On the existence of global, bounded, periodic and almost-periodic solutions of nonlinear parabolic equations

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1. Introduction

Let \( \Omega \) be an open bounded set in the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \) with boundary \( \partial \Omega \). Consider the following partial differential equation of parabolic type:

\[
(1.1) \quad u_t - \sum_{i=1}^{n} a_{ii}(x, t) u_{xx_i} + \sum_{i=1}^{n} b_i(x, t) u_{x_i} + c(x, t) u + \beta(x, t, u) + f(x, t) = 0, \\
(x, t) \in \Omega \times (t_0, \infty),
\]

together with boundary condition

\[
(1.2) \quad e(x, t) \frac{\partial u}{\partial n} + h(x, t) u = 0, \quad (x, t) \in \partial \Omega \times (t_0, \infty),
\]

where \( \nu \) is the unit outward normal vector on \( \partial \Omega \).

The existence theory concerning global solutions to the initial-boundary value problem, i.e., (1.1)-(1.2) with additional condition \( u(x, t_0) = u_0(x) \), \( t_0 \neq -\infty \), has been developed by a number of authors. But in most cases the following hypothesis on the nonlinear term is made:

\[
(1.3) \quad -u(c(x, t) u + \beta(x, t, u) + f(x, t)) \leq Au^2 + B,
\]

where \( A, B \) are positive constants. For instance, see O. A. Oleinik and S. N. Kruzkov [9], A. Friedman [4], O. A. Ladyzenskaja, V. A. Solonikov and N. N. Uralceva [7], D. H. Sattinger [13], C. V. Pao [10] and D. W. Bange [2]. Especially, in the case \( \beta(x, t, u) \) is monotonically increasing in \( u \), the condition (1.3) is satisfied. For example \( \beta(x, t, u) = |u|^{\theta} u, \ \theta > 0 \), is a typical one.

Bounded, periodic and almost-periodic solutions to the problem (1.1)-(1.2) with \( t_0 = -\infty \) have been considered also by several people: I. I. Smulev [11], [12], J. S. Kolesov [6], L. Amerio-G. Prouse [1], M. Biroli
[3] and others. However the assumptions of the type (1.3) are still assumed in their works.

The object of this paper is to establish existence theorems concerning global bounded, periodic and almost-periodic solutions to the problem (1.1)-(1.2) under different assumptions which permit (1.3) to be broken. Roughly speaking we try to replace the condition (1.3) by local ones. For example, if \( c(x, t) \geq c_0 > 0 \), \( \beta(x, t, u) = -|u|^\theta u, \theta \geq 0 \), and \( \sup_{x,t}|f(x, t)| \) is sufficiently small, our assumptions will be satisfied. As is seen by this example, for our purpose it must be assumed in many cases that the initial values \( u(x, 0) = u_0(x) \) and the inhomogeneous terms \( f(x, t) \) are small. But these restrictions are not too artificial as is easily conjectured by the work of H. Fujita [5], where the initial-boundary value problem:

\[
(1.4) \quad u_t - \Delta u = u^\alpha, \quad \alpha > 0, \quad u \geq 0,
\]

with

\[
(1.5) \quad u|_{t=0} = 0, \quad u(x, 0) = u_o(x) \geq 0,
\]

is investigated (there the case \( \Omega = \mathbb{R}^n \) is also considered). It is established there that if \( u_o(x) \) is small, then a global solution of (1.4)-(1.5) exists and if \( u_o(x) \) is large the solution blows up in a finite time. See also M. Tsutsumi [14] where the result of H. Fujita is generalized.

Recently similar problem has been considered by one of the present authors [8] by use of so-called \( L^2 \)-method. Here we employ a classical one and the solutions obtained are classical.

As a corollary of our result an existence theorem of elliptic boundary value problem is given.

2. Preliminaries

Following a standard notation we introduce the following norms on a function \( u(x, t) \) defined on a set \( \Omega \subset \mathbb{R}^{n+1} \).

\[
|u|^2_\infty = \sup_{P \in \Omega} |u(P)|,
\]

\[
|u|^2_\infty = |u|^2_\infty + \sup_{P, P'} \frac{|u(P) - u(P')|}{d(P, P')^a},
\]

\[
|u|^2_{\ast, \alpha} = |u|^2_\infty + \sum_{i=1}^n |u_i|^2_\alpha,
\]

\[
|u|^2_{\ast, \alpha} = |u|^2_{\ast, \alpha} + |u_t|^2_\alpha + \sum_{i=1}^n |u_{ii}^\alpha|^2_{\ast, \alpha}
\]
where $0 < \alpha < 1$ and $d(P, P') = (|x - x'|^2 + |t - t'|)^{\frac{1}{2}}$ for points $P = (x, t), P' = (x', t') \in \mathbb{R}^n \times \mathbb{R}$.

We say $u(x, t)$ is in Hölder class $C^\alpha(\mathcal{O})$ if $|u|^{2+\alpha} + q = 0, \alpha, 1+\alpha, 2+\alpha$.

Before stating our hypotheses we give several definitions.

**DEFINITION 2.1.**

By a bounded solution to the initial-boundary value problem (1.1)-(1.2), $t_0 \neq -\infty$, with initial value $u_0(x)$ we mean a function $u(x, t)$ defined on $\mathcal{O} \times [t_0, \infty)$ such that $u \in C^2_{\alpha}(\mathcal{O} \times [t_0, T])$ for any $T > t_0$ and for some $\alpha \geq 0$, $u \in C_0(\mathcal{O} \times [t_0, \infty))$ and the equations (1.1), (1.2) and the initial condition $u(x, 0) = u_0(t)$ are satisfied.

**DEFINITION 2.2.**

We say a function $u(x, t)$ on $\mathcal{O} \times (-\infty, \infty)$ is bounded solution to (1.1)-(1.2), $t_0 = -\infty$, if $u \in C^2_{\alpha}(\mathcal{O} \times [-T, T])$ for any $T > 0$, $u \in C_0(\mathcal{O} \times (-\infty, \infty))$ and (1.1), (1.2) are satisfied by $u(x, t)$.

**DEFINITION 2.3.**

Let $V$ be a Banach space with norm $\|\cdot\|_V$. A continuous function $u(t) : \mathbb{R} \rightarrow V$ is said to be almost-periodic with respect to $\|\cdot\|_V$ or $V$-almost-periodic if to every $\varepsilon > 0$ there corresponds a relatively dense set $(\tau)_\varepsilon \subset \mathbb{R}$ such that

$$\sup_{t} \|u(t+\tau) - u(t)\|_V \leq \varepsilon, \tau \in (\tau)_\varepsilon.$$ 

Definition 2.3 is equivalent to the following (see [1]).

**DEFINITION 2.3’ (Bochner).**

A continuous function $u(t) : \mathbb{R} \rightarrow V$ is $V$-almost periodic if, and only if, having taken an arbitrary real sequence $(s_n)$ there exists a subsequence $(s_{n'})$ such that the sequence $(u(t+s_{n'}))$ converges uniformly in $t$.

Now we state our hypotheses. For $A = [t_0, \infty), t_0 \neq -\infty$, or $A = (-\infty, \infty)$ we impose:

**(H_i).** (2.1) $c(x, t) > c_0$ and $\sum_{i=1}^{n} a_i(x, t) \xi_i \xi_i \geq a_0 \sum_{i=1}^{n} \xi_i^2$,

for $(x, t) \in \mathcal{O} \times A$ and $\xi \in \mathbb{R}^n$. 

Nonlinear parabolic equation
where $a_0, c_0$ are positive, nonnegative, constants respectively. Moreover

$$
\begin{align*}
(1 & ) \quad e(x, t) \geq \delta_0 > 0 \quad \text{and} \quad h(x, t) > 0, \quad \text{or} \\
(2 & ) \quad e(x, t) \equiv 0 \quad \text{and} \quad h(x, t) > 0.
\end{align*}
$$

(H2). $a_0(x, t), b_0(x, t), c(x, t)$ and $f(x, t)$ are in class $C_a(\mathcal{Q} \times I)$ and $e(x, t), h(x, t)$ are in $C_{1, a}(\mathcal{Q} \times I)$, where $I$ denotes every compact set in $A$.

(H3). $\mathcal{Q}$ is of class $C_{a+1}$, i.e., for every point $p \in \mathcal{Q} \times (-\infty, \infty)$ there exists an $(n+1)$-dimensional neighborhood $U$ such that $U \cap \mathcal{Q} \times (-\infty, \infty)$ can be represented in the form

$$
x_k = g(x_1, x_2, \ldots, x_{k-1}, x_{k-1}, \ldots, x_n, t)
$$

for some $k$, and the derivatives of $g(x, t)$ with orders $\leq 2$ are all Hölder continuous with exponent $\alpha$, $0 < \alpha < 1$.

Regarding the nonlinear term we require the following hypotheses.

(H4) There exists a positive constant $\gamma_0$ such that

$$
|\beta(x, t, u_1) - \beta(x, t, u_2)| \leq k(\gamma, T)(|x_1 - x_2|^\alpha + |t_1 - t_2|^\alpha + |u_1 - u_2|)
$$

for $(x_1, t_1), (x_2, t_2) \in \mathcal{Q} \times [-T, T]$ and $|u_1|, |u_2| \leq \gamma \leq \gamma_0$ ($T$: arbitrary).

(H5) There exists a positive constant $\gamma_1$ such that

$$
-\alpha \beta(x, t, u)u \leq K_0(|u|, t)|u|^2 \quad \text{if} \quad |u| \leq \gamma_0, \quad t \in T
$$

and

$$
-\alpha c_0 + M + K_1(\gamma_0, r_1) < 0,
$$

where $k(\gamma, T)$ and $K_0(\gamma, t)$ are functions of $(\gamma, T)$ and $(\gamma, t)$, respectively, and

$$
M \equiv \sup_{\mathcal{Q} \times I}|f(x, t)|.
$$

(H6). There exists a positive constant $r_i$ such that

$$
-\alpha (u_i - u_2)(\beta(x, t, u_i) - \beta(x, t, u_2)) \leq K_1(|u_i - u_2|, \gamma_1)|u_i - u_2|^2
$$

if $|u_i|, i = 1, 2, \leq r_i$, and

$$
-\alpha c_0 + K_i(\gamma, \gamma_1) < 0 \quad \text{if} \quad 0 < \gamma \leq 2 \gamma_1,
$$

where $K_1(\gamma, \gamma_1)$ is a nonincreasing function in $\gamma$ defined on $[0, 2 \gamma_1]$.

The hypothesis (H5) is used for the uniqueness of bounded solution.

We need the following lemmas (see A. Friedman [4], D. A. Ladyzenskaja, V. A. Solonnikov and N. N. Uralceva [7]).
LEMMA 2.1. ([Ths. 5.2, 5.3; 7])

Consider the initial-boundary problem for the linear equation:

\[
(2.8) \quad u_t - \sum_{i=1}^{n} a_{ii}(x, t) u_{x_i} + \sum_{i=1}^{n} b_{i}(x, t) u_{x_i} + c(x, t) = f(x, t)
\]

on \( \Omega \times (t_0, \infty) \) with conditions (1.2) and \( u(x, t_0) = u_0(x) (t_0 \neq - \infty) \). In addition to the assumptions \((\text{H}_1)-(\text{H}_3)\), suppose that \( u_0(x) \) is in \( C^{2+a}(\bar{\Omega}) \) and satisfies the compatibility condition

\[
(2.9) \quad e(x, t_0) \frac{\partial}{\partial \nu} u_0(x) + h(x, t_0) u_0(x) = 0 \quad \text{on} \quad \partial \Omega.
\]

Then there exists one and only one solution \( u(x, t) \) to the initial-boundary problem satisfying the \((2+a)\)-estimate for each \( T > t_0 \):

\[
(2.10) \quad \|u\|_{C^{2+a}[t_0, T]} \leq c_1(T-t_0) (\|f\|_{C^{2+a}[t_0, T]} + \|u_0\|_{C^{2+a}}),
\]

where \( c_1(T-t_0) \) is a constant depending on \( T-t_0 \) and other various constants, but independent of \( \|f\|_{C^{2+a}[t_0, T]} \) and \( \|u_0\|_{C^{2+a}} \).

LEMMA 2.2. ([Ths. 4 and 4'; 4])

Under the same conditions of Lemma 2.1, any solution \( u(x, t) \) to the initial-boundary problem satisfies, for any \( \delta \in (0, 1) \), the \((\delta)\)-estimate

\[
(2.11) \quad \|u\|_{C^{2}([t_0, T])} \leq c_2(T-t_0) (\|f\|_{C^{2}([t_0, T])} + \|u_0\|_{C^{2+a}}),
\]

where \( c_2(T-t_0) \) is a similar constant to \( c_1 \) independent of \( \|f\|_{C^{2}([t_0, T])} \), \( \|u_0\|_{C^{2+a}} \).

By the use of above lemmas, a standard argument shows (cf. C. V. Pao [10]):

LEMMA 2.3.

In addition to the assumptions of Lemma 2.1, suppose \((2.3)\) is valid. Then for \( u_0(x) \in C_{1+a}(\bar{\Omega}) \) with \( \|u_0\|_{C^{2+a}} < \gamma_0 \), the problem \((1.1)-(1.2)\) with \( u(x, t_0) = u_0(x) \), \( t_0 \neq - \infty \), has a unique "local solution" \( u(x, t) \) in the sense that for some \( T_0 > t_0 \) \( u(x, t) \) satisfies \((1.1)-(1.2)\) and \( u(x, t_0) = u_0(x) \) for \( x \in \Omega \) and \( t \in [t_0, T_0] \).

The value \( T_0 \) is determined by the largest interval \([t_0, T_0]\) on which \( \|u(x, t)\| \leq \gamma_0 \) on \( \Omega \). If \( \beta \) satisfies \((2.3)\) for every finite \( \gamma \) with \( k(\gamma, I) < + \infty \) for \( \gamma, I \), then \( u(x, t) \) can be continued as long as it remains bounded on \( \Omega \).
3. Initial-Boundary Value Problem

In this section we prove the following existence theorem of the initial-boundary value problem.

**Theorem 3.1.**

In addition to hypotheses (H1)-(H4), we assume that \( u_0(x) \) is in \( c_{2+a}(\bar{\Omega}) \) with \( |u_0| \leq \gamma_0 \) and the compatibility condition (2.9) is fulfilled. Then the initial-boundary value problem (1.1), (1.2) with \( u(x, t_0) = u_0(x) \) has a unique bounded solution \( u \) with \( |u| \leq \gamma_0 \).

In order to prove the theorem we need the following standard lemma.

**Lemma 3.1.**

Let \( u(x, t) \) be a solution of the problem (1.1)-(1.2) on \( \bar{\Omega} \times [t_0, T] \) for some \( T > t_0 \), and let for \( \bar{t} \in (t_0, T) \), \( x = x(\bar{t}) \) be a maximum point of \( |u(x, \bar{t})| \), i.e.,

\[
|u(x, \bar{t})| = \max_{x \in \bar{\Omega}} |u(x, \bar{t})|
\]

Then, if \( u(x, \bar{t}) \neq 0 \), we have \( x \in \bar{\Omega} \) and

\[
(3.1) \quad u_x(x, \bar{t}) = 0 \quad \text{and} \quad u(x, \bar{t}) \sum_{i,j=1}^n a_{ij}(x, \bar{t}) u_{x_i x_j}(x, \bar{t}) \leq 0.
\]

**Proof.**

If \( e \equiv 0, h > 0 \), then \( u(x, t) = 0 \) on \( \partial \bar{\Omega} \), and hence \( x \notin \partial \bar{\Omega} \). If \( e(x, \bar{t}) \geq \delta_0 > 0 \) and \( h > 0 \), then

\[
\frac{\partial u}{\partial \nu}(x, \bar{t}) = \frac{-h(x, \bar{t})/e(x, \bar{t})}{u(x, \bar{t})} u(x, \bar{t}).
\]

Since \( u(x, \bar{t}) \) is either a positive maximum or a negative minimum, we must have \( \frac{\partial u}{\partial \nu} u(x, \bar{t}) \geq 0 \) or \( \frac{\partial u}{\partial \nu} u(x, \bar{t}) \leq 0 \), respectively. This contradicts above equality. The proof of (3.1) is standard and omitted. Q.E.D.

**Proof of Theorem 3.1.**

By Lemma 2.3 the solution exists uniquely on \([t_0, T]\) for some \( T > t_0 \). For the proof of Theorem it suffices again by Lemma 2.3 to show that

\[
\max_{x \in \bar{\Omega}} |u(x, \bar{t})| \leq \gamma_0 \quad \text{for} \quad \bar{t} \in (t_0, T).
\]

We prove this by contradiction. Suppose that our assertion is false. Then there would exist a time \( \bar{t} \in [t_0, T] \) such that
Here we may assume \( \bar{t} < T \) since the solution is continued as long as it is bounded. Let \( \bar{x} \) be a maximum point of \( |u(x, \bar{t})| \). Then \( \frac{d}{dt} |u(x, \bar{t})| \) exists in the neighborhood of \( \bar{t} \) where \( |u(x, \bar{t})| \neq 0 \). Thus multiplying (1.1) by \( u(x, \bar{t}) \), we obtain, with the aid of Lemma 3.1,

\[
|u(\bar{x}, \bar{t})| \frac{d}{dt} |u(\bar{x}, \bar{t})| = u(\bar{x}, \bar{t}) \frac{d}{dt} u(\bar{x}, \bar{t})
\leq -c(\bar{x}, \bar{t})u(\bar{x}, \bar{t})u(\bar{x}, \bar{t}) - \beta(\bar{x}, \bar{t}, u(\bar{x}, \bar{t}))u(\bar{x}, \bar{t}) - f(\bar{x}, \bar{t})u(\bar{x}, \bar{t})
\leq -c_0 \bar{t}^2 + K_0(\bar{t}, \bar{t})\bar{t}^2 + M\bar{t} \quad \text{(by (2.4) and (3.2))}
\leq 0 \quad \text{(by (2.5)).}
\]

This implies that the function \( \max_{x \in \partial} |u(x, t)| \) of \( t \) begins to decrease just before it should reach to the value \( \gamma_0 \) from below. Therefore we can conclude

\[
|u(x, t)| < \gamma_0 \quad \text{for all} \quad t \in [t_0, \bar{t}].
\]

which contradicts (3.2).

4. Bounded Solution on \(-\infty < t < \infty\)

On the basis of the result in the previous section we prove here:

**Theorem 4.1.**

*Under the hypotheses \( (H_1)-(H_4) \), the problem (1.1)-(1.2) with \( t_0 = -\infty \) has a bounded solution \( u \) on \(-\infty < t < \infty \) with \( |u|_{\bar{\partial} \times (-\infty, \infty)} \leq \gamma_0 \).*

**Proof.**

Let us introduce a smooth function \( \zeta(t) \) on \((-\infty, \infty)\) for technical reason such that

\[
0 \leq \zeta(t) \leq 1, \quad \zeta(0) = 0 \quad \text{for} \quad t \leq 0 \quad \text{and} \quad \zeta(t) = 1 \quad \text{for} \quad t \geq 1.
\]

Put \( f_\alpha(x, t) = \zeta(t+m)f(x, t), \ \beta_\alpha(x, t, u) = \zeta(t+m)\beta(x, t, u) \), and consider the initial-boundary problem on \( \bar{\partial} \times (-m, \infty) \):

\[
(4.1) \quad u_t - \sum_{i,j=1}^n a_{ij}(x, t)u_{x_ix_j} + \sum_{i=1}^n b_i(x, t)u_{x_i} + c(x, t)u + \beta_\alpha(x, t, u) + f_\alpha(x, t) = 0 \quad \text{on} \quad \bar{\partial} \times (-m, \infty),
\]
\( e(x, t) \frac{\partial}{\partial \nu} u(x, t) + h(x, t) u = 0, \quad (x, t) \in \Omega \times (-m, \infty) \),

\( u(x, -m) = 0, \quad x \in \Omega \).

Then by Theorem 3.1, (4.1)-(4.3) has a unique bounded solution \( u_m \)
on \( \bar{\Omega} \times [-m, \infty) \), for each \( m=1, 2, 3, \ldots \), with

\( |u_m|_{\bar{\Omega} \times [-m, \infty)} < \gamma_0 \).

By trivial modification we may assume \( u_n(x, t) \) is a solution of (4.1)-(4.2)
on \( \bar{\Omega} \times (-\infty, \infty) \) and (4.4) is replaced by

\( |u_n|_{\bar{\Omega} \times [-m, \infty)} < \gamma_0 \).

We show that the sequence \( \{u_n\}_{n=1, 2, \ldots} \) contains a subsequence whichis convergent in \( C^\infty_\bar{\Omega}([-T, T]) \) for every \( T > 0 \). For this purpose consider thefunctions

\( \hat{u}_n(x, t) = \zeta(t + T + 1) u_n(x, t), \quad m=1, 2, 3, \ldots \).

Then \( \hat{u}_n(x, t), \quad m=1, 2, 3, \ldots, \) become the solutions of the problems

\( \hat{u}_n = \sum_{i=1}^n a_i(x, t) \hat{u}_n(x, t) + \sum_{i=1}^n b_i(x, t) \hat{u}_{n+1} + c(x, t) \hat{u}_n \)

\( = \zeta(t + T + 1) u_n(x, t) - \zeta(t + T + 1) (\beta_n(x, t, u) + f_n(x, t)) \)

with

\( \hat{u}_n(x, t) = 0 \) for \((x, t) \in \Omega \times [-T, T]\)

and

\( \hat{u}_n(x, -T-1) = 0 \).

Hence we obtain by Lemma 2.2

\( |\hat{u}_n|_{\bar{\Omega} \times [-T, T]} \leq c_1(2T + 1) |\zeta(t + T + 1) \hat{u}_n - \zeta(t + T + 1) (\beta_n(x, t, u) + f_n)| \)

\( \leq c_2(T, \gamma_0, M) < +\infty, \quad m=1, 2, 3, \ldots, \quad 0 < \delta < 1, \)

and also by Lemma 2.1 together above estimate (with \( T \) replaced by \( T+1 \))

\( |\tilde{u}_n|_{\bar{\Omega} \times [-T, T]} \leq c_1(2T + 1) \) the right hand side of (4.6)\( |\tilde{u}_n|_{\bar{\Omega} \times [-T, T]} \)

\( \leq c_2(T, k_{\gamma_0, T}, \gamma_0) < +\infty. \)

Since \( \tilde{u}_n(x, t) = u_n(x, t) \) if \( t > -T \), we obtain by (4.7)

\( |u_n|_{\bar{\Omega} \times [-T, T]} \leq c_4(T, \gamma_0) < +\infty, \quad m=1, 2, 3, \ldots. \)

Since a bounded set in \( C^\infty_\bar{\Omega}([-T, T]) \) is compact with respect to the uniform
topology on $\partial \times [-T, T]$, the estimate (4.8) shows that we can extract a subsequence of $\{u_n\}$, which will be denoted by the same notation for brevity, converging uniformly on $\partial \times [-T, T]$. Combining this fact, a priori estimates in Lemmas 2.1, 2.2 and the usual compactness argument we conclude that $\{u_n(x, t)\}$, more precisely a subsequence of it, converges to a function $u(x, t)$ in the norm $\| \frac{\partial u}{\partial t} \|_{L^2([-T, T])}$ for every $T > 0$. Taking the limit in (4.1), (4.2) with $u$ replaced by $u_n$ we see easily $u(x, t)$ is a required bounded solution. The boundedness of $\|u\|_{L^\infty([-T, T])}$ follows from (4.4). This completes the proof.

5. Uniqueness, Periodicity and Almost-periodicity of Bounded Solution

In this section we show that bounded solutions on $-\infty < t < +\infty$ are determined uniquely in a small ball centered at the origin and moreover that they become periodic and almost-periodic solutions if the data are so, respectively.

THEOREM 5.1.

Under the assumptions (2.1), (2.2) and (H1), the problem (1.1)-(1.2) with $t_0 = -\infty$ has at most one bounded solution with $\|u\|_{L^\infty([-\infty, +\infty])} \leq \gamma_1$.

PROOF.

Let $u_1$ and $u_2$ be two bounded solutions with $\|u_i\|_{L^\infty([-\infty, +\infty])} \leq \gamma_i$, $i = 1, 2$. Then for $w(x, t) = u_1(x, t) - u_2(x, t)$ we see at $(\tilde{x}, \tilde{t})$ such that $\|w(\tilde{x}, \tilde{t})\| = \max_{x \in \Omega} \|w(x, \tilde{t})\|$ (note that $\tilde{x} \in \Omega$),

$$\left| w(\tilde{x}, \tilde{t}) \right| \frac{d}{dt} w(\tilde{x}, \tilde{t}) = w(\tilde{x}, \tilde{t}) \frac{d}{dt} - w(\tilde{x}, \tilde{t})$$

$$\leq -c_1 \left| w(\tilde{x}, \tilde{t}) \right|^2 - w(\beta(\tilde{x}, \tilde{t}, u_1) - \beta(\tilde{x}, \tilde{t}, u_2))$$

(5.1) $$\leq (-c_1 + K_i(\|u_1(x, t) - u_2(x, t)\|, \gamma_1)) \left| w(\tilde{x}, \tilde{t}) \right|^2 \leq 0,$$

which implies that $\max_{x \in \Omega} \|w(x, t)\|$ is nonincreasing in $t$. Therefore if $\max_{x \in \Omega} \|w(x, t)\| \neq 0$ at some time $t = t^*$, we have $\max_{x \in \Omega} \|w(x, t)\| \geq \max_{x \in \Omega} \|w(x, t^*)\| > 0$ for $t \leq t^*$, and hence by (5.1) and the assumption that $K_i(\gamma, \gamma_1)$ is nonincreasing in $\gamma$,

(5.2) $$\left| w(\tilde{x}, \tilde{t}) \right| \frac{d}{dt} \left| w(\tilde{x}, \tilde{t}) \right| \leq (-c_1 + K_i(\|w(x^*, t^*)\|, \gamma_1)) \left| w(\tilde{x}, \tilde{t}) \right|^2$$

$$\equiv -c_1 \left| w(\tilde{x}, \tilde{t}) \right|^2 < 0 \quad \text{for} \quad \tilde{t} \leq t^* \quad \text{(by (2.7))},$$
where \( x^* \) is a maximum point of \( |w(x, t^*)| \). For \( \varepsilon_i \) with \( 0 < \varepsilon_i < \varepsilon_0 \) we have by (5.2)

\[
\frac{d}{dt} |w(\bar{x}, \bar{t})|^2 e^{2\varepsilon_1 t} |_{\bar{t} = \bar{t}} = -2(\varepsilon_0 - \varepsilon_i)e^{2\varepsilon_1 \bar{t}} |w(\bar{x}, \bar{t})|^2 < 0, \forall \bar{t} \leq t^*,
\]

which gives

\[
|w(\bar{x}, \bar{t})|^2 e^{2\varepsilon_1 \bar{t}} \geq |w(x^*, t^*)|^2 e^{2\varepsilon_1 t^*}, \bar{t} \leq t^*,
\]

or

(5.3) \[ |w(\bar{x}, \bar{t})|^2 \geq |w(x^*, t^*)|^2 e^{2\varepsilon_1 (t^* - \bar{t})} \]

This is a contradiction since the right hand side of (5.3) tends to \( \infty \) as \( \bar{t} \to -\infty \), while the left hand side is bounded by \( \gamma_l \). Q. E. D.

Combining Theorems 4.1 and 5.1 we get immediately the following:

**Corollary 5.1.**

Suppose (H1)-(H5) with \( t_0 = -\infty \) and \( \gamma_0 = \gamma_1 \). Furthermore we let \( a_1(x, t), b_1(x, t), c(x, t), \beta(x, t, u) \) and \( f(x, t) \) be periodic in \( t \) (with period \( T \)). Then the bounded solution \( u(x, t) \) with \( |u|_{\mathcal{B}^\infty(\mathfrak{I}_0, -\infty, m)} \leq \gamma_0 \), which exists uniquely by Theorems 4.1 and 5.1, is a periodic function with period \( T \).

**Proof.**

It is obvious from the fact that \( u(x, t) \) and \( u(x, t+T) \) are both bounded solutions with bounds \( \leq \gamma_0 \). Q. E. D.

Finally in this section we give a theorem concerning the almost-periodicity of bounded solution.

**Theorem 5.2.**

In addition to the hypotheses (H1)-(H5) with \( t_0 = -\infty \) and with \( \gamma_0 = \gamma_1 \), we assume that \( a_1, b_1, c, \beta, f \) are in \( C_a^\mathcal{B}^\infty(\mathfrak{I}_0, -\infty, m) \), \( e, h \) are in \( C_{1-a}^\mathcal{B}^\infty(\mathfrak{I}_0, -\infty, m) \) and \( \beta(\cdot, \cdot, \cdot) \) satisfies

\[
|\beta(x_1, t_1, u_1) - \beta(x_2, t_2, u_2)| \leq k(\gamma_0)(|x_1 - x_2|^a + |t_1 - t_2|^a + |u_1 - u_2|)
\]

for \( (x_1, t_1), (x_2, t_2) \in \mathbb{R}^{n+1} \) and for \( |u_1|, |u_2| \leq \gamma_0 \). Moreover we assume \( a_1(\cdot, t), b_1(\cdot, t), c(\cdot, t), f(\cdot, t) \) are almost-periodic with respect to \( \cdot \), \( e(\cdot, t) \) and \( h(\cdot, t) \) are almost-periodic with respect to \( \cdot \). and \( \beta(\cdot, t, \cdot) \) is almost-periodic with respect to \( \cdot \).
Then the bounded solution $u$ of (1.1)-(1.2) with $|u|_{\tilde{\mathcal{B}}_{a}\times\mathcal{Y}_{0}} \leq \gamma$ is also almost-periodic with respect to $\cdot |_{\tilde{\mathcal{B}}_{a}}$ and $u$, is so with respect to $|\cdot |_{\tilde{\mathcal{B}}_{a}}$

**PROOF.**

Let $(s_n)$ be any real sequence. By Bohner's criterion it suffices to prove that $(u(\cdot, t+s_n))$ contains a subsequence which converges uniformly in $t$ in the norm $|\cdot |_{\tilde{\mathcal{B}}_{a}}$. Suppose that this is false. Then there exist sequences $(t_n)$, $(s_{n_1})$, $(s_{n_2})$ such that

$$|u(\cdot, t+sn_1) - u(\cdot, t+sn_2)|_{\tilde{\mathcal{B}}_{a}} > \delta > 0$$

for some $\delta > 0$.

Put $t_n + s_{n_1} = t_{n_1}$ and $t_n + s_{n_2} = t_{n_2}$. Here we may assume by the almost periodicity assumption,

$$a_i(x, t + t_n) \to A_i(x, t) \text{ in } C_{\tilde{\mathcal{B}}_{a}\times(-\infty, \infty)} \text{ for } k = 1, 2,$$

$$b(x, t + t_n) \to B(x, t) \text{ in } C_{\tilde{\mathcal{B}}_{a}} \text{ for } k = 1, 2,$$

$$c(x, t + t_n) \to C(x, t) \text{ in } C_{\tilde{\mathcal{B}}_{a}},$$

$$f(x, t + t_n) \to F(x, t) \text{ in } C_{\tilde{\mathcal{B}}_{a}},$$

$$e(x, t + t_n) \to E(x, t) \text{ in } C_{\tilde{\mathcal{B}}_{a}},$$

$$h(x, t + t_n) \to H(x, t) \text{ in } C_{\tilde{\mathcal{B}}_{a}},$$

and

$$\beta(x, t + t_n) \to B(x, t, u) \text{ in } C_{\tilde{\mathcal{B}}_{a}\times(-\infty, \infty)} \times C_{\tilde{\mathcal{B}}_{a}},$$

for $k = 1, 2$.

Now consider the sequences $(u_{n_k}(x, t)) = (u(x, t + t_{n_k}))$, $k = 1, 2$. Then by the similar argument in the proof of Theorem 4.1 we can see that there exist subsequences of $(u_{n_k})$, $k = 1, 2$, which are denoted by the same notations, such that

$$u(x, t + t_{n_k}) \to U_k(x, t) \text{ in } C_{\tilde{\mathcal{B}}_{a}\times(-T, T)} \text{ for every } T > 0.$$ (Notice that the assumptions $a_i$, $b_i \in C_{\tilde{\mathcal{B}}_{a}\times(-\infty, \infty)}$, etc. are used in this procedure.)

By (5.5) and (5.6) it is easily seen that $U_k(x, t)$, $k = 1, 2$, are both the solutions of (1.1)-(1.2) with $t_0 = -\infty$ and with $a_i$, $b_i$, $c$, $\beta$, $f$, $e$, and $h$ replaced by $A_i$, $B$, $C$, $B$, $F$, $E$ and $H$, respectively. Moreover we have by (5.6)

$$|U_k|_{\tilde{\mathcal{B}}_{a}\times(-\infty, \infty)} \leq \Gamma_k, \text{ } k = 1, 2.$$
Therefore we obtain, by the uniqueness theorem of bounded solutions (note that for uniqueness it is not needed that the coefficients are Hölder continuous),

\[ U_1(x, t) \equiv U_2(x, t), \]

in particular

\[ (5.7) \quad U_1(x, 0) = U_2(x, 0). \]

But, on the other hand, (5.4) and (5.6) yield

\[ |U_1(x, 0) - U_2(x, 0)| \leq 0, \]

which contradicts (5.7). Finally the almost-periodicity of \( u_t \) follows easily by the equation (1.1) itself. Q. E. D.

6. An Application to Elliptic Boundary Problem

Let us consider the elliptic boundary problem:

\[ (6.1) \quad -\sum_{i,j=1}^n a_{ij}(x)u_{x_ix_j} + \sum_{i=1}^n b_i(x)u_{x_i} + c(x)u + \beta(x, u) + f(x) = 0 \quad \text{on } \Omega \]

with

\[ (6.2) \quad e(x) \frac{\partial}{\partial \nu} u + h(x)u = 0 \quad \text{on } \partial \Omega. \]

By the result of previous section we can obtain easily an existence theorem for (6.1)–(6.2).

THEOREM 6.1.

Suppose (H1)–(H5) with \( \gamma_0 = \gamma_1 \) where the functions should be replaced by the ones independent of \( t \). Then the problem (6.1)–(6.2) has a unique classical solution \( u \) with \( |u|_0^\infty \leq \gamma_0 \).

PROOF.

By the assumptions, the problem (1.1)–(1.2) with \( t_0 = -\infty \) and with \( a_i(x, t), b_i(x, t), c(x, t), \beta(x, t, u), f(x, t), e(x, t) \) and \( h(x, t) \) replaced by \( a_i(x), b_i(x), c(x), \beta(x, u), f(x), e(x) \) and \( h(x) \), respectively, admits a unique bounded solution \( u(x, t) \) with \( |u|_0^{\infty} \leq \gamma_0 \). Moreover by corollary 5.1, \( u(x, t) \) has an arbitrary period in \( t \), since the coefficients are all so, in other words \( u(x, t) \) is independent of \( t \). Hence \( u(x, t) \equiv u(x) \) becomes the required
solution of (6.1)-(6.2).

Q. E. D.

7. Examples

For illustrations, we give some typical examples.

1. \(c(x, t) = 0, \beta(x, t, u) = |u|^\theta u, \theta \geq 0\).

In this case we can take \(K_0(\tau, t) = -\tau^\theta\) in (2.4) and \(K_1(\tau, \tau_1) = -2^{-\theta+1}\tau^\theta\)
in (2.6) for arbitrary numbers \(\tau_0, \tau_1 > 0\). Hence, for any bounded function \(f(x, t)\), we can choose \(\tau_0\) and \(\tau_1\) so that the hypotheses \((H_1)\) and \((H_2)\) may be fulfilled. Consequently under the natural conditions \((H_1)-(H_3)\), Theorems 3.1 and 4.1 are valid for arbitrary \(\tau_0 > 0\), and Theorem 5.1, corollary 5.1 are so for any \(\tau_1\) and \(\tau_0 = \tau_1\), respectively. Of course, Theorems 5.2, 6.1 are valid for any \(\tau_0 = \tau_1\).

2. \(c(x, t) = 1, \beta(x, t, u) = -|u|^\theta u, \theta > 0\).

In this case we can take \(K_0(\tau, t) = -\tau^\theta\) and \(K_1(\tau, \tau_1) = (\theta+1)\tau^\theta\). Thus if we choose \(\tau_0\) and \(M\) so that \(\tau_0 < 1\) and \(M \leq \tau_0(1-\tau_0)\), Theorems 3.1, 4.1 are valid for such \(\tau_0\). Moreover if we choose \(\tau_1 < (\theta+1)^{-\frac{1}{\theta}}\), then Theorem 5.1 holds for such \(\tau_1\). If \(\tau_0 = \tau_1\), Theorems 5.2, 6.1 and 5.1 hold.

3. \(\beta(x, t, u) = |u|^\theta u \cos u, \theta > 0\).

In this case we can take \(K_0(\tau, t) = -\tau^\theta \cos \gamma\) and \(K_1(\tau, \tau_1) = (\theta+1)(1+\tau_1)\gamma_1^\theta\). Hence \((H_1)\) and \((H_2)\) are satisfied if we choose \(\tau_0\) and \(\tau_1\) so that

\[
\begin{align*}
(7.1) & \quad -c_0\tau_0 + M + \tau_0^\theta \cos \tau_0 < 0, \\
(7.2) & \quad -c_0 + (\theta+1)(1+\tau_1)\gamma_1^\theta < 0,
\end{align*}
\]

respectively.

The choice of \(\tau_0\) in (7.1) is possible for any \(M\), and (7.2) is possible if \(c_0 > 0\) and \(\gamma_1\) is small.

Final remark.

Using the method in this paper we can obtain the nonnegative solutions for the problem (1.1)-(1.2) without essential changes.

References

Van Nostrand (1971).


