{x \in R^n} |u(x, t)| < \gamma_0 \quad \text{if } t < \bar{t}.$$

Let x_0 be a maximum point of $|u(x, \bar{t})|$.

Then we obtain

$$\frac{1}{2} \frac{d}{dt} u^2(x_0, \bar{t}) \leqq [-\beta(u(x_0, \bar{t})) + f(x_0, \bar{t})] u(x_0, \bar{t})$$

$$\leqq -K_0 u^2(x_0, \bar{t}) + M |u(x_0, \bar{t})| < 0.$$

This implies that the function $\operatorname{Max}_{x \in R^n} |u(x, t)|$ of t begins to decrease just before it should reach to the value γ_0 from below. This is contradiction to (3.6). Therefore we have (3.4).

Now, we define the mappings F: $R \to R$ and E: $X_2^1 \to R$ by setting

(3.7) $$F(z) = \int_0^z \beta(s) ds \qquad (z \in R),$$

(3.8) $$E(\phi) = \int_{R^n} \{\frac{1}{2} |\nabla_x \phi(x)|^2 + F(\phi)(x)\} dx (\phi \in X_2^1).$$

Multiplying (1.1) by $u_t(x,t)$, integrating over R^n and using Lemma 2.3, we have

$$(3.9)\quad \int_{R^n} u_t(x,t)dx+\frac{d}{dt}E(u(t))=\int_{R^n} f(x,t)u_t(x,t)dx$$
$$\leq |||f(t)|||_2\,|||u_t(t)|||_2 \quad (0<t<s(u_0)).^{(**)}$$

From this, integrating it over $[0,t)\subset[0,s(u_0))$, we have

$$E(u(t))\leqq E(u_0)+1/4\int_0^{s(u_0)}|||f(t)|||_2^2\,dt.$$

Since $|u|<\gamma_0$, the right hand of above inequlity has a lower bound

$$|||\nabla_x u(t)|||_2^2+k_0/2\,|||u(t)|||_2^2$$

and for $t\leqq \mathrm{Min}\ (1, s(u_0))$

$$|||\nabla_x u(t)|||_2^2+|||u(t)|||_2^2\leqq C(E(u_0)+\mathop{\mathrm{Max}}_{0\leq s\leq 1}|||f(s)|||_2^2).$$

Hence we may assume $s(u_0)>1$.

Next, integrating (3.9) over $[t_1, t_2]\subset[0, s(u_0))$, we obtain

$$(3.10)\quad \int_{t_1}^{t_2}|||u_t(s)|||_2^2ds+2(E(u(t_2))-E(u(t_1)))\leqq\int_{t_1}^{t_2}|||f(s)|||_2^2\,ds.$$

Now, taking $t_2=t+1$ and $t_1=t$, we have

$$(3.11)\quad 0\leqq\int_t^{t+1}|||u_t(s)|||_2^2\,ds\leqq 2[E(u(t))-E(u(t+1))]+\int_t^{t+1}|||f(s)|||_2^2\,ds$$
$$\equiv 2[E(u(t))-E(n(t+1))]+\delta^2(t)\equiv D^2(t).$$

Multiplyiug (1.1) by u, integrating over $R^n\times(t, t+1)$ and using Lemma 2.3, we have

$$\int_t^{t+1}[\,|||\nabla_x u(s)|||_2^2+K_0\,|||u(s)|||_2^2]\,ds$$
$$\leqq\int_t^{t+1}|||\nabla_x u(s)|||_2^2\,ds+\int_s^{t+1}\int_{R^n}\beta(u(x,s))u(x,s)dxds$$
$$=\int_t^{t+1}\int_{R^n}[-u_t(x,s)u(x,s)-f(x,s)u(x,s)]\,dxds$$
$$\leqq D(t)\left(\int_t^{t+1}|||u(s)|||_2^2\,ds\right)^{1/2}+\delta(t)\left(\int_t^{t+1}|||u(s)|||_2^2\,ds\right)^{1/2}.$$

Therefore we obtain

$$(3.12)\quad \int_t^{t+1}|||u(s)|||_2^2\,ds\leqq(2/K_0)^2(D^2(t)+\delta^2(t)).$$

(**) For brevity, we will use notation $|||v(s)|||_2^2$ instead of $\int_{R^n}v(x,s)^2dx$.

From (3.11) and (3.12), it follows that

$$\int_t^{t+1} [\,|||\nabla_x u(s)|||_2^2 + K_0 |||u(s)|||_2^2 + |||u_t(s)|||_2^2]\, ds$$
$$\leqq 4/K_0(D^2(t)+\delta^2(t)) + D^2(t)$$
$$\leqq (4/K_0+1)(D^2(t)+\delta^2(t))$$
$$= C_3(D^2(t)+\delta^2(t)) .$$

Therefore we see that there exists $t^* \in [t, t+1]$ such that

$$|||\nabla_x u(t^*)|||_2^2 + |||u(t^*)|||_2^2 + |||u_t(t^*)|||_2^2 \leqq C_3(D^2(t)+\delta^2(t)).$$

Using (3.8), (3.10), (3.11) and above inequality, we have easily

$$\underset{s\in[t,t+1]}{\mathrm{Max}}\, 2\,E(u(s)) \leqq 2\,E(u(t^*)) + \delta^2(t) + \int_t^{t+1} |||u_t(s)|||_2^2\, ds$$
$$\leqq (C_3+1)(D^2(t)+\delta^2(t)).$$

Applying Lemma 2.4 to the above, we obtain (3.5) immediately. The estimate (3.3) follows readily from (3.5). Q.E.D.

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