Invariant variation and covariant differentiation

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§ 1. Introduction.

First the concept of parallel displacement was formulated by Levi-Civita. But he considered the Riemannian space R_n as a surface in a $n(n+1)/2$-dimensional Euclidean space.

Weyl showed later that this concept can be defined in R_n directly without a support of the Euclidean space. Furthermore he succeeded to define axiomatically the concept of parallel displacement in a manifold and built up the concept of affine connection.

In the usual textbooks of general relativity the covariant differentiation of any vector field is introduced by using the concept of parallel displacement.

On the other hand, Cartan arrived at the concept of a more general affine connection than the Weyl's affine connection by extending the notion of the equivalent systems of reference of Galilee.

From this point of view, the concept of parallel displacement and the covariant differentiation are introduced by the help of this affine connection.

Utiyama obtained "the covariant differentiation" from the invariance postulate of the Lagrangian under the Lorentz transformation and deduced the covariant differentiation of any tensor field on the pseudo-Riemannian manifold.

Hayashi attained to the same result as Utiyama by starting from the invariance postulate of action integral under the Lorentz transformation.

The main purpose of the present paper is to investigate the following problem: (1) to obtain the gauge covariant differentiation by starting from the invariance postulate of action integral under the affine transformation, (2) to formulate the assumptions in the gauge theory, (3) to deduce the Cartan's affine connection and the connection of spinor fields in the Weyl's geometry as a special case, (4) to show that vierbein can be considered as a gauge field.

The solution of these problems will be state in §4, §3, §5~§7 and §4 respectively.

§2. Manifold with vierbein.

Let $\mathscr{M}$ be a Hausdorff space. A local chart on $\mathscr{M}$ is a pair (U, φ) where U is an open subset of $\mathscr{M}$ and φ is a homeomorphism of U onto an open subset of the Minkowski space M. A global chart ϕ on $\mathscr{M}$ is a homeomorphism of $\mathscr{M}$ onto M.

A four-dimensional differentiable manifold $\mathscr{M}$ with vierbein is a Hausdorff space with the collections of local charts $(U_\alpha, \varphi_\alpha)_{\alpha\in A}$ and global charts $\phi_{\beta\in B}$ such that the following conditions are satisfied:

$(\mathscr{M}_1)$ $\mathscr{M}=\bigcup_{\alpha\in A} U_\alpha$.

$(\mathscr{M}_2)$ For each pair (α, β) the mapping $\varphi_\beta\circ\varphi_\alpha^{-1}$ is a differentiable mapping of $\varphi_\alpha(U_\alpha\cap U_\beta)$ onto $\varphi_\beta(U_\alpha\cap U_\beta)$ and the mapping $\phi_\beta\circ\phi_\alpha^{-1}$ is a differentiable mapping of M onto M.

$(\mathscr{M}_3)$ For each pair (α, β) the mapping $\varphi_\beta\circ\phi_\alpha^{-1}$ is a diffeomorphism of $\phi_\alpha(U_\beta)$ onto $\varphi_\beta(U_\beta)$.

If $p\in U_\alpha$, $\varphi_\alpha(p)=(x^\mu(p))$ $(\mu=0,1,2,3)$ and $\phi_\beta(p)=(X^k(p))$ $(k=0,1,2,3)$, then the numbers $x^\mu(p)$ and $X^k(p)$ are called the local coordinates and the global coordinates of p respectively.

A vierbein $h_k^\mu(p)$ at a point $p\in\mathscr{M}$ is a Jacobian $J_p\{\varphi_\alpha\circ\phi_\beta^{-1}\}$. The vierbein $h_k^\mu(p)$ is transformed contragradiently under the transformations of local coordinate systems and cogradiently under the transformations of global coordinate systems.

Let us introduce the fields $h^k{}_\mu(p)$ which are defined by the orthogonal relations:

$$h^k{}_\mu(p)h_l{}^\mu(p)=\delta_l^k, \qquad h^k{}_\mu(p)h_k{}^\nu(p)=\delta_\mu^\nu,$$

where δ_l^k, δ_ν^μ are the Kronecker's δ. Then we have $\partial_\mu=h^k{}_\mu\partial_k$ and $d^4X=h\, d^4x$, where we have put $h=\det(h^k{}_\mu)$.

If we represent a vector field $X(p)$ on $\mathscr{M}$ as follows

$$X(p)=X^\mu(p)(\partial_\mu)_p=X^k(p)(\partial_k)_p,$$

we have $X^\mu(p)=h_k^\mu(p)X^k(p)$.

§ 3. Postulate and requirements.

Let us consider an infinitesimal affine transformation in M:

$$X^k \longrightarrow X^k + \delta X^k, \qquad \delta X^k = \omega^k{}_j X^j + \epsilon^k, \tag{3.1}$$

and a set of fields $Q^\alpha(X)$ ($\alpha = 1, 2, 3, \cdots\cdots, N$) defined on M, with the Lagrangian density

$$L(Q^\alpha, Q^\alpha{}_{,k}, X^k), \qquad Q^\alpha{}_{,k} = \partial_k Q^\alpha.$$

Under this transformation (3.1), these fields $Q^\alpha(X)$ are assumed to be transformed as

$$Q^\alpha(X) \longrightarrow Q^\alpha(X) + \delta Q^\alpha(X), \qquad \delta Q^\alpha(X) = \omega^\alpha{}_\beta Q^\beta(X), \tag{3.2}$$

where we have put $\omega^\alpha{}_\beta = \omega^k{}_j T^j{}_k{}^\alpha{}_\beta$. The constant coefficients $T^j{}_k{}^\alpha{}_\beta$ are the (α, β) element of (N, N) matrices $T^j{}_k$ and the matrices $T^j{}_k$ are subject to the following commutation relations:

$$[T^k{}_j, T^l{}_m] = C^{klp}_{jmq} T^q{}_p . \tag{3.3}$$

These constants, C^{klp}_{jmq}, have the following properties:

$$C^{klp}_{jmq} C^{qrt}_{psu} + C^{lrp}_{msq} C^{qkt}_{pju} + C^{rkp}_{sjq} C^{qlt}_{pmu} = 0, \qquad C^{klp}_{jmq} = -C^{lkp}_{mjq} .$$

Now let us postulate that the action integral I referred to an arbitrary four-dimentional domain $D \subset M$,

$$I = \int_D L(Q^\alpha, Q^\alpha{}_{,k}, x^k) d^4x,$$

is invariant under the transformation (3.1).

From the invariant character of I under the transformation (3.1) and the fact that this invariance is always preserved for an arbitrary domain D, we have

$$\delta L + (\delta X^k)_{,k} L \equiv 0.$$

By using the explicit expressions δX^k, δQ^α and $\delta(Q^\alpha{}_{,k}) = (\delta Q^\alpha)_{,k} - (\delta X^l)_{,k} Q^\alpha{}_{,l}$, we get the following conditions for L:

$$I_{,k} \equiv 0, \tag{3.4}$$

$$\frac{\partial L}{\partial Q^\alpha} \omega^\alpha{}_\beta Q^\beta + \frac{\partial L}{\partial (Q^\alpha{}_{,k})} \{\omega^\alpha{}_\beta Q^\beta{}_{,k} - \omega^l{}_k Q^\alpha{}_{,l}\} + \omega^k{}_k L \equiv 0. \tag{3.5}$$

For each point $p \in \mathscr{M}$, let $U_\alpha(p)$ be a neighborhood of p and D an arbitrary domain contained in $U_\alpha(p)$. Putting $D = \phi_\alpha(\mathfrak{D})$, $\mathscr{D} = \varphi_\alpha(\mathfrak{D})$ and considering the conditions (3.4), the action integral I becomes as follows

$$I=\int_D h(x)L^*(Q^\alpha(x), Q^\alpha{}_{,\mu}(x))d^4x, \tag{3.6}$$

where we have put

$$L^*(Q^\alpha, Q^\alpha{}_{,\mu})=L(Q^\alpha, h_k{}^\mu Q^\alpha{}_{,\mu}),$$

$$Q^\alpha{}_{,\mu}=\frac{\partial Q^\alpha}{\partial x^\mu}=h^k{}_\mu Q^\alpha{}_{,k}.$$

Replacing the parameters $\omega^k{}_j$ and ϵ^μ by the differentiable functions $\omega^k{}_j(x)$ and $\epsilon^\mu(x)$ of a local coordinate of a point $p\in\mathscr{M}$, we consider the extended transformations of infinitesimal affine transformation (3.1):

$$x^\mu \longrightarrow x^\mu+\delta x^\mu, \qquad \delta x^\mu=\epsilon^\mu(x), \tag{3.7}$$

$$X^k \longrightarrow X^k+\delta X^k, \qquad \delta X^k=\omega^k{}_j(X)X^j, \tag{3.8}$$

where $\epsilon^\mu(x)$ and $\omega^k{}_j(x)$ are the infinitesimal parameters. Under these extended transformations (3.7) and (3.8), the fiields $Q^\alpha(x)$ and $h_k{}^\mu(x)$ are transformed as follows

$$\begin{aligned} &Q^\alpha(x) \longrightarrow Q^\alpha(x)+\delta Q^\alpha(x),\\ &\delta Q^\alpha(x)=\omega^\alpha{}_\beta(x)Q^\beta(x),\\ &h_k{}^\mu(x) \longrightarrow h_k{}^\mu(x)+\delta h_k{}^\mu(x), \end{aligned} \tag{3.9}$$

$$\delta h_k{}^\mu(x)=\epsilon^\mu{}_{,\nu}(x)h_k{}^\nu(x)-\omega^l{}_k(x)h_l{}^\mu(x), \tag{3.10}$$

where we have put

$$\omega^\alpha{}_\beta(x)=\omega^k{}_j(x)T^j{}_k{}^\alpha{}_\beta. \tag{3.11}$$

Now we require three assumptions.

1. *Replacement of $Q^\alpha{}_{,\mu}(x)$ by $\nabla_\mu Q^\alpha(x)$.*

We replace the differentiations $Q^\alpha{}_{,\mu}(x)$ in the Lagrangian density $L^*(Q^\alpha(x), Q^\alpha{}_{,\mu}(x))$ by the gauge covariant differentiations $\nabla_\mu Q^\alpha(x)$. Then the action integral (3.6) becomes

$$I=\int_{\mathscr{D}} h(x)L^*(Q^\alpha(x), \nabla_\mu Q^\alpha(x))d^4x, \tag{3.12}$$

where we have put

$$L^*(Q^\alpha, \nabla_\mu Q^\alpha)=L(Q^\alpha, h_k{}^\mu \nabla_\mu Q^\alpha).$$

2. *Covariant character of $\nabla_\mu Q^\alpha(x)$.*

Under the extended transformations (3.7) and (3.8), $\nabla_\mu Q^\alpha(x)$ are transformed as follows

$$\nabla_\mu Q^\alpha \longrightarrow \nabla_\mu Q^\alpha + \delta(\nabla_\mu Q^\alpha),$$
$$\delta(\nabla_\mu Q^\alpha) = -\epsilon^\nu{}_{,\mu}(x)\nabla_\nu Q^\alpha + \omega^\alpha{}_\beta(x)\nabla_\mu Q^\beta. \tag{3.13}$$

3. *Introduction of the new fields $b^I(x)$ and $A^J(x)$.*

We introduce the new fields $b^I(x)$ $(I=1, 2, \cdots\cdots, n_I)$, $A^J(x)$ $(J=1, 2, \cdots\cdots, n_J)$ which are transformed under the transformations (3.7) and (3.8) as follows

$$b^I \longrightarrow b^I + \delta b^I,$$
$$\delta b^I = \epsilon^\nu{}_{,\mu}(x)\bar{B}^I{}_\nu{}^\mu(x) + \omega^k{}_j(x)\tilde{B}^I{}_k{}^j(x), \tag{3.14}$$
$$A^J \longrightarrow A^J + \delta A^J,$$

$$\delta A^J = \epsilon^\nu{}_{,\mu}(x)\bar{C}^J{}_\mu{}^\nu(x) + \omega^k{}_j(x)\tilde{C}^J{}_k{}^j(x) + \omega^k{}_{j,\mu}(x)C^J{}_k{}^{j\mu}(x), \tag{3.15}$$

where $\bar{B}^I{}_\nu{}^\mu(x)$, $\tilde{B}^I{}_k{}^j(x)$, $\bar{C}^J{}_\nu{}^\mu(x)$, $\tilde{C}^J{}_k{}^j(x)$ and $C^J{}_k{}^{j\mu}(x)$ are the fields defined on $\mathscr{M}$.

In addition, we require that $\nabla_\mu Q^\alpha$ depend on the fields $Q^\alpha(x)$, $Q^\alpha{}_{,\mu}(x)$, $b^I(x)$ and $A^J(x)$:

$$\nabla_\mu Q^\alpha = f_\mu{}^\alpha(Q^\alpha,\ Q^\alpha{}_{,\mu},\ b^I,\ A^J).$$

The first fact which is deduced from a postulate and three requirements is that the action integral (3.12) *is invariant under the transformations* (3.7) *and* (3.8).

In fact, the transformation character of $h=\det(h^k{}_\mu)$ is obtained from the orthogonal relations for $h^k{}_\mu$ and the transformation character (3.10) of $h_k{}^\mu$:

$$\delta h = -\epsilon^\mu{}_{,\mu}h + \omega^k{}_k h. \tag{3.16}$$

Thus we have

$$\delta(hL^*) + (\delta x^\mu)_{,\mu}hL^* = h\{\delta L^* + \omega^k{}_k L^*\}.$$

From $L^*(Q^\alpha, \nabla_\mu Q^\alpha) = L(Q^\alpha, h_k{}^\mu\nabla_\mu Q^\alpha)$, L^* is transformed as follows

$$\delta L^* = \frac{\partial L}{\partial Q^\alpha}\delta Q^\alpha + \frac{\partial L}{\partial(\nabla_k Q^\alpha)}\delta(\nabla_k Q^\alpha), \tag{3.17}$$

where we have put $\nabla_k Q^\alpha = h_k{}^\mu\nabla_\mu Q^\alpha$. By using (3.10) and (3.13), the transformation character of $\nabla_k Q^\alpha$ becomes

$$\delta(\nabla_k Q^\alpha) = \omega^\alpha{}_\beta \nabla_k Q^\beta - \omega^l{}_k \nabla_l Q^\alpha. \tag{3.18}$$

Inserting (3.9) and (3.18) into (3.17), we have

$$\delta L^* = \frac{\partial L}{\partial Q^\alpha}\omega^\alpha{}_\beta Q^\beta + \frac{\partial L}{\partial(\nabla_k Q^\alpha)}(\omega^\alpha{}_\beta \nabla_k Q^\beta - \omega^l{}_k \nabla_l Q^\alpha).$$

By considering the identity (3.5), it follows that

$$\delta L^* + \omega^k{}_k L^* = \frac{\partial L}{\partial Q^\alpha}\omega^\alpha{}_\beta Q^\beta + \frac{\partial L}{\partial(\nabla_k Q^\alpha)}(\omega^\alpha{}_\beta \nabla_k Q^\beta - \omega^l{}_k \nabla_l Q^\alpha) + \omega^k{}_k L \equiv 0.$$

Therefore we have obtained the following condition for the invariant character of action integral (3.12):

$$\delta(hL^*) + (\delta x^\mu)_{,\mu} hL^* = h(\delta L^* + \omega^k{}_k L^*) \equiv 0. \tag{3.19}$$

§ 4. Gauge covariant differentiation.

From (3.19), we obtain the following identity:

$$\frac{\partial L^*}{\partial Q^\alpha}\delta Q^\alpha + \frac{\partial L^*}{\partial(\nabla_\mu Q^\gamma)}\left\{\frac{\partial f_\mu{}^\gamma}{\partial Q^\alpha}\delta Q^\alpha + \frac{\partial f_\mu{}^\gamma}{\partial Q^\alpha{}_{,\nu}}\delta(Q^\alpha{}_{,\nu}) + \frac{\partial f_\mu{}^\gamma}{\partial b^I}\delta b^I + \frac{\partial f_\mu{}^\gamma}{\partial A^J}\delta A^J\right\} + \omega^k_k L^* \equiv 0.$$

Inserting $\delta(Q^\alpha{}_{,\nu}) = (\delta Q^\alpha)_{,\nu} - (\delta x^\rho)_{,\nu} Q^\alpha{}_{,\rho}$, (3.9), (3.14) and (3.15) into the above identity, and taking account of the arbitrariness of choosings of $\omega^k{}_j$, $\omega^k{}_{j,\nu}$ and $\epsilon^\rho{}_{,\nu}$, we see that each coefficients of $\omega^k{}_j$, $\omega^k{}_{j,\nu}$ and $\epsilon^\rho{}_{,\nu}$ must vanish independently. Namely, we have the identities

$$\frac{\partial L^*}{\partial Q^\alpha}T^j{}_k{}^\alpha{}_\beta Q^\beta + \frac{\partial L^*}{\partial(\nabla_\mu Q^\gamma)}\left\{\frac{\partial f_\mu{}^\gamma}{\partial Q^\alpha}T^j{}_k{}^\alpha{}_\beta Q^\beta + \frac{\partial f_\mu{}^\gamma}{\partial Q^\alpha{}_{,\nu}}T^j{}_k{}^\alpha{}_\beta Q^\beta{}_{,\nu} + \frac{\partial f_\mu{}^\gamma}{\partial b^I}\tilde{B}^I{}_k{}^j + \frac{\partial f_\mu{}^\gamma}{\partial A^J}\tilde{C}^J{}_k{}^j\right\} + \delta^j_k L^* \equiv 0, \tag{4.1}$$

$$\frac{\partial f_\mu{}^\gamma}{\partial Q^\alpha{}_{,\nu}}T^j{}_k{}^\alpha{}_\beta Q^\beta + \frac{\partial f_\mu{}^\gamma}{\partial A^J}C^J{}_k{}^{j\nu} \equiv 0, \tag{4.2}$$

$$\frac{\partial f_\mu{}^\gamma}{\partial Q^\alpha{}_{,\nu}}Q^\alpha{}_{,\rho} - \frac{\partial f_\mu{}^\gamma}{\partial b^I}\bar{B}^I{}_\rho{}^\nu - \frac{\partial f_\mu{}^\gamma}{\partial A^J}\bar{C}^J{}_\rho{}^\nu \equiv 0. \tag{4.3}$$

Now, in order to be able to determine the A^J-dependence of $f_\mu{}^\gamma$, the number of the components of A^J should be equal to the number of equations (4.2), namely, $J = 1, 2, \cdots\cdots, 12$ should hold. In addition, the matrix $C^J{}_k{}^{j\nu}$ must be nonsingular. Thus there is the inverse of C defined by

$$C^J{}_k{}^{j\nu}C^{-1}{}_K{}^k{}_{j\nu} = \delta^J_K, \qquad C^{-1}{}_J{}^i{}_{j\mu}C^J{}_l{}^{m\nu} = \delta^i_l\,\delta^m_j\,\delta^\nu_\mu.$$

By the similar consideration to the above, it is obtained from the equations (4.3) that $I=1, 2, \cdots\cdots, 8$ should hold. Then (4.2) can be rewritten as

$$\frac{\partial f_\mu{}^\tau}{\partial Q^\alpha{}_{,\nu}} T^j{}_k{}^\alpha{}_\beta\, Q^\beta + \frac{\partial f_\mu{}^\tau}{\partial A^k{}_{j\nu}} \equiv 0,$$

where we have put

$$A^k{}_{j\nu} = C^{-1}{}_j{}^k{}_{j\nu}\, A^I. \tag{4.4}$$

Thus $A^k{}_{j\nu}$ should be contained in $f_\mu{}^\tau$ through the combination

$$f_\mu{}^\tau = Q^\tau{}_{,\mu} - A^\tau{}_{\beta\mu} Q^\beta, \tag{4.5}$$

where we have put

$$A^\tau{}_{\beta\mu} = T^j{}_k{}^\tau{}_\beta\, A^k{}_{j\mu}\,. \tag{4.6}$$

Next, we consider the fields $B_I{}^k{}_\mu(x)$ so that the matrix $B_I{}^k{}_\mu(x)$ is nonsingular at each point $p \in \mathscr{M}$. Let us make the new fields $b^l{}_\mu(x)$ from the fields $b^I(x)$ as follows

$$b^l{}_\mu(x) = B_I{}^l{}_\mu(x) b^I(x). \tag{4.7}$$

In addition, we introduce the fields $b_k{}^\mu(x)$ which are determined by the orthogonal relations

$$b_k{}^\mu(x) b^l{}_\mu(x) = \delta^l_k\,, \qquad b_k{}^\mu(x) b^k{}_\nu(x) = \delta^\mu_\nu\,,$$

and let us put

$$Q^\tau{}_{,k} = b_k{}^\mu\, Q^\tau{}_{,\mu}\,, \qquad A^k{}_{jl} = b_l{}^\mu\, A^k{}_{j\mu}\,. \tag{4.8}$$

Thus (4.5) can be rewritten as

$$f_\mu{}^\tau = b^l{}_\mu (Q^\tau{}_{,l} - A^\tau{}_{\beta l}\, Q^\beta), \tag{4.9}$$

where we have put

$$A^\tau{}_{\beta l} = T^j{}_k{}^\tau{}_\beta\, A^k{}_{jl}\,. \tag{4.10}$$

Before the considerations of the identities (4.1) and (4.3), we shall calculate the following derivatives by using the relations from (4.4) to (4.9):

$$\frac{\partial f_\mu{}^\tau}{\partial Q^\alpha} = -A^\tau_{\alpha\mu}\,, \qquad \frac{\partial f_\mu{}^\tau}{\partial Q^\alpha{}_{,\nu}} = \delta^\tau_\alpha\, \delta^\nu_\mu\,,$$

$$\frac{\partial f_\mu{}^\tau}{\partial b^I} = \frac{\partial f_\mu{}^\tau}{\partial b^l{}_\nu} \frac{\partial b^l{}_\nu}{\partial b^I} = b_l{}^\nu\, \nabla_\nu\, Q^\tau\, b_I{}^l{}_\mu\,,$$

$$\frac{\partial f_\mu{}^\tau}{\partial A^J}=\frac{\partial f_\mu{}^\tau}{\partial A^\lambda{}_{\beta\nu}}\frac{\partial A^\lambda{}_{\beta\nu}}{\partial A^J}=-Q^\beta T^j{}_k{}^\tau{}_\beta C^{-1}{}_J{}^k{}_{j\mu}\,.$$

Now, inserting the above relations into (4.1) and using the commutation relations (3.3), we get

$$\frac{\partial L^*}{\partial Q^\alpha}T^j{}_k{}^\alpha{}_\beta Q^\beta+\frac{\partial L^*}{\partial(\nabla_\mu Q^\tau)}\{T^j{}_k{}^\tau{}_\beta \nabla_\mu Q^\beta+b_l{}^\nu \nabla_\nu Q^\tau B_I{}^l{}_\mu \tilde{B}^I{}_k{}^j$$

$$+T^q{}_p{}^\tau{}_\beta(C^{jlp}_{kmq}A^m{}_{l\mu}-C^{-1}{}_J{}^p{}_{q\mu}\tilde{C}^J{}_k{}^j)Q^\beta\}+\delta^j_k L\equiv 0.$$

If we select the fields $\tilde{B}^I{}_k{}^j(x)$ and $\tilde{C}^J{}_k{}^j(x)$ as follows

$$B_I{}^l{}_\mu \tilde{B}^I{}_k{}^j=-\delta^l_k b^j{}_\mu\,, \tag{4.11}$$

$$\tilde{C}^J{}_k{}^j=C^J{}_p{}^{q\mu}C^{jlp}_{kmq}A^m{}_{l\mu}\,, \tag{4.12}$$

then the above identities become

$$\frac{\partial L^*}{\partial Q^\alpha}T^j{}_k{}^\alpha{}_\beta Q^\beta+\frac{\partial L^*}{\partial(\nabla_\mu Q^\tau)}\{T^j{}_k{}^\tau{}_\beta \nabla_\mu Q^\beta-b_k{}^\nu b^j{}_\mu \nabla_\nu Q^\tau\}+\delta^j_k L^*\equiv 0. \tag{4.13}$$

Using the relations which are obtained from a relation $L^*(Q^\alpha, \nabla_\mu Q^\alpha)=L(Q^\alpha, h_k{}^\mu \nabla_\mu Q^\alpha)$, namely,

$$\frac{\partial L^*}{\partial Q^\alpha}=\frac{\partial L}{\partial Q^\alpha}\,,\qquad \frac{\partial L^*}{\partial(\nabla_\mu Q^\tau)}=\frac{\partial L}{\partial(\nabla_k Q^\tau)}h_k{}^\mu,$$

and multiplying $\omega^k{}_j$ to the both sides of (4.13), and taking the sum for k, j, we have the following identity:

$$\frac{\partial L}{\partial Q^\alpha}\omega^\alpha{}_\beta Q^\beta+\frac{\partial L}{\partial(\nabla_k Q^\tau)}\{\omega^\tau{}_\beta \nabla_k Q^\beta-\omega^l{}_k \nabla_l Q^\tau\}+\omega^k{}_k L$$

$$+\frac{\partial L^*}{\partial(\nabla_\mu Q^\tau)}\omega^k{}_j(h_k{}^\nu h^j{}_\mu-b_k{}^\nu b^j{}_\mu)\nabla_\nu Q^\tau\equiv 0.$$

Because of a condition (3.5) for L, we get the conditions for the new fields $b_k{}^\mu(x)$:

$$h_k{}^\nu h^j{}_\mu-b_k{}^\nu b^j{}_\mu\equiv 0. \tag{4.14}$$

Next, let us discuss the identities (4.3). According to the same insertion with case of the consideration of identities (4.1), the identities (4.3) become

$$\delta^\nu_\mu Q^\tau{}_{,\rho}-b_l{}^\nu \nabla_\nu Q^\tau B_I{}^l{}_\mu \bar{B}^I{}_\rho{}^\nu+Q^\beta T^j{}_k{}^\tau{}_\beta C^{-1}{}_J{}^k{}_{j\mu}\bar{C}^J{}_\rho{}^\nu\equiv 0.$$

If we select the fields $\bar{B}^I{}_\rho{}^\nu(x)$ and $\bar{C}^J{}_\rho{}^\nu(x)$ as follows

$$B_I{}^l{}_\mu \bar{B}^I{}_\rho{}^\nu=\delta^\nu_\mu b^l{}_\rho\,, \tag{4.15}$$

$$\bar{C}^J{}_\rho{}^\nu = -C^J{}_k{}^{j\nu} A^k{}_{j\rho}\,, \tag{4.16}$$

then the above identities are satisfied.

Finally, let us go to a consideration of the transformation character of fields $A^\alpha{}_{\beta\mu}(x)$ and $b^k{}_\mu(x)$ under the extended transformations (3.7) and (3.8). For this purpose, we shall use the requirement of the covariant character of $\nabla_\mu Q^\alpha$.

If we use the relations $\nabla_\mu Q^\alpha = Q^\alpha{}_{,\mu} - A^\alpha{}_{\beta\mu} Q^\beta$, $\delta(Q^\alpha{}_{,\mu}) = (\delta Q^\alpha)_{,\mu} - (\delta x^\nu)_{,\mu} Q^\alpha{}_{,\nu}$, and if we utilize the transformation characters (3.7) and (3.9), then we obtain the following transformation character of $A^\alpha{}_{\beta\mu}$ from the covariant character (3.13) of $\nabla_\mu Q^\alpha$:

$$\begin{aligned} &A^\alpha{}_{\beta\mu} \longrightarrow A^\alpha{}_{\beta\mu} + \delta A^\alpha{}_{\beta\mu}\,, \\ &\delta A^\alpha{}_{\beta\mu} = -\epsilon^\nu{}_{,\mu} A^\alpha{}_{\beta\nu} + \omega^\alpha{}_\gamma A^\gamma{}_{\beta\mu} - \omega^\gamma{}_\beta A^\alpha{}_{\gamma\mu} + \omega^\alpha{}_{\beta,\mu}\,. \end{aligned} \tag{4.17}$$

On the other hand, from (4.6), (4.4) and the transformation character (3.15) of A^J, we get the following infinitesimal transformation of $A^\alpha{}_{\beta\mu}$:

$$\begin{aligned} \delta A^\alpha{}_{\beta\mu} = {} & \delta(T^j{}_k{}^\alpha{}_\beta\, C^{-1}{}_J{}^k{}_{j\mu}) A^J \\ & + T^j{}_k{}^\alpha{}_\beta\, C^{-1}{}_J{}^k{}_{j\mu} (\epsilon^\nu{}_{,\rho} \bar{C}^J{}_\nu{}^\rho + \omega^l{}_m \tilde{C}^J{}_l{}^m + \omega^l{}_{m,\nu}\, C^J{}_l{}^{m\nu}). \end{aligned}$$

By taking the use of the relations (4.12) and (4.16) for $\tilde{C}^J{}_l{}^m$ and $\bar{C}^J{}_\nu{}^\rho$ respectively, the above transformation becomes as follows

$$\delta A^\alpha{}_{\beta\mu} = \delta(T^j{}_k{}^\alpha{}_\beta C^{-1}{}_J{}^k{}_{j\mu}) A^J - \epsilon^\nu{}_{,\mu} A^\alpha{}_{\beta\nu} + \omega^l{}_m\, T^j{}_k{}^\alpha{}_\beta\, C^{mpk}_{lqj}\, A^q{}_{p\mu} + \omega^\alpha{}_{\beta,\mu}\,. \tag{4.18}$$

Comparing (4.17) and (4.18), we get the conditions for the fields $C^J{}_k{}^{l\mu}(x)$:

$$\delta C^{-1}{}_J{}^\alpha{}_{\beta\mu} = \omega^\alpha{}_\gamma\, C^{-1}{}_J{}^\gamma{}_{\beta\mu} - \omega^\gamma{}_\beta\, C^{-1}{}_J{}^\alpha{}_{\gamma\mu} - T^j{}_k{}^\alpha{}_\beta (\omega^l{}_m\, C^{mpk}_{lqj}\, C^{-1}{}_J{}^q{}_{p\mu}),$$

where we have used (4.4) and (4.6), and we have put $C^{-1}{}_J{}^\alpha{}_{\beta\mu} = T^j{}_k{}^\alpha{}_\beta C^{-1}{}_J{}^k{}_{j\mu}$.

If we consider the transformation character of $T^j{}_k{}^\alpha{}_\beta$:

$$\delta T^j{}_k{}^\alpha{}_\beta = \omega^\alpha{}_\gamma\, T^j{}_k{}^\gamma{}_\beta - \omega^\gamma{}_\beta\, T^j{}_k{}^\alpha{}_\gamma\,,$$

we obtain the following transformation character of $C^{-1}{}_J{}^k{}_{j\mu}$:

$$\delta C^{-1}{}_J{}^k{}_{j\mu} = -\omega^l{}_m\, C^{mpk}_{lqj}\, C^{-1}{}_J{}^q{}_{p\mu}\,.$$

Namely, the fields $C^J{}_k{}^{j\mu}(x)$ must be transformed as follows

$$\begin{aligned} &C^J{}_k{}^{j\mu} \longrightarrow C^J{}_k{}^{j\mu} + \delta C^J{}_k{}^{j\mu}, \\ &\delta C^J{}_k{}^{j\mu} = \omega^l{}_m\, C^{mjp}_{lkq}\, C^J{}_p{}^{q\mu}. \end{aligned} \tag{4.19}$$

Next let us go to discuss a transformation character of $b^k{}_\mu$. From

(4.7) and the transformation character (3.14) of b^l, we get the following infinitesimal transformation of $b^k{}_\mu$:

$$\delta b^k{}_\mu = (\delta B_l{}^k{}_\mu) b^l + \epsilon^\nu{}_{,\rho}\, B_l{}^k{}_\mu\, \bar{B}^l{}_\nu{}^\rho + \omega^j{}_l\, B_l{}^k{}_\mu\, \tilde{B}^l{}_j{}^l .$$

Using the selections (4.11) and (4.15) for $\tilde{B}^l{}_j{}^l$ and $\bar{B}^l{}_\nu{}^\rho$ respectively, the above transformation becomes

$$\delta b^k{}_\mu = (\delta B_l{}^k{}_\mu) b^l + \epsilon^\nu{}_{,\mu}\, b^k{}_\nu - \omega^k{}_j\, b^j{}_\mu .$$

Here let us remember the conditions (4.14) for $b^k{}_\mu$. For these conditions are satisfied, it is sufficient if we make the fields $b^k{}_\mu(x)$ to identify with vierbein $h^k{}_\mu(x)$ locally. If the fields $B_l{}^k{}_\mu(x)$ are transformed as follows

$$\delta B_l{}^k{}_\mu = 2(-\epsilon^\nu{}_{,\mu}\, B_l{}^k{}_\nu + \omega^k{}_l\, B_l{}^l{}_\mu), \tag{4.20}$$

the fields $b^k{}_\mu(x)$ are transformed in the same way as vierbein $h^k{}_\mu(x)$, namely,

$$\begin{aligned} & b^k{}_\mu \longrightarrow b^k{}_\mu + \delta b^k{}_\mu , \\ & \delta b^k{}_\mu = -\epsilon^\nu{}_{,\mu}\, b^k{}_\nu + \omega^k{}_l\, b^l{}_\mu . \end{aligned} \tag{4.21}$$

Consequently, if we select suitably the fields $\bar{B}^l{}_\nu{}^\mu(x)$, $\tilde{B}^l{}_k{}^j(x)$, $B_l{}^k{}_\mu(x)$, $\bar{C}^l{}_\nu{}^\mu(x)$, $\tilde{C}^l{}_k{}^j(x)$ *and* $C^l{}_k{}^{j\mu}(x)$ on $\mathscr{M}$, *then the new fields* $b_k{}^\mu(x)$ *can be identified with vierbein* $h_k{}^\mu(x)$ *locally.*

The transformation characters of new fields $b_k{}^\mu(x)$ and $A^\alpha{}_{\beta\mu}(x)$ under the extended infinitesimal transformations (3.7) and (3.8), are given by the following:

$$\begin{aligned} & b_k{}^\mu \longrightarrow b_k{}^\mu + \delta b_k{}^\mu , \\ & \delta b_k{}^\mu = \epsilon^\mu{}_{,\nu}\, b_k{}^\nu - \omega^l{}_k\, b_l{}^\mu , \end{aligned} \tag{4.22}$$

$$\begin{aligned} & A^\alpha{}_{\beta\mu} \longrightarrow A^\alpha{}_{\beta\mu} + \delta A^\alpha{}_{\beta\mu} , \\ & \delta A^\alpha{}_{\beta\mu} = -\epsilon^\nu{}_{,\mu}\, A^\alpha{}_{\beta\nu} + \omega^\alpha{}_\gamma\, A^\gamma{}_{\beta\mu} - \omega^\gamma{}_\beta\, A^\alpha{}_{\gamma\mu} + \omega^\alpha{}_{\beta,\mu} . \end{aligned} \tag{4.23}$$

We call $\nabla_\mu Q^\alpha$, *which is given by* (4.5), *the gauge covariant differentiation of fields* $Q^\alpha(x)$ *and the new fields* $b_k{}^\mu(x)$, $A^\alpha{}_{\beta\mu}(x)$ *are called the gauge field.*

We can think so that a connection $\Gamma^\alpha_{\beta\mu} = -A^\alpha{}_{\beta\mu}(x)$ *have been introduced on the manifold which was made by the fields* $Q^\alpha(x)$, *where the fields* $Q^\alpha(x)$ *have the transformation character* (3.9) *under the extended infinitesimal transformations* (3.7) and (3.8).

§ 5. Connection on a manifold $\mathscr{M}$.

When the field $Q^\alpha(X)$ is a vector field $Q^i(X)$ on M, the transformation character of $Q^l(x)$ under the extended transformation (3.7) and (3.8) must become as follows

$$Q^l(x) \longrightarrow Q^l(x)+\delta Q^l(x),$$
$$\delta Q^l(x)=\omega^k{}_j(x)T^j{}_k{}^l{}_m Q^m(x)=\omega^l{}_m(x)Q^m(x),$$

where we have used (3.9).
Hence we get

$$T^j{}_k{}^l{}_m=\delta^j_m\,\delta^l_k, \qquad C^{klp}_{jmq}=\delta^k_m\,\delta^l_q\,\delta^p_j-\delta^k_q\,\delta^l_j\,\delta^p_m. \tag{5.1}$$

Thus the gauge covariant differentiation $\nabla_\mu Q^l$ of a vector field $Q^l(x)$ becomes

$$\nabla_\mu Q^l=Q^l{}_{,\mu}-A^l{}_{m\mu}Q^m.$$

Now we introduce a covariant differentiation $\nabla_\mu Q^\nu$ of a vector field $Q^\nu(x)$ as follows

$$\nabla_\mu Q^\nu=b_l{}^\nu\,\nabla_\mu Q^l, \tag{5.2}$$

where we have put

$$Q^\nu(x)=b_l{}^\nu(x)Q^l(x). \tag{5.3}$$

From this definition, we obtain the usual formula of covariant differentiation

$$\nabla_\mu Q^\nu=Q^\nu{}_{,\mu}+\Gamma^\nu_{\tau\mu}Q^\tau, \tag{5.4}$$

where we have put

$$\Gamma^\nu_{\tau\mu}=-(b^l{}_\tau\,b_l{}^\nu{}_{,\mu}+A^\nu{}_{\tau\mu}), \tag{5.5}$$

$$A^\nu{}_{\tau\mu}=b_k{}^\nu b^j{}_\tau\,A^k{}_{j\mu}. \tag{5.6}$$

The covariant character of $\nabla_\mu Q^\nu$ under the extended transformation is obtained easily, if we notice the following transformations of $\nabla_\mu Q^l$ and $b_k{}^\nu$:

$$\delta(\nabla_\mu Q^l)=-\epsilon^\tau{}_{,\mu}\nabla_\tau Q^l+\omega^l{}_m\,\nabla_\mu Q^m,$$
$$\delta b_k{}^\nu=\epsilon^\nu{}_{,\tau}\,b_k{}^\tau-\omega^l{}_k\,b_l{}^\nu.$$

The covariant differentiation of a mixed tensor of rank n with k covariant and j contravariant indices is obtained through the same process as the above discussion, namely,

$$\nabla_\mu Q^{\nu_1\cdots\cdots\nu_j}{}_{\rho_1\cdots\cdots\rho_k}=Q^{\nu_1\cdots\cdots\nu_j}{}_{\rho_1\cdots\cdots\rho_k,\mu}$$

$$+\Gamma^{\nu_1}_{\tau\mu} Q^{\tau\nu_2\cdots\cdots\nu}{}_{\rho_1\cdots\cdots\rho_k}+\cdots\cdots+\Gamma^{\nu_j}_{\tau\mu} Q^{\nu_1\cdots\cdots\nu_{j-1}\tau}{}_{\rho_1\cdots\cdots\rho_k}$$
$$-\Gamma^{\tau}_{\rho_1\mu} Q^{\nu_1\cdots\cdots\nu_j}{}_{\tau\rho_2\cdots\cdots\rho_k}-\cdots\cdots-\Gamma^{\tau}_{\rho_k\mu} Q^{\nu_1\cdots\cdots\nu_j}{}_{\rho_1\cdots\cdots\rho_{k-1}\tau}\,. \tag{5.7}$$

We call $\Gamma^{\nu}_{\tau\mu}$ which is given by (5.5) the connection on a manifold $\mathscr{M}$.

The covariant differentiation has the following properties: for the tensor fields Q_1 and Q_2,

(1) $\nabla_\mu(a_1 Q_1+a_2 Q_2)=a_1 \nabla_\mu Q_1+a_2 \nabla_\mu Q_2$,

where a_1 and a_2 are constants,

(2) $\nabla_\mu(Q_1 Q_2)=(\nabla_\mu Q_1)Q_2+Q_1(\nabla_\mu Q_2)$.

(3) $\nabla_\mu Q=Q_{,\mu}$ for the scalar field $Q(x)$.

Next we proceed to discuss the transformation character of connection $\Gamma^{\nu}_{\tau\mu}$.

From the relations (5.5), we have

$$\delta\Gamma^{\nu}_{\tau\mu}=-\{\delta(b^l{}_\tau\, b_l{}^\nu{}_{,\mu})+\delta A^\nu{}_{\tau\mu}\}. \tag{5.8}$$

Paying our attention to the relations (5.1) and the transformation (4.23), we get

$$\delta A^k{}_{j\mu}=-\epsilon^\tau{}_{,\mu}\, A^k{}_{j\tau}+\omega^k{}_m\, A^m{}_{j\mu}-\omega^m{}_j\, A^k{}_{m\mu}+\omega^k{}_{j,\mu}\,. \tag{5.9}$$

Hence it is easy to deduce the following formula:

$$\delta A^\nu{}_{\tau\mu}=\epsilon^\nu{}_{,\rho}\, A^\rho{}_{\tau\mu}-\epsilon^\rho{}_{,\tau}\, A^\nu{}_{\rho\mu}-\epsilon^\rho{}_{,\mu}\, A^\nu{}_{\tau\rho}+b_k{}^\nu\, b^j{}_\tau\, \omega^k{}_{j,\mu}\,, \tag{5.10}$$

if we use the relations (5.6) and the transformation character (4.22) of $b_k{}^\nu$. Further, from (4.22) and $\delta(b_l{}^\nu{}_{,\mu})=(\delta b_l{}^\nu)_{,\mu}-\epsilon^\rho{}_{,\mu}\, b_l{}^\nu{}_{,\rho}$, we have

$$\delta(b^l{}_\tau\, b_l{}^\nu{}_{,\mu})=\epsilon^\nu{}_{,\rho}\, b^l{}_\tau\, b_l{}^\rho{}_{,\mu}-\epsilon^\rho{}_{,\tau} b^l{}_\rho\, b_l{}^\nu{}_{,\mu}-\epsilon^\rho{}_{,\mu} b^l{}_\tau\, b_l{}^\nu{}_{,\rho}+\epsilon^\nu{}_{,\tau\mu}-b_k{}^\nu\, b^j{}_\tau\, \omega^k{}_{j,\mu}\,. \tag{5.11}$$

Inserting (5.10) and (5.11) into (5.8), we obtain

$$\delta\Gamma^{\nu}_{\tau\mu}=\epsilon^\nu{}_{,\rho}\, \Gamma^{\rho}_{\tau\mu}-\epsilon^\rho{}_{,\tau}\, \Gamma^{\nu}_{\rho\mu}-\epsilon^\rho{}_{,\mu}\, \Gamma^{\nu}_{\tau\rho}-\epsilon^\nu{}_{,\tau\mu}\,.$$

The result is that the transformation character of connection $\Gamma^{\nu}_{\tau\mu}$ is given by the following:

$$\Gamma^{\nu}_{\tau\mu}\longrightarrow \Gamma^{\nu}_{\tau\mu}+\delta\Gamma^{\nu}_{\tau\mu},$$
$$\delta\Gamma^{\nu}_{\tau\mu}=\epsilon^\nu{}_{,\rho}\, \Gamma^{\rho}_{\tau\mu}-\epsilon^\rho{}_{,\tau}\, \Gamma^{\nu}_{\rho\mu}-\epsilon^\rho{}_{,\mu}\, \Gamma^{\nu}_{\tau\rho}-\epsilon^\nu{}_{,\tau\mu}\,. \tag{5.12}$$

This transformation character coincides with the infinitesimal transformation character of affine connection.

Finaly we shall investigate a relation between the connection $\Gamma^{\nu}_{\tau\mu}$ and the Christoffel symbol $\left\{ {\nu \atop \tau\mu} \right\}$.

For this purpose we must introduce a metric tensor $g_{\mu\nu}(x)$ on $\mathscr{M}$.

Let U be a neighborhood of a point $p \in \mathscr{M}$ and X^k a global coordinate of p and x^μ a local coordinate of p.

The square of the invariant length of the infinitesimal line element in the Minkowski space M is given by

$$ds^2 = \eta_{kj}\, dX^k\, dX^j,$$

where η_{kj} is a covariant tensor of degree 2 and when ϕ is a Lorentz coordinate system, we have

$$-\eta_{00} = \eta_{11} = \eta_{22} = \eta_{33} = 1, \qquad \eta_{kj} = 0 \text{ for } k \neq j.$$

By the transformation formula $dX^k = h^k{}_\mu(x) dx^\mu$, we get

$$ds^2 = \eta_{kj}\, h^k{}_\mu(x)\, h^j{}_\nu(x)\, dx^\mu\, dx^\nu.$$

Since the vierbein $h^k{}_\mu(x)$ could be identified with the gauge field $b^k{}_\mu(x)$ locally, the above formula becomes

$$ds^2 = g_{\mu\nu}(x)\, dx^\mu\, dx^\nu, \tag{5.13}$$

where we have put

$$g_{\mu\nu}(x) = \eta_{kj}\, b^k{}_\mu(x)\, b^j{}_\nu(x). \tag{5.14}$$

The field $g_{\mu\nu}(x)$ is obviously a symmetrical covariant tensor of degree 2 on $\mathscr{M}$.

We accept that the tensor field $g_{\mu\nu}(x)$ is utilized as a metric tensor on $\mathscr{M}$.

Now we proceed to discuss a relation between the connection $\Gamma^\nu_{\tau\mu}$ and the Christoffel symbol $\left\{ {\nu \atop \tau\mu} \right\}$.

From the definition of the gauge covariant differentiation and the covariant differentiation, we get

$$\nabla_\rho g_{\mu\nu} = b^k{}_\mu\, b^j{}_\nu (\eta_{jl}\, A^l{}_{k\rho} + \eta_{kl}\, A^l{}_{j\rho}), \tag{5.15}$$

where we have used

$$\nabla_\rho\, \eta_{kj} = \eta_{jl}\, A^l{}_{k\rho} + \eta_{kl}\, A^l{}_{j\rho}\ .$$

On the other hand, by using the formula (5.7), the covariant differentiation of $g_{\mu\nu}(x)$ becomes as follows

$$\nabla_\rho\, g_{\mu\nu} = g_{\mu\nu,\rho} - \Gamma^\sigma_{\mu\rho}\, g_{\sigma\nu} - \Gamma^\sigma_{\nu\rho}\, g_{\mu\sigma}\ .$$

Thus we have the following relations:

$$g_{\mu\nu;\rho}-\Gamma^{\sigma}_{\mu\rho}\,g_{\sigma\nu}-\Gamma^{\sigma}_{\nu\rho}\,g_{\mu\sigma}=K_{\mu\nu\rho}\,, \tag{a}$$

where we have put

$$K_{\mu\nu\rho}=b^{k}{}_{\mu}\,b^{j}{}_{\nu}(\eta_{jl}\,A^{l}{}_{k\rho}+\eta_{kl}\,A^{l}{}_{j\rho}). \tag{5.16}$$

Exchanging μ for ρ and ν for ρ in the formula (a), we have

$$g_{\rho\nu;\mu}-\Gamma^{\sigma}_{\rho\mu}\,g_{\sigma\nu}-\Gamma^{\sigma}_{\nu\mu}\,g_{\rho\sigma}=K_{\rho\nu\mu}\,, \tag{b}$$

$$g_{\mu\rho;\nu}-\Gamma^{\sigma}_{\mu\nu}\,g_{\sigma\rho}-\Gamma^{\sigma}_{\rho\nu}\,g_{\mu\sigma}=K_{\mu\rho\nu}\,. \tag{c}$$

Carring out (b)+(c)−(a) and some calculation, *we obtain the familiar relation between the connection* $\Gamma^{\nu}_{\tau\mu}$ *and the Cristoffel symbol* $\begin{Bmatrix}\nu\\ \tau\mu\end{Bmatrix}$:

$$\Gamma^{\tau}_{\mu\nu}=\begin{Bmatrix}\tau\\ \mu\nu\end{Bmatrix}+g^{\tau\rho}(\Gamma_{[\mu\rho]\nu}+\Gamma_{[\nu\rho]\mu}+\Gamma_{[\mu\nu]\rho})-\frac{1}{2}g^{\tau\rho}(K_{\rho\mu\nu}+K_{\rho\nu\mu}-K_{\mu\nu\rho}), \tag{5.17}$$

where we have put

$$\Gamma_{[\mu\rho]\nu}=g_{\nu\sigma}\,\Gamma^{\sigma}_{[\mu\rho]}\,, \tag{5.18}$$

$$\Gamma^{\sigma}_{[\mu\rho]}=\frac{1}{2}(\Gamma^{\sigma}_{\mu\rho}-\Gamma^{\sigma}_{\rho\mu}). \tag{5.19}$$

§ 6. Weyl's reine infinitesimalgeometrie.

First of all, we shall show that if the infinitesimal affine transformation (3.1) is an infinitesimal Lorentz transformation:

$$\delta X^{k}=\omega^{k}{}_{j}\,X^{j}+\epsilon^{k},\qquad \omega^{kj}+\omega^{jk}=0, \tag{6.1}$$

then the gauge fields $A^{k}{}_{j\mu}(x)$ are the antisymmetric fields:

$$A^{kj}{}_{\mu}(x)+A^{jk}{}_{\mu}(x)=0,$$

where we have used $\omega^{kj}=\eta^{jl}\omega^{k}{}_{l}$ and $A^{kj}{}_{\mu}(x)=\eta^{jl}A^{k}{}_{l\mu}(x)$.

In fact, under this transformation (6.1), the fields $Q^{\alpha}(X)$ must be transformed as follows

$$Q^{\alpha}(X)\longrightarrow Q^{\alpha}(X)+\delta Q^{\alpha}(X),$$

$$\delta Q^{\alpha}(X)=\omega^{kj}\,T_{[jk]}{}^{\alpha}{}_{\beta}\,Q^{\beta}(X),$$

where we have put

$$T_{[kj]}{}^{\alpha}{}_{\beta}=\frac{1}{2}(T_{kj}{}^{\alpha}{}_{\beta}-T_{jk}{}^{\alpha}{}_{\beta}),$$

$$T_{kj}{}^{\alpha}{}_{\beta}=\eta_{kl}T^{l}{}_{j}{}^{\alpha}{}_{\beta}.$$

Thus the extended transformations of infinitesimal Lorentz transformation become

$$x^{\mu}\longrightarrow x^{\mu}+\delta x^{\mu},\qquad \delta x^{\mu}=\epsilon^{\mu}(x),\tag{6.2}$$

$$X^{k}\longrightarrow X^{k}+\delta X^{k},\qquad \delta X^{k}=\omega^{k}{}_{j}(x)X^{j},\qquad \omega^{kj}(x)+\omega^{jk}(x)=0,\tag{6.3}$$

Under this extended transformations (6.2) and (6.3), the fields $Q^{\alpha}(x)$ are transformed as follows

$$Q^{\alpha}(x)\longrightarrow Q^{\alpha}(x)+\delta Q^{\alpha}(x),$$
$$\delta Q^{\alpha}(x)=\omega^{kj}(x)T_{[jk]}{}^{\alpha}{}_{\beta}\,Q^{\beta}(x).\tag{6.4}$$

In addition, the new fields $b^{I}(x)$ and $A^{J}(x)$ have the following transformation character:

$$\delta b^{I}(x)=\epsilon^{\nu}{}_{,\mu}(x)\bar{B}^{I}{}_{\nu}{}^{\mu}(x)+\omega^{kj}(x)\tilde{B}^{I}{}_{[kj]}(x),\tag{6.5}$$

$$\delta A^{J}(x)=\epsilon^{\nu}{}_{,\mu}(x)\bar{C}^{J}{}_{\nu}{}^{\mu}(x)+\omega^{kj}(x)\tilde{C}^{J}{}_{[kj]}(x)+\omega^{kj}{}_{,\mu}(x)C^{J}{}_{[kj]}{}^{\mu}(x).\tag{6.6}$$

Therefore if we assume $T^{j}{}_{k}{}^{\alpha}{}_{\beta}$, $\tilde{B}^{I}{}_{k}{}^{j}(x)$, $\tilde{C}^{J}{}_{k}{}^{j}(x)$, $C^{J}{}_{k}{}^{j\mu}(x)$, as the antisymmetric quantities $T_{[jk]}{}^{\alpha}{}_{\beta}$, $\tilde{B}^{I}{}_{[kj]}(x)$, $\tilde{C}^{J}{}_{[kj]}(x)$, $C^{J}{}_{[kj]}{}^{\mu}(x)$ respectively, all discussions in § 4 are valid.

Thus we have the antisymmetric fields $A^{kj}{}_{\mu}(x)$ from the formula (4.4):

$$A^{kj}{}_{\mu}=C^{-1}{}_{J}{}^{[kj]}A^{J}.$$

At the starting point, we shall consider a special case of the infinitesimal affine transformation (3.1), namely, the following infinitesimal transformation:

$$X^{k}\longrightarrow X^{k}+\delta X^{k},\qquad \delta X^{k}=\lambda X^{k}+\omega^{k}{}_{j}X^{j}+\epsilon^{k},\qquad \omega^{kj}+\omega^{jk}=0.\tag{6.7}$$

Under this transformation (6.7), the fields $Q^{l}(X)$ are assumed to be transformed as

$$Q^{l}(X)+\delta Q^{l}(X),$$
$$\delta Q^{l}(X)=\lambda Q^{l}(X)+\omega^{l}{}_{m}Q^{m}(X).$$

Thus the constant coefficients $T^{j}{}_{k}{}^{l}{}_{m}$ and the constants C^{klp}_{jmq} are given by (5.1) and the extended transformations of infinitesimal transformation (6.7) become as follows

$$x^{\mu}\longrightarrow x^{\mu}+\delta x^{\mu},\qquad \delta x^{\mu}=\epsilon^{\mu}(x),$$

$$X^k \longrightarrow X^k+\delta X^k, \quad \delta X^k=\lambda(x)X^k+\omega^k{}_j(x)X^j, \quad \omega^{kj}(x)+\omega^{jk}(x)=0. \tag{6.8}$$

Under these extended transformations, the fields $Q^l(x)$ are transformed as

$$\begin{aligned}&Q^l(x)\longrightarrow Q^l(x)+\delta Q^l(x),\\ &\delta Q^l(x)=\lambda(x)Q^l(x)+\omega^{kj}(x)T_{[jk]}{}^l{}_m Q^m(x).\end{aligned} \tag{6.9}$$

where we have

$$T_{[kj]}{}^l{}_m=\frac{1}{2}(\eta_{km}\,\delta^l_j-\eta_{jm}\,\delta^l_k).$$

Therefore the gauge fields $A^k{}_{j\mu}(x)$ satisfy the following relations:

$$A^{kj}{}_\mu(x)+A^{jk}{}_\mu(x)=0 \qquad \text{for } k \neq j,$$

$$-A^{00}{}_\mu(x)=A^{11}{}_\mu(x)=A^{22}{}_\mu(x)=A^{33}{}_\mu(x)=-\frac{1}{2}\varphi_\mu(x).$$

In addition, if we take into account the relations $\nabla_\rho g_{\mu\nu}=K_{\mu\nu\rho}$ and the formula (5.16), the covariant differentiation of metric tensor $g_{\mu\nu}(x)$ becomes as follows

$$\nabla_\rho g_{\mu\nu}=K_{\mu\nu\rho}=-g_{\mu\nu}\,\varphi_\rho\,. \tag{6.10}$$

Further if we append the conditions of the symmetric property for the indices μ, ν of connection $\Gamma^\tau_{\mu\nu}$, namely,

$$A^\tau{}_{\mu\nu}-A^\tau{}_{\nu\mu}=b_k{}^\tau(b^k{}_{\mu,\nu}-b^k{}_{\nu,\mu}), \tag{6.11}$$

then we obtain the following formula for the connection $\Gamma^\tau_{\mu\nu}$:

$$\Gamma^\tau_{\mu\nu}=\frac{1}{2}g^{\tau\rho}\{D_\nu\, g_{\rho\mu}+D_\mu\, g_{\rho\nu}-D_\rho\, g_{\mu\nu}\}, \tag{6.12}$$

where we have put $D_\mu=\partial_\mu+\varphi_\mu$.

The connection $\Gamma^\tau_{\mu\nu}$ *which is given by* (6.12) *agree with the connection in the weyl's infinitesimalgeometrie.*

Next we investigate the transformation character of $\varphi_\mu(x)$.

By using the transformation character (5.9) of $A^k{}_{j\mu}(x)$, we obtain the transformation character of$\varphi_\mu(x)$:

$$\begin{aligned}&\varphi_\mu(x)\longrightarrow\varphi_\mu(x)+\delta\varphi_\mu(x),\\ &\delta\varphi_\mu(x)=-\epsilon^\nu{}_{,\mu}(x)\varphi_\nu(x)-2\,\lambda_{,\mu}(x).\end{aligned} \tag{6.13}$$

The second term of (6.13) *gives the Weyl's infinitesimal gauge transformation.*

Finally, by using the formula (5.17) for $\Gamma^\tau_{\mu\nu}$ and the formula (5.16) for $K_{\mu\nu\rho}$, *it is easy to show that* (1) *if we start from the infinitesimal Lorentz transformation* (6.1), *we have* $\nabla_\rho g_{\mu\nu}=K_{\mu\nu\rho}=0$, *namely*,

$$\Gamma^\tau_{\mu\nu}=\begin{Bmatrix}\tau\\ \mu\nu\end{Bmatrix}+g^{\tau\rho}(\Gamma_{[\mu\rho]\nu}+\Gamma_{[\nu\rho]\mu}+\Gamma_{[\mu\nu]\rho}),$$

(*we no have only the transference of distance, therefore we have the torsion and the transference of direction*), *and* (2) *if we give additionaly the symmetric conditions* (6.11) *for* $\Gamma^\tau_{\mu\nu}$, *we have*

$$\Gamma^\tau_{\mu\nu}=\begin{Bmatrix}\tau\\ \mu\nu\end{Bmatrix},$$

(*we have only the transference of direction*).

§7. Covariant differentiation of semi-spinor fields.

We shall start with an infinitesimal transformation of the same type as the transformation in §6:

$$\begin{aligned}&X^k\longrightarrow X^k+\delta X^k,\\ &\delta X^k=\lambda X^k+\omega^k{}_j X^j+\epsilon^k, \qquad \omega^{kj}+\omega^{jk}=0.\end{aligned}\tag{7.1}$$

Under this transformation, the fields $Q^\alpha(X)$ $(\alpha=0,1,2,3)$ are assumed to be transformed as

$$\begin{aligned}&Q^\alpha(X)\longrightarrow Q^\alpha(X)+\delta Q^\alpha(X),\\ &\delta Q^\alpha(X)=\lambda Q^\alpha(X)+\omega^{kj}T_{[jk]}{}^\alpha{}_\beta Q^\beta(X),\end{aligned}\tag{7.2}$$

where $T_{[jk]}$ are given by

$$T_{[jk]}=\frac{1}{4}(\gamma_j\gamma_k-\gamma_k\gamma_j),\tag{7.3}$$

and γ_k are the Dirac's matrices which satisfy the following relations:

$$\gamma_k\gamma_j+\gamma_j\gamma_k=2\eta_{kj}.\tag{7.4}$$

Under the extended tıansformation (6.8), the fields $Q^\alpha(x)$ are transformed as

$$\begin{aligned}&Q^\alpha(x)\longrightarrow Q^\alpha(x)+\delta Q^\alpha(x),\\ &\delta Q^\alpha(x)=\lambda(x)\,Q^\alpha(x)+\omega^{kj}(x)\,T_{[jk]}{}^\alpha{}_\beta\,Q^\beta(x).\end{aligned}\tag{7.5}$$

We call the fields $Q^\alpha(x)$ *the semi-spinor fields which have the transformation character* (7.5).

This transformation is obtained from the transformation (3.9), if we consider in (3.9) as follows

$$-\omega^{00}(x)=\omega^{11}(x)=\omega^{22}(x)=\omega^{33}(x)=\frac{1}{2}\lambda(x), \qquad T_{kj}=\frac{1}{2}\gamma_k\gamma_j.$$

The matrices T_{kj} satisfy the commutation relations:

$$[T_{kj}, T_{lm}]=\frac{1}{2}(\eta_{km}T_{jl}-\eta_{mk}T_{lj}+\eta_{jl}T_{km}-\eta_{lj}T_{mk}-\eta_{kl}T_{jm}$$
$$+\eta_{lk}T_{mj}-\eta_{jm}T_{kl}+\eta_{mj}T_{lk}). \tag{7.6}$$

Thus the constants C^{klp}_{jmq} are given by

$$C^{klp}_{jmq}=\frac{1}{2}(\eta_{km}\delta_{jq}\delta_{lp}-\eta_{mk}\delta_{lq}\delta_{jp}+\eta_{jl}\delta_{kq}\delta_{mp}-\eta_{lj}\delta_{mq}\delta_{kp}$$
$$-\eta_{kl}\delta_{jq}\delta_{mp}+\eta_{lk}\delta_{mq}\delta_{jp}-\eta_{jm}\delta_{kq}\delta_{lp}+\eta_{mj}\delta_{lq}\delta_{kp}). \tag{7.7}$$

The new fields $A^k{}_{j\mu}(x)$ satisfy the relations:

$$A^{kj}{}_\mu(x)+A^{jk}{}_\mu(x)=0 \qquad \text{for } k \neq j,$$

$$-A^{00}{}_\mu(x)=A^{11}{}_\mu(x)=A^{22}{}_\mu(x)=A^{33}{}_\mu(x)=-\frac{1}{2}\varphi_\mu(x),$$

and the gauge fields $A^\alpha{}_{\beta\mu}(x)$ are obtained from (4.6):

$$A^\alpha{}_{\beta\mu}(x)=T_{[jk]}{}^\alpha{}_\beta A^{kj}{}_\mu(x)-\delta^\alpha_\beta\,\varphi_\mu(x). \tag{7.8}$$

Thus we have the following gauge covariant differentiation of semi-spinor fields $Q^\alpha(x)$:

$$\nabla_\mu Q^\alpha(x)=D_\mu Q^\alpha(x)-T_{[jk]}{}^\alpha{}_\beta\,A^{kj}{}_\mu(x)Q^\beta(x). \tag{7.9}$$

In § 6 *we introduced a connection* (6.12) *into a manifold* $\mathscr{M}$ *by starting with the infinitesimal transformation* (7.1). *Therefore we may consider so that the spinor fields in the Wely's geometry had a connection* (7.8).

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