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On the existence of global, bounded, periodic and almost-periodic solutions of nonlinear parabolic equations

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1. Introduction

Let Ω be an open bounded set in the n-dimensional Euclidean space R^n with boundary $\partial\Omega$. Consider the following partial differential equation of parabolic type:

$$(1.\ 1)\qquad u_t-\sum_{i,j=1}^{n} a_{ij}(x,t)u_{x_ix_j}+\sum_{i=1}^{n} b_i(x.\ t)u_{x_i}+c(x,t)u+\beta(x,t,u)+f(x,t)=0\,,\qquad (x,t)\in\Omega\times(t_0,\infty)\,,$$

together with boundary condition

$$(1.\ 2)\qquad e(x,t)\frac{\partial u}{\partial\nu}+h(x,t)u=0\,,\ (x,t)\in\partial\Omega\times(t_0,\infty)\,,$$

where ν is the unit outward normal vector on $\partial\Omega$.

The existence theory concering global solutions to the initial-boundary value problem, i.e., (1. 1)-(1. 2) with additional condition $u(x,t_0)=u_0(x)$, $t_0\neq-\infty$, has been developed by a number of authors. But in most cases the following hypothesis on the nonlinear term is made:

$$(1.\ 3)\qquad -u(c(x,t)u+\beta(x,t,u)+f(x,t))\leqq Au^2+B\,,$$

where A, B are positive constants. For instance, see O. A. Oleinik and S. N. Kruzkov [9], A. Friedman [4], O. A. Ladyzenskaja, V. A. Solonikov and N. N. Uralceva [7], D. H. Sattinger [13], C. V. Pao [10] and D. W. Bange [2]. Especially, in the case $\beta(x,t,u)$ is monotonically increasing in u, the condition (1. 3) is satisfied. For example $\beta(x,t,u)=|u|^\theta u$, $\theta>0$, is a typical one.

Bounded, periodic and almost-periodic solutions to the problem (1. 1)-(1. 2) with $t_0=-\infty$ have been considered also by several people: I. I. Smulev [11], [12], J. S. Kolesov [6], L. Amerio-G. Prouse [1], M. Biroli

[3] and others. However the assumptions of the type (1. 3) are still assumed in their works.

The object of this paper is to establish existence theorems concerning global bounded, periodic and almost-periodic solutions to the problem (1. 1)-(1. 2) under different assumptions which permit (1. 3) to be broken. Roughly speaking we try to replace the condition (1. 3) by local ones. For example, if $c(x, t) \geq c_0 > 0$, $\beta(x, t, u) = -|u|^{\theta} u$, $\theta \geq 0$, and $\sup_{x,t}|f(x, t)|$ is sufficiently small, our assumptions will be satisfied. As is seen by this example, for our purpose it must be assumed in many cases that the initial values $u(x, 0) = u_0(x)$ and the inhomogeneous terms $f(x, t)$ are small. But these restrictions are not too artificial as is easily conjectured by the work of H. Fujita [5], where the initial-bounary value problem:

(1. 4) $\quad u_t - \Delta u = u^{1+\alpha}, \ \alpha > 0, \ u \geq 0,$

with

(1. 5 $\quad u|_{\partial\Omega} = 0, \ u(x, 0) = u_0(x) \geqq 0,$

is investigated (there the case $\Omega = R^n$ is also considered). It is established there that if $u_0(x)$ is small, then a global solution of (1. 4)-(1. 5) exists and if $u_0(x)$ is large the solution blows up in a finite time. See also M. Tsutsumi [14] where the result of H. Fujita is generalized.

Recently similar problem has been considered by one of the present authors [8] by use of so-called L^2-method. Here we employ a classical one and the solutions obtained are classical.

As a corollary of our result an existence theorem of elliptic boundary value problem is given.

2. Preliminaries

Following a standerd notation we introduce the following norms on a function $u(x, t)$ defined on a set $Q \subset R^{n+1}$.

$$|u|_0^Q = \sup_{P \in Q} |u(P)|,$$

$$|u|_\alpha^Q = |u|_0^Q + \sup_{\substack{P, P' \in Q \\ P \neq P'}} \frac{|u(P) - u(P')|}{d(P, P')^\alpha},$$

$$|u|_{1+\alpha}^Q = |u|_\alpha^Q + \sum_{i=1}^n |u_{x_i}|_\alpha^Q,$$

$$|u|_{2+\alpha}^Q = |u|_{1+\alpha}^Q + |u_t|_{1+\alpha}^Q + \sum_{i=1}^n |u_{x_i}|_{1+\alpha}^Q$$

where $0<\alpha<1$ and $d(P, P')=(|x-x'|^2+|t-t'|)^{\frac{1}{2}}$ for points $P=(x, t)$, $P'=(x', t')\in R^n\times R$.

We say $u(x, t)$ is in Hölder class $C_q(Q)$ if $|u|_q^Q<+\infty$, $q=0$, α, $1+\alpha$, $2+\alpha$.

Before stating our hypotheses we give several definitions.

DEFINITION 2. 1.

By a bounded solution to the initial-boundary value problem (1. 1)-(1. 2), $t_0\neq-\infty$, with initial value $u_0(x)$ we mean a function $u(x, t)$ defined on $\bar{\Omega}\times[t_0, \infty)$ such that $u\in C_{2+\alpha}(\bar{\Omega}\times[t_0, T])$ for any $T>t_0$ and for some $\alpha>0$, $u\in C_0(\bar{\Omega}\times[t_0, \infty))$ and the equations (1. 1), (1. 2) and the initial condition $u(x, 0)=u_0(t)$ are satisfied.

DEFINITION 2. 2.

We say a function $u(x, t)$ on $\bar{\Omega}\times(-\infty, \infty)$ is bounded solution to (1. 1)-(1. 2), $t_0=-\infty$, if $u\in C_{2+\alpha}(\bar{\Omega}\times[-T, T])$ for any $T>0, u\in C_0(\bar{\Omega}\times(-\infty, \infty))$ and (1. 1), (1. 2) are satisfied by $u(x, t)$.

DEFINITION 2. 3.

Let V be a Banach space with norm $\|\cdot\|_V$. A continuous function $u(t)$: $R\to V$ is said to be almost-periodic with respect to $\|\cdot\|_V$ or V-almost-periodic if to every $\varepsilon>0$ there corresponds a relatively dense set $\{\tau\}_\varepsilon\subset$R such that

$$\sup_t\|u(t+\tau)-u(t)\|_V\leq\varepsilon\,,\ \tau\in\{\tau\}_\varepsilon\,.$$

Definition 2. 3 is equivalent to the following (see [1]).

DEFINITIO 2. 3′ (Bochner).

A continuous function $u(t)$; $R\to V$ is V-almost periodic if, and only if, having taken an arbitrary real sequence $\{s_n\}$ there exists a subsequence $\{s_n'\}$ such that the sequence $\{u(t+s_n')\}$ converges uniformly in t.

Now we state our hypotheses. For $A=[t_0, \infty)$, $t_0\neq-\infty$, or $A=(-\infty, \infty)$ we impose:

(H$_1$). (2. 1) $c(x, t)>c_0$ and $\sum_{i,j=1}^{n}a_{ij}(x, t)\xi_i\xi_j\geq a_0\sum_{i=1}^{n}\xi_i^2$,

for $(x, t)\in\bar{\Omega}\times A$ and $\xi\in R^n$, where a_0, c_0 are positive, nonnegative, constants respectively. Moreover

$$(2.2)\qquad \begin{aligned} &(\mathrm{i})\quad e(x,t)\geq\delta_0>0 \text{ and } h(x,t)>0, \text{ or}\\ &(\mathrm{ii})\quad e(x,t)\equiv 0 \text{ and } h(x,t)>0. \end{aligned}$$

(H_2). $a_{ij}(x,t)$, $b_i(x,t)$, $c(x,t)$ and $f(x,t)$ are in class $C_\alpha(\bar{\Omega}\times I)$ and $e(x,t)$, $h(x,t)$ are in $C_{1+\alpha}(\partial\Omega\times I)$, where I denotes every compact set in A.
(H_3). $\partial\Omega$ is of class $C_{\alpha+2}$, i.e., for every point $p\in\partial\Omega\times(-\infty,\infty)$ there exists an $(n+1)$-dimensional neighborhood U such that $U\cap\partial\Omega\times(-\infty,\infty)$ can be represented in the form

$$x_k=g(x_1,x_2,\cdots,x_{k-1},x_{k-1},\cdots,x_n,t)$$

for some k, and the derivatives of $g(x,t)$ with orders ≤ 2 are all Hölder continuous with exponent α, $0<\alpha<1$.

Regarding the nonlinear term we require the following hypotheses.
(H_4). There exists a positive constant γ_0 such that

$$(2.3)\qquad |\beta(x_1,t_1,u_1)-\beta(x_2,t_2,u_2)|\leq k(\gamma,T)(|x_1-x_2|^\alpha+|t_1-t_2|^{\alpha/2}+|u_1-u_2|)$$

for (x_1,t_1), $(x_2,t_2)\in\Omega\times[-T,T]$ and $|u_1|$, $|u_2|\leq\gamma\leq\gamma_0$ (T: arbitrary),

$$(2.4)\qquad -\beta(x,t,u)u\leq K_0(|u|,t)|u|^2 \text{ if } |u|\leq\gamma_0,\ t\in T$$

and

$$(2.5)\qquad -c_0\gamma_0+M+K_0(\gamma_0,t)\gamma_0<0,$$

where $k(\gamma,T)$ and $K_0(\gamma,t)$ are functions of (γ,T) and (γ,t), respectively, and

$$M\equiv\sup_{\bar{\Omega}\times A}|f(x,t)|.$$

(H_5). There exists a positive constant γ_1 such that

$$(2.6)\qquad -(u_1-u_2)(\beta(x,t,u_1)-\beta(x,t,u_2))\leq K_1(|u_1-u_2|,\gamma_1)|u_1-u_2|^2$$

if $|u_i|$, $i=1,2,\leq\gamma_1$, and

$$(2.7)\qquad -c_0+K_1(\gamma,\gamma_1)<0 \text{ if } 0<\gamma\leqq 2\gamma_1,$$

where $K_1(\gamma,\gamma_1)$ is a nonincreasing function in γ defined on $[0, 2\gamma_1]$.

The hypothesis (H_5) is used for the uniqueness of bounded solution.

We need the following lemmas (see A. Friedman [4], D. A. Ladyzenskaja, V. A. Solonikov and N. N. Uralceva [7]).

LEMMA 2.1. ([Ths. 5.2, 5.3; 7])

Consider the initial-boundery problem for the linear equation:

$$(2.8)\quad u_t-\sum_{i,j=1}^{n} a_{ij}(x,t)u_{x_i x_j}+\sum_{i=1}^{n} b_i(x,t)u_{x_i}+c(x,t) = f(x,t)$$

on $\Omega\times(t_0,\infty)$ *with conditions* (1.2) *and* $u(x,t_0)=u_0(x)(t_0\neq-\infty)$. *In addtition to the assumptions* (H_1)-(H_3), *suppose that* $u_0(x)$ *is in* $C_{2+\alpha}(\bar{\Omega})$ *and satisfies the compatibility condition*

$$(2.9)\quad e(x,t_0)\frac{\partial}{\partial\nu}u_0(x)+h(x,t_0)u_0(x)=0 \quad on \quad \partial\Omega.$$

Then there exists one and only one solution $u(x,t)$ *to the initial-boundary problem satisfying the* $(2+\alpha)$*-estimate for each* $T>t_0$:

$$(2.10)\quad |u|_{2+\alpha}^{\bar{\Omega}\times[t_0,T]}\leq c_1(T-t_0)\ (|f|_{\alpha}^{\bar{\Omega}\times[t_0,T]}+|u_0|_{2+\alpha}^{\bar{\Omega}}),$$

where $c_1(T-t_0)$ *is a constant depending on* $T-t_0$ *and other various constants, but independent of* $|f|_{\alpha}^{\bar{\Omega}\times[t_0,T]}$ *and* $|u_0|_{2+\alpha}^{\bar{\Omega}}$.

LEMMA 2.2. ([Ths. 4 and 4′; 4])

Under the same conditions of Lemma 2.1, *any solution* $u(x,t)$ *to the initial-boundary problem satisfies, for any* $\delta\in(0,1)$, *the* (δ)*-estimate*

$$(2.11)\quad |u|_{\delta}^{\bar{\Omega}\times t_0,T]}\leq c_2(T-t_0)(|f|_{0}^{\bar{\Omega}\times[t_0,T]}+|u_0|_{2+\alpha}^{\bar{\Omega}}),$$

where $c_2(T-t_0)$ *is a similar constant to* c_1 *independent of* $|f|_{0}^{\bar{\Omega}\times[t_0,T]}$, $|u_0|_{2+\alpha}^{\bar{\Omega}}$.

By the use of above lemmas, a standard argument shows (cf. C. V. Pao [10]):

LEMMA 2.3.

In addition to the assumptions of Lemma 2.1, *suppose* (2.3) *is valid. Then for* $u_0(x)\in C_{2+\alpha}(\bar{\Omega})$ *with* $|u_0|_0^{\bar{\Omega}}<\gamma_0$, *the problem* (1.1)-(1.2) *with* $u(x,t_0)=u_0(x)$, $t_0\neq-\infty$, *has a unique "local solution"* $u(x,t)$ *in the sense that for some* $T_0>t_0$ $u(x,t)$ *satisfies* (1.1), (1.2) *and* $u(x,t_0)=u_0(x)$ *for* $x\in\bar{\Omega}$ *and* $t\in[t_0,T_0]$. *The value* T_0 *is determined by the largest interval* $[t_0,T_0]$ *on which* $|u(x,t)|\leq\gamma_0$ *on* $\bar{\Omega}$. *If* β *satisfies* (2.3) *for every finite* γ *with* $k(\gamma,I)<+\infty$ *for* $^{\forall}I$, *then* $u(x,t)$ *can be continued as long as it remains bounded on* $\bar{\Omega}$.

3. Initial-Boundary Value Problem

In this section we prove the following existence theorem of the initial-boundary value problem.

THEOREM 3.1.

In addition to hypotheses (H_1)-(H_4), *we assume that* $u_0(x)$ *is in* $c_{2+\alpha}(\bar{\Omega})$ *with* $|u_0|_0^{\Omega}<\gamma_0$ *and the compatibility condition* (2.9) *is fulfilled. Then the initial-boundary value problem* (1.1), (1.2) *with* $u(x, t_0)=u_0(x)$ *has a unique bounded solution* u *with* $|u|_0^{\bar{\Omega}\times[t_0,\infty)}\leq\gamma_0$.

In order to prove the thorem we need the following standard lemma.

LEMMA 3.1.

Let $u(x, t)$ *be a solution of the problem* (1.1)-(1.2) *on* $\bar{\Omega}\times[t_0, T]$ *for some* $T>t_0$, *and let for* $\bar{t}\in(t_0, T]$, $\bar{x}=\bar{x}(\bar{t})$ *be a maximum point of* $|u(x, \bar{t})|$, *i.e.*, $|u(\bar{x}, \bar{t})|=\max_{x\in\Omega}|u(x, \bar{t})|$, *Then, if* $u(\bar{x}, \bar{t})\neq 0$, *we have* $\bar{x}\in\Omega$ *and*

$$
u_{x_i}(\bar{x}, \bar{t})=0 \quad \textit{and} \quad u(\bar{x}, \bar{t})\sum_{i,j=1}^{n}a_{ij}(\bar{x}, \bar{t})u_{x_ix_j}(\bar{x}, \bar{t})\leq 0 . \tag{3.1}
$$

PROOF.

If $e\equiv 0$, $h>0$, then $u(x, t)=0$ on $\partial\Omega$, and hence $\bar{x}\notin\partial\Omega$. If $e(x, t)\geq\delta_0>0$ and $h>0$, then

$$
\frac{\partial u}{\partial\nu}(\bar{x}, \bar{t})=(-h(\bar{x}, \bar{t})/e(\bar{x}, \bar{t}))\ u(\bar{x}, \bar{t}) .
$$

Since $u(\bar{x}, \bar{t})$ is either a positive maximum or a negative minimum, we must have $\frac{\partial u}{\partial\nu}u(\bar{x}, \bar{t})\geq 0$ or $\frac{\partial u}{\partial\nu}u(\bar{x}, \bar{t})\leq 0$, respectively. This contradicts above equality. The proof of (3.1) is standard and omitted. Q.E.D.

PROOF of Theorem 3.1.

By Lemma 2.3 the solution exists uniquely on $[t_0, T]$ for some $T>t_0$. For the proof of Theorem it suffices again by Lemma 2.3 to show that $\max_{x\in\bar{\Omega}}|u(x, t)|<\gamma_0$ for $^{\forall}t\in[t_0, T]$. We prove this by contradiction. Suppose that our assertion is false. Then there would exist a time $\bar{t}\in[t_0, T]$ such that

(3. 2) $\quad \max_{x \in \Omega} |u(x, \bar{t})| = \gamma_0$ and $\max_{x \in \bar{\Omega}} |u(x, t)| < \gamma_0$ if $t < \bar{t}$.

Here we may assume $\bar{t} < T$ since the solution is continued as long as it is bounded. Let $\bar{x}$ be a maximum point of $|u(x, \bar{t})|$. Then $\frac{d}{dt}|u(\bar{x}, \bar{t})|$ exists in the neighborhood of $\bar{t}$ where $|u(\bar{x}, \bar{t})| \neq 0$. Thus multiplying (1. 1) by $u(\bar{x}, \bar{t})$, we obtain, with the aid of Lemma 3. 1,

$$|u(\bar{x}, \bar{t})| \frac{d}{dt} |u(\bar{x}, \bar{t})| = u(\bar{x}, \bar{t}) \frac{d}{dt} u(\bar{x}, \bar{t})$$

$$\leq -c(\bar{x}, \bar{t}) u(\bar{x}, \bar{t}) u(\bar{x}, \bar{t}) - \beta(\bar{x}, \bar{t}, u(\bar{x}, \bar{t})) u(\bar{x}, \bar{t}) - f(\bar{x}, \bar{t}) u(\bar{x}, \bar{t})$$

$$\leq -c_0 \gamma_0{}^2 + K_0(\gamma_0, \bar{t}) \gamma_0^2 + M \gamma_0 \qquad \text{(by (2. 4) and (3. 2))} \tag{3. 3}$$

$$< 0 \qquad \text{(by (2. 5))}.$$

This implies that the function $\max_{x \in \Omega} |u(x, t)|$ of t begins to decrease just before it should reach to the value γ_0 from below. Therefore we can conclude

(3. 4) $\quad \max_{x \in \bar{\Omega}} |u(x, t)| < \gamma_0$ for all $t \in [t_0, \bar{t}]$.

which contradicts (3. 2). Q. E. D.

4. Bounded Solution on $-\infty < t < \infty$

On the basis of the result in the previous section we prove here:

THEOREM 4. 1.

Under the hypotheses (H_1)-(H_4), *the problem* (1. 1)-(1. 2) *with* $t_0 = -\infty$ *has a bounded solution* u *on* $-\infty < t < +\infty$ *with* $|u|_0^{\bar{\Omega} \times (-\infty, \infty)} \leq \gamma_0$

PROOF.

Let us introduce a smooth function $\zeta(t)$ on $(-\infty, \infty)$ for technical reason such that

$$0 \leq \zeta(t) \leq 1, \ \zeta(t) = 0 \ \text{for} \ t \leq 0 \ \text{and} \ \zeta(t) = 1 \ \text{for} \ t \geq 1.$$

Put $f_m(x, t) = \zeta(t+m) f(x, t)$, $\beta_m(x, t, u) = \zeta(t+m) \beta(x, t, u)$, and consider the initial-boundary problem on $\bar{\Omega} \times [-m, \infty)$:

$$u_t - \sum_{i,j=1}^{r} a_{ij}(x, t) u_{x_i x_j} + \sum_{i=1} b_i(x, t) u_{x_i} + c(x, t) u$$
$$+ \beta_m(x, t, u) + f_m(x, t) = 0 \ \text{on} \ \Omega \times (-m, \infty), \tag{4. 1}$$

(4. 2) $\quad e(x, t)\dfrac{\partial}{\partial\nu}u(x, t)+h(x, t)u=0,\ (x, t)\in\partial\Omega\times(-m, \infty)$,

(4. 3) $\quad u(x, -m)=0\,,\ x\in\Omega$.

Then by Theorem 3. 1, (4. 1)-(4. 3) has a unique bounded solution u_m on $\bar{\Omega}\times[-m, \infty)$, for each $m=1, 2, 3,\cdots$, with

(4. 4) $\quad |u_m|_0^{\bar{\Omega}\times[-m,\infty)}<\gamma_0$.

By trivial modification we may assume $u_m(x, t)$ is a solution of (4. 1)-(4. 2) on $\bar{\Omega}\times(-\infty, \infty)$ and (4. 4) is replaced by

(4. 4)$'$ $\quad |u_m|_0^{\bar{\Omega}\times(-\infty,\infty)}<\gamma_0$.

We show that the sequence $\{u_m\}_{m=1,2,\cdots,}$ contains a subsequence which is convergent in $C_{2+\alpha}^{\bar{\Omega}\times[-T,T]}$ for every $T>0$. For this purpose consider the functions

(4. 5) $\quad \tilde{u}_m(x, t)\equiv\zeta(t+T+1)u_m(x, t),\ m=1, 2, 3, \cdots$.

Then $\tilde{u}_m(x, t)$, $m=1, 2, 3,\cdots$, become the solutions of the problems

(4. 6)
$$\tilde{u}_{mt}-\sum_{i,j=1}^n a_{ij}(x, t)\tilde{u}_{mx_ix_j}+\sum_{i=1}^n b_i(x, t)\tilde{u}_{mx_i}+c(x, t)\tilde{u}_m$$
$$=\zeta'(t+T+1)u_m(x, t)-\zeta(t+T+1)(\beta_m(x, t, u)+f_m(x, t))$$

with

$$\tilde{u}_m(x, t)=0\ \text{ for }\ (x, t)\in\partial\Omega\times[-T-1, T]$$

and with

$$\tilde{u}_m(x, -T-1)=0\,.$$

Hence we obtain by Lemma 2. 2

$$|\tilde{u}_m|_\delta^{\bar{\Omega}\times[-T-1,T]}\le c_2(2T+1)\,|\zeta'(t+T+1)u_m-\zeta(t+T+1)(\beta_m(x, t, u)+f_m)|$$
$$\le c_3(T, \gamma_0, M)<+\infty,\ m=1, 2, 3, \cdots,\ 0<\delta<1\,,$$

and also by Lemma 2. 1 together above estimate (with T replaced by $T+1$)

$$|\tilde{u}_m|_{2+\alpha}^{\bar{\Omega}\times[-T-1,T]}\le c_1(2T+1)|\text{ the right hand side of (4. 6)}|_\alpha^{\bar{\Omega}\times[-T-1,T]}$$
$$\le c_4(T, k(\gamma_0, T),\ \gamma_0)<+\infty\,.$$

Since $\tilde{u}_m(x, t)=u_m(x, t)$ if $t>-T$, we obtain by (4. 7)

(4. 8) $\quad |u_m|_{2+\alpha}^{\bar{\Omega}\times[-T,T]}\le c_4(T, \gamma_0)<+\infty,\ m=1, 2, 3, \dots$.

Since a bounded set in $C_{2+\alpha}^{\bar{\Omega}\times[-T,T[}$ is compact with respect to the uniform topology on $\bar{\Omega}\times[-T, T]$, the estimate (4. 8) shows that we can extract a subsequence of $\{u_m\}$, which will be denoted by the same notation for brevity, converging uniformly on $\bar{\Omega}\times[-T, T]$. Combinning this fact, a priori estimates in Lemmas 2. 1, 2. 2 and the usual compactness argument we conclude that $\{u_m(x, t)\}$, more precisely a subsequence of it, converges to a function $u(x, t)$ in the norm $|\cdot|_{\alpha}^{\bar{\Omega}\times[-T,T]}$ for every $T>0$. Taking the limit in (4. 1), (4. 2) with u replaced by u_m we see easily $u(x, t)$ is a required bounded solution. The boundedness of $|u|_0^{\bar{\Omega}\times(-\infty,\infty)}$ follows from (4. 4)$'$. This completes the proof.

5. Uniqueness, Periodicity and Almost-periodicity of Bounded Solution

In this section we show that bounded solutions on $-\infty<t<+\infty$ are determined uniquely in a small ball centered at the origin and moreover that they become periodic and almost-periodec solutions if the data are so, respectively.

THEOREM 5. 1.

Under the assumptions (2. 1), (2. 2) *and* (H_5), *the problem* (1. 1)-(1. 2) *with* $t_0=-\infty$ *has at most one bounded solution with* $|u|_0^{\bar{\Omega}\times(-\infty,\infty)}\leq\gamma_1$.

PROOF.

Let u_1 and u_2 be two bounded solutions with $|u_i|_0^{\bar{\Omega}\times(-\infty,\infty)}\leq\gamma_1$, $i=1, 2$. Then for $w(x, t)=u_1(x, t)-u_2(x, t)$ we see at $(\bar{x}, \bar{t})$ such that $|w(\bar{x}, \bar{t})|=\max_{x\in\Omega}|w(x, t)|$ (note that $\bar{x}\in\Omega$),

$$\begin{aligned}
&|w(\bar{x}, \bar{t})|\frac{d^+}{dt}|w(\bar{x}, \bar{t})|=w(\bar{x}, \bar{t})\frac{d}{dt}w(\bar{x}, \bar{t})\\
&\leq -c_0|w(\bar{x}, \bar{t})|^2-w(\beta(\bar{x}, \bar{t}, u_1)-\beta(\bar{x}, \bar{t}, u_2))\\
&\leq(-c_0+K_1(|u_1(x, t)-u_2(x, t)|, \gamma_1))|w(\bar{x}, \bar{t})|^2\leq 0,
\end{aligned}\tag{5. 1}$$

which implies that $\max_{x\in\Omega}|w(x, t)|$ is nonicreasing in t. Therefore if $\max_{x\in\Omega}|w(x, t)|\neq 0$ at some time $t=t^*$, we have $\max_{x\in\Omega}|w(x, t)|\geq\max_{x\in\Omega}|w(x, t^*)|>0$ for $t\leq t^*$, and hence by (5. 1) and the assumption that $K_1(\gamma, \gamma_1)$ is nonincreasing in γ,

$$\begin{aligned}
&|w(\bar{x}, \bar{t})|\frac{d}{dt}|w(\bar{x}, \bar{t})|\leq(-c_0+K_1(|w(x^*, t^*)|, \gamma_1)|w(\bar{x}, \bar{t})|^2\\
&\equiv -\varepsilon_0|w(\bar{x}, \bar{t})|^2<0 \quad \text{for } \bar{t}\leq t^* \quad \text{(by (2. 7))},
\end{aligned}\tag{5. 2}$$

where x^* is a maximum point of $|w(x, t^*)|$. For ε_1 with $0<\varepsilon_1<\varepsilon_0$ we have by (5.2)

$$\frac{d}{dt}|w(\bar{x}, t)|^2 e^{2\varepsilon_1 t}|_{t=\bar{t}} = -2(\varepsilon_0-\varepsilon_1)e^{2\varepsilon_1\bar{t}}|w(\bar{x}, \bar{t})|^2<0, \quad {}^{\forall}\bar{t}\leq t^*,$$

which gives

$$|w(\bar{x}, \bar{t})|^2 e^{2\varepsilon_1\bar{t}}\geq |w(x^*, t^*)|^2 e^{2\varepsilon_1 t^*}, \quad \bar{t}\leq t^*,$$

or

$$(5.3) \qquad |w(\bar{x}, \bar{t})|^2\geq |w(x^*, t^*)|^2 e^{2\varepsilon_1(t^*-\bar{t})}.$$

This is a contradiction since the right hand side of (5.3) tends to ∞ as $\bar{t}\to -\infty$, while the left hand side is bounded by γ_1^2. Q.E.D.

Combinning Theorems 4.1 and 5.1 we get immediately the following:

COROLLARY 5.1.

Suppose (H_1)-(H_5) *with* $t_0=-\infty$ *and* $\gamma_0=\gamma_1$. *Furthermore we let* $a_{ij}(x, t)$, $b_i(x, t)$, $c(x, t)$, $\beta(x, t, u)$ *and* $f(x, t)$ *be periodic in* t *(with period* T*). Then the bounded solution* $u(x, t)$ *with* $|u|_0^{\bar{\Omega}\times(-\infty,\infty)}\leq\gamma_0$, *which exists uniquely by Theorems* 4.1 *and* 5.1, *is a periodic function with period* T.

PROOF.

It is obvious from the fact that $u(x, t)$ and $u(x, t+T)$ are both bounded solutions with bounds $\leq\gamma_0$. Q.E.D.

Finally in this section we give a theorem concerning the almost-periodicity of bounded solution.

THEOREM 5.2.

In addition to the hypothses (H_1)-(H_5) *with* $t_0=-\infty$ *and with* $\gamma_0=\gamma_1$, *we assume that* a_{ij}, b_i, c, f *are in* $C_\alpha^{\bar{\Omega}\times(-\infty,\infty)}$, e, h *are in* $C_{1+\alpha}^{\bar{\Omega}\times(-\infty,\infty)}$ *and* $\beta(\cdot,\cdot,\cdot)$ *satisfies*

$$|\beta(x_1, t_1, u_1)-\beta(x_2, t_2, u_2)|\leq k(\gamma_0)(|x_1-x_2|^\alpha+|t_1-t_2|^{\frac{\alpha}{2}}+|u_1-u_2|)$$

for (x_1, t_1), $(x_2, t_2)\in R^{n+1}$ *and for* $|u_1|$, $|u_2|\leq\gamma_0$. *Moreover we assume* $a_{ij}(\cdot, t)$, $b_i(\cdot, t)$, $c(\cdot, t)$, $f(\cdot, t)$ *are almost-periodic with respect to* $|\cdot|_0^{\bar{\Omega}}$, $e(\cdot, t)$ *and* $h(\cdot, t)$ *are almost-periodic with respect to* $|\cdot|_0^{\partial\Omega}$ *and* $\beta(\cdot, t, \cdot)$ *is almost-periodic with respect to* $|\cdot|_0^{\bar{\Omega}\times[-\gamma_0,\gamma_0]}$.

Then the bounded solution u of (1. 1)-(1. 2) *with* $|u|_0^{\bar{\Omega}\times(\infty,\infty)}\leq\gamma_0$ *is also almost-periodic with respect to* $|\cdot|_{2+\alpha}^{\bar{\Omega}}$ *and* u_t *is so with respect to* $|\cdot|_0^{\bar{\Omega}}$

PROOF.

Let $\{s_n\}$ be any real sequence. By Bohner's criterion it suffices to prove that $\{u(\cdot, t+s_n)\}$ contains a subsequence which converges uniformly in t in the norm $|\cdot|_{2+\alpha}^{\bar{\Omega}}$. Suppose that this is false. Then there exist sequences $\{t_n\}$, $\{s_{n_1}\}$, $\{s_{n_2}\}$ such that

(5. 4) $\quad |u(\cdot, t_n+s_{n_1})-u(\cdot, t_n+s_{n_2})|_{2+\alpha}^{\bar{\Omega}}>\delta>0$

for some $\delta>0$.

Put $t_n+s_{n_1}=t_{n_1}$ and $t_n+s_{n_2}=t_{n_2}$. Here we may assume by the almost periodicity assumption,

$$
(5.5)\qquad
\begin{array}{llll}
a_{ij}(x, t+t_{n_k}) & \to A_{ij}(x, t) & \text{in } C^{\bar{\Omega}\times(-\infty,\infty)} & \text{for } k=1, 2, \\
b_i(x, t+t_{n_k}) & \to B_i(x, t) & \quad\quad " & , \\
c(x, t+t_{n_k}) & \to C(x, t) & \quad\quad " & , \\
f(x, t+t_{n_k}) & \to F(x, t) & \quad\quad " & , \\
e(x, t+t_{n_k}) & \to E(x, t) & \quad\quad " & , \\
h(x, t+t_{n_k}) & \to H(x, t) & \quad\quad " & ,
\end{array}
$$

and

$$\beta(x, t+t_{n_k}) \to B(x, t,u) \quad \text{in} \quad C_0^{\bar{\Omega}\times(-\infty,\infty)\times[-\gamma_0,\gamma_0]}$$

for $k=1, 2$.

Now consider the sequences $\{u_{n_k}(x, t)\}\equiv\{u(x, t+t_{n_k})\}$, $k=1, 2$. Then by the similar argument in the proof of Theorem 4. 1 we can see that there exist subsequences of $\{u_{n_k}\}$, $k=1, 2$, which are denoted by the same notations, such that

(5. 6) $\quad u(x, t+t_{n_k}) \to U_k(x, t)$ in $C_{2+\alpha}^{\bar{\Omega}\times[-T,T]}$

for every $T>0$. (Notice that the assumptions $a_{ij}, b_i\in C_\alpha^{\bar{\Omega}\times(-\infty,\infty)}$, etc. are used in this procedure.)

By (5. 5) and (5. 6) it is easily seen that $U_k(x, t)$, $k=1, 2$, are both the solutions of (1. 1)-(1. 2) with $t_0=-\infty$ and with a_{ij}, b_i, c, β, f, e, and h replaced by A_{ij}, B_i, C, B, F, E and H, respectively. Moreover we have by (5. 6)

$$|U_k|_0^{\bar{\Omega}\times(-\infty,\infty)}\leq\gamma_0, \quad k=1, 2.$$

Therefore we obtain, by the uniqueness theorem of bounded solutions (note that for uniqueness it is not needed that the coefficients are Hölder continuous),

$$U_1(x, t) \equiv U_2(x, t) ,$$

in particular

$$U_1(x, 0) = U_2(x, 0) . \tag{5. 7}$$

But, on the other hand, (5. 4) and (5. 6) yield

$$|U_1(x, 0) - U_2(x, 0)|_{2+\alpha}^{\bar{\Omega}} \geq \delta > 0 ,$$

which contradicts (5. 7). Finally the almost-periodicity of u_t follows easily by the equation (1. 1) itself. Q. E. D.

6. An Application to Elliptic Boundary Problem

Let us consider the elliptic boundary problem:

$$-\sum_{i,j=1} a_{ij}(x) u_{x_i x_j} + \sum_{i=1}^{n} b_i(x) u_{x_i} + c(x) u + \beta(x, u) + f(x) = 0 \quad \text{on} \quad \Omega \tag{6. 1}$$

with

$$e(x) \frac{\partial}{\partial \nu} u + h(x) u = 0 \quad \text{on} \quad \partial\Omega. \tag{6. 2}$$

By the result of previous section we can obtain easily an existence theorem for (6. 1)-(6. 2).

THEOREM 6. 1.

Suppose (H_1)-(H_5) *with* $\gamma_0 = \gamma_1$ *where the functions should be replaced by the ones independent of* t. *Then the problem* (6. 1)-(6. 2) *has a unique classical solution* u *with* $|u|_0^{\bar{\Omega}} \leq \gamma_0$.

PROOF.

By the assumptions, the problem (1, 1)-(1, 2) with $t_0 = -\infty$ and with $a_{ij}(x, t)$, $b_i(x, t)$, $c(x, t)$, $\beta(x, t, u)$, $f(x, t)$, $e(x, t)$ and $h(x, t)$ replaced by $a_{ij}(x)$, $b_i(x)$, $c(x)$, $\beta(x, u)$, $f(x)$, $e(x)$ and $h(x)$, respectively, adimits a unique bounded solution $u(x, t)$ with $|u|_0^{\bar{\Omega} \times (-\infty, \infty)} \leq \gamma_0$. Moreover by corollary 5. 1, $u(x, t)$ has an arbitrary period in t, since the coefficients are all so, in other words $u(x, t)$ is independent of t. Hence $u(x, t) \equiv u(x)$ becomes the required solution of (6. 1)-(6. 2). Q. E. D.

7. Examples

For illustrations, we give some typical examples.

1. $c(x, t) \equiv 0,\ \beta(x, t, u) = |u|^{\theta}u,\ \theta \geq 0.$

In this case we can take $K_0(\gamma, t) = -\gamma^{\theta}$ in (2. 4) and $K_1(\gamma, \gamma_1) = -2^{-(\theta+1)}\gamma^{\theta}$ in (2. 6) for arbitrary numbers $\gamma_0, \gamma_1 > 0$. Hence, for any bounded function $f(x, t)$, we can choose γ_0 and γ_1 so that the hypotheses (H_4) and (H_5) may be fulfilled. Consequently under the natural conditions (H_1)-(H_3), Theorems 3. 1 and 4. 1 are valid for arbitrary $\gamma_0 > 0$, and Theorem 5. 1, corollary 5. 1 are so for any γ_1 and $\gamma_0 = \gamma_1$, respectively. Of course, Theorems 5. 2, 6. 1 are valid for any $\gamma_0 = \gamma_1$.

2. $c(x, t) \equiv 1,\ \beta(x, t, u) = -|u|^{\theta}u,\ \theta > 0,$

In this case we can take $K_0(\gamma, t) = \gamma^{\theta}$ and $K_1(\gamma, \gamma_1) = (\theta+1)\gamma_1^{\theta}$. Thus if we choose γ_0 and M so that $\gamma_0 < 1$ and $M \leq \gamma_0(1-\gamma_0^{\theta})$, Theorems 3. 1, 4. 1 are valid for such γ_0. Moreover if we choose $\gamma_1 < (\theta+1)^{-1/\theta}$, then Theorem 5. 1 holds for such γ_1. If $\gamma_0 = \gamma_1$, Theorems 5. 2, 6. 1 and 5. 1 hold.

3. $\beta(x, t, u) = |u|^{\theta}u \cos u,\ \theta > 0.$

In this case we can take $K_0(\gamma, t) = \gamma^{\theta} \cos \gamma$ and $K_1(\gamma, \gamma_1) = (\theta+1)(1+\gamma_1)\gamma_1^{\theta}$. Hence (H_4) and (H_5) are satisfied if we choose γ_0 and γ_1 so that

$$-c_0\gamma_0 + M + \gamma_0^{\theta+1} \cos \gamma_0 < 0, \tag{7. 1}$$

and

$$-c_0 + (\theta+1)(1+\gamma_1)\gamma_1^{\theta} < 0, \tag{7. 2}$$

respectively.

The choice of γ_0 in (7. 1) is possible for any M, and (7. 2) is possible if $c_0 > 0$ and γ_1 is small.

Final remark.

Using the method in this paper we can obtain the nonnegative solutions for the problem (1. 1)-(1. 2) without essential changes.

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