On the global classical solutions of nonlinear hyperbolic-parabolic systems

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On the global classical solutions of nonlinear hyperbolic-parabolic systems

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§ 0. Introduction

Let Ω be a smooth bounded domain in $\boldsymbol{R}^n$. Points in Ω are denoted by $x=(x_1, x_2, \cdots, x_n)$ and the time variable by t.

In this article we consider the Initial-Boundary Value Problem;

$$\left.\begin{array}{ll}(1) & u''-\triangle u+u'(\gamma+(u)^{p_1}+(u')^{q_1}+(v)^{r_1})=0 \\ & v'-\triangle v+(u)^{p_2}+(u')^{q_2}+(v)^{r_2}=0 \\ & x\in\Omega,\ t>0 \\ (2) & u(x,\ 0)=u_0(x),\ u'(x,\ 0)=u_1(x),\ v(x,\ 0)=v_0(x) \\ & x\in\Omega \\ (3) & u|_{\partial\Omega}=v|_{\partial\Omega}=0 \qquad t\geqq 0\end{array}\right\}$$

where $\triangle$ is the Laplacian in $\boldsymbol{R}^n$, $'=\frac{\partial}{\partial t}$, p_i, q_i, r_i, are positive integers and γ is a positive constant.

A question of a global existence of a classical solution of (1) - (3) is investigated in this article.

Previously, B. K. Kalantarov [3] has obtained classical solutions for more complicated equations with some growth restrictions to nonlinear terms.

It seems to be impossible to obtain a global classical solution for (1) -(3) with no conditions of initial values u_0, u_1, v_0 if we do not put such restrictions to nonlinear terms but here we can see that if the initial values are sufficiently smooth and have small norms then it admits a global classical solution.

The aim of the article is to give such sufficient condition under which (1)-(3) is globally solved.

The method is the analogous one used in Y. Ebihara [2] to obtain classical solutions for systems of equations;

$$\begin{cases} u''-\triangle u+u'(1+(u)^{p_1}+(v)^{q_1}+(u')^{r_1}+(v')^{s_1})=0 \\ v''-\triangle v+v'(1+(u)^{p_2}+(v)^{q_2}+(u')^{r_2}+(v')^{s_2})=0 \end{cases}$$

$$\begin{cases} u'-\triangle u+u^{p_1}+v^{q_1}=0 \\ v'-\triangle v+u^{p_2}+v^{q_2}=0 \end{cases}$$

§ 1. Auxiliary Concepts

Notations of function spaces are as usual.

Let us fix positive integer m as $m \geqq \left[\frac{n}{2}\right]+1$.

we know from the positivity of $-\triangle$ in $\overset{\circ}{H}{}^1(\Omega)$,

(1. 1) $$(\cdot, \cdot\cdot)_k=<(-\triangle)^k\cdot, \cdot\cdot>, \ |\cdot|_k^2=(\cdot, \cdot)_k$$

defines equivalent inner product of the space $\overset{\circ}{H}{}^k(\Omega)$ where k is a positive integer and $<\cdot, \cdot\cdot>$ is the duality bracket of $H^{-k}(\Omega)\times\overset{\circ}{H}{}^k(\Omega)$.
In this article we identify this space equipped with the inner product as $\overset{\circ}{H}{}^k(\Omega)$.
Then we have by Sobolev lemma;

LEMMA 1. *It holds for $u \in \overset{\circ}{H}{}^m(\Omega)$ that*

(1) $$|u|\mathcal{B}_{m_0}(\bar{\Omega}) \leqq c(n, m)|u|_m \qquad \left(m=\left[\frac{n}{2}\right]+1+m_0\right)$$

(2) $$|(u)^p\cdot(v)^q|_m \leqq c(n, m, p, q)|u|_m^p\cdot|v|_m^q$$

where p, q are positive integers.

Now, we consider a system of differential inequalities;

(1. 2) $$\begin{cases} \varphi'(t) \leqq f(\varphi(t), \psi(t))\{-\gamma+\varphi^p(t)+\psi^q(t)\} \\ \psi'(t) \leqq g(\varphi(t), \psi(t))\{-\psi(t)+\varphi^r(t)+\psi^s(t)\} \end{cases} \quad t\in[0, \infty)$$

where $\varphi(t)$, ψ(t) are unknown nonnegative functions and γ, p, q, r, s are positive numbers with $s>1$ and $f(\cdot, \cdot\cdot)$, $g(\cdot, \cdot\cdot)$ are given functions which are nonnegative, continuous in $\boldsymbol{R}^2$.
This plays an important role to the Problem (1) - (3) and the following Lemma 2 is a key estimate to obtain our theorem in § 2.

LEMMA 2. *For $\varphi(t)$, $\psi(t)$* in (1. 2), *there exists a positive number δ such that if $\varphi(0)+\psi(0)<\delta$, then $\varphi(t)$ should be decreasing and it holds that $\varphi(t)+\psi(t)<K(\delta)$ $(t\in[0, \infty))$*
where $K(\delta)$ is some constant depending only on δ.

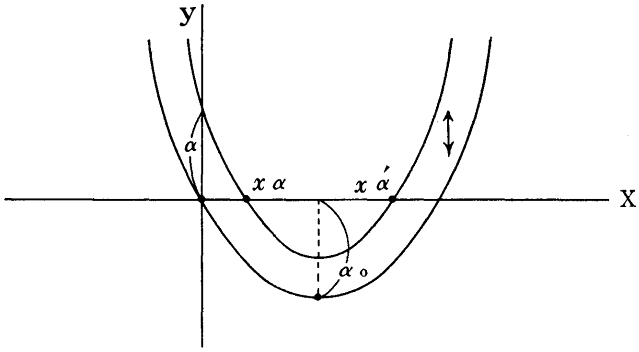
PROOF.

At first we consider the curve:

$$y(\alpha,\ x)=x^s-x+\alpha$$

where $0<\alpha<\alpha_0=\left(\frac{1}{s}\right)^{\frac{1}{s-1}}\left(1-\frac{1}{s}\right)$ (α_0: depth).

Put x_α, x_α' as minimum and maximum root of the equation $y(\alpha,\ x)=0$



Now we divide into two cases.

① If we assume

$$\text{(1. 3)} \qquad \varphi^p(0)+\psi^q(0)<\gamma,\quad \varphi^r(0)<\alpha,\quad x_\alpha \leqq \psi(0)<x_\alpha',$$

then $\varphi(t)$ should be decreasing and

$$\psi(t)<x_\alpha' \qquad \text{for} \quad t\in[0,\ \infty)$$

that is,

$$\varphi(t)+\psi(t)<\varphi(0)+x_\alpha'.$$

In fact, for some neighborhood of $t=0$, it holds from (1. 3)

$$\varphi'(t)<0,\ \psi'(t)<0$$

that is, $\varphi(t)$, $\psi(t)$ are decreasing.

Therefore for some $t_0>0$ we have

$$\text{(1. 4)} \qquad \begin{cases} y(\varphi^r(t_0),\ \psi(t_0))=0 \\ \varphi^p(t)+\psi^p(t) \leqq \varphi^p(0)+\psi^q(0)<\gamma \quad (t\leqq t_0) \\ \psi'(t_0)=0,\ \varphi'(t_0)<0 \end{cases}$$

At this time, since $\varphi(t_0)<\varphi(0)<\alpha^{\frac{1}{r}}$, it should hold $\psi(t_0)<x_\alpha$.

And moreover since $\varphi'(t_0)<0$, the curve $y(\varphi^r(t),\ x)$ goes down.

Therefore for every $t\geqq t_0$, we have

$$\psi(t)\leqq\psi(t_0)<x_\alpha.$$

Thus observing these considerations we can conclude for $t\in[0,\ \infty)$

$$\varphi'(t) < 0, \quad \psi(t) < x_\alpha'.$$

② If we assume,

$$\varphi^p(0) + \psi^q(0) < \varphi^p(0) + x_a^q < \gamma, \ \varphi^r(0) < \alpha,$$

(This is possible by taking α sufficiently small for γ.)
then $\varphi(t)$ should be decreasing and

$$\psi(t) < x_\alpha, \quad \text{that is,}$$

$$\varphi(t) + \psi(t) < \varphi(0) + x_\alpha \text{ for } t \in [0, \infty).$$

In fact, since $\varphi'(t) < 0$ for a neighborhood of $t=0$, the curve $y(\varphi^r(t), x)$ should get down, so even though $\psi'(t) > 0$ in this neighborhood $\psi(t)$ can not cross over x_α i.e. $\psi(t) < x_\alpha$. Therefore it holds that

$$\varphi^p(t) + \psi^q(t) < \gamma.$$

This shows that the situation continues for any t in $[0, \infty)$.
Thus we have

$$\varphi'(t) < 0, \ \psi(t) < x_\alpha.$$

Consequently, from ①, ② we have the statement of the lemma.

(q. e. d.)

§ 2. Theorem.

In this section we prove the following theorem by the aid of the preliminary concepts of section 1 and the theorems in [1], [2].

THEOREM. *If the initial values u_0, u_1 and v_0 satisfy the following conditions:*

(2. 1) $\quad u_0, \ v_0 \in \overset{\circ}{H}{}^{m+3}(\Omega), \qquad u_1 \in \overset{\circ}{H}{}^{m+2}(\Omega)$

(2. 2) $\quad |u_0|_{m+1} + |v_0|_m + |u_1|_m < \delta$

for some $\delta > 0$,

then we have a pair of solutions $\{u(x, t), v(x, t)\}$ of (1)—(3) *satisfying*

(2. 3) $\quad u(x, t) \in \mathcal{E}^0_{[0,\infty)}[\overset{\circ}{H}{}^{m+1}(\Omega) \cap H^{m+2}(\Omega)] \cap \mathcal{E}^1_{[0,\infty)}[\overset{\circ}{H}{}^{m+1}(\Omega)] \cap \mathcal{E}^2_{[0,\infty)}[\overset{\circ}{H}{}^m(\Omega)]$

(2. 4) $\quad v(x, t) \in \mathcal{E}^0_{[0,\infty)}[\overset{\circ}{H}{}^{m+1}(\Omega) \cap H^{m+2}(\Omega)] \cap \mathcal{E}^1_{[0,\infty)}[\overset{\circ}{H}{}^m(\Omega)]$.

PROOF.

Put $\{\varphi_j\}$ as a system of eigen functions of $(-\triangle)^{m+3}$ considered in the space $\overset{\circ}{H}{}^{m+3}(\Omega)$.
Then since, u_0, $v_0 \in \overset{\circ}{H}{}^{m+3}(\Omega)$, $u_1 \in \overset{\circ}{H}{}^{m+2}(\Omega)$ we have sequences of numbers $\{A_j\}$, $\{B_j\}$ and $\{D_j\}$ with

$$(2.5)\quad \begin{cases} u_{0.} = \sum_{j=1}^{k} A_j\varphi_j \longrightarrow u_0\ (s)\ in\ \overset{\circ}{H}{}^{m+3}(\Omega) \\ v_{0,k} = \sum_{j=1}^{k} B_j\varphi_j \longrightarrow v_0\ (s)\ in\ \overset{\circ}{H}{}^{m+3}(\Omega) \\ u_{1,k} = \sum_{j=1}^{k} D_j\varphi_j \longrightarrow u_1\ (s)\ in\ \overset{\circ}{H}{}^{m+2}(\Omega). \end{cases}$$

Here we put $u_k(t)=\sum_{j=1}^{k}\lambda_{kj}(t)\varphi_j$, $v_k(t)=\sum_{j=1}^{k}\mu_{kj}(t)\varphi_j$ where $\{\lambda_{kj}(t)\}$, $\{\mu_{kj}(t)\}$ are solutions of the systems of ordinary differential equations:

$$(2.6)\quad \begin{cases} (u_k'', \varphi_j)_m + (u_k, \varphi_j)_{m+1} + (u_k'(\gamma + (u_k)^{p_1} + (u_k')^{q_1} + (v_k)^{r_1}), \varphi_j)_m = 0 \\ (v_k', \varphi_j)_m + (v_k, \varphi_j)_{m+1} + ((u_k)^{p_2} + (u_k')^{q_2} + (v_k)^{r_2}, \varphi_j)_m = 0 \\ \qquad j=1, 2, \cdots, k \\ u_k(0) = \sum_{j=1}^{k} A_j\varphi_j,\ u_k'(0) = \sum_{j=1}^{k} D_j\varphi_j,\ v_k(0) = \sum_{j=1}^{k} B_j\varphi_j. \end{cases}$$

It suffices to verify the following:

$$(2.7)\quad \sup_{k\ge k_0}\ \sup_{t\in[0,T]} \{|u_k(t)|_{m+1} + |u_k'(t)|_m + |v_k(t)|_m\} < C(\mathrm{T})$$

for large k_0 and for every fixed positive number T.
(The remaining part of the proof is done by the quite analogous reasoning of the proofs of the theorems in [1], [2].)

Now we prove (2.7).
From (2.6), we have for each t in existence interval $[0, \varepsilon_k)$,

$$\begin{cases} \frac{1}{2}\frac{d}{dt}\left\{|u_k'|_m^2 + |u_k|_{m+1}^2\right\} + \gamma|u_k'|_m^2 + \\ \qquad (u_k'((u_k)^{p_1} + (u_k')^{q} + (v_k)^{r_1}), u_k')_m = 0 \\ \frac{1}{2}\frac{d}{dt}|v_k|_m^2 + |v_k|_{m+}^2 + ((u_k)^{p_2} + (u_k')^{q_2} + (v_k)^{r_2}, v_k)_m = 0. \end{cases}$$

Moreover from (2) of Lemma 1, it follows that

$$\begin{cases} \frac{d}{dt}\left\{|u_k'|_m^2 + |u_k|_{m+1}^2\right\} \\ \le 2|u_k'|_m^2\{-\gamma + c_1|u_k|_{m+1}^{p_1} + c_2|u_k'|_m^{q_1} + c_3|v_k|_m^{r_1}\} \\ \frac{d}{dt}|v_k|_m^2 \le |v_k|_m\{-c_4|v_k|_m + c_5|u_k|_{m+1}^{p_2} + c_6|u_k'|_m^{q_1} + c_7|v_k|_m^{r_2}\}. \end{cases}$$

Therefore if we put

$$\varphi_k(t) = |u_k'(t)|_m^2 + |u_k(t)|_{m+1}^2$$
$$\psi_k(t) = |v_k(t)|_m$$

then they satisfy:

$$\begin{cases}\varphi_k'(t) \leqq 2|u_k'|_m^2\{-\gamma+c_8\varphi_k^p(t)(1+\varphi_k^{p'}(t))+c_9\psi_k^q(t)\} \\ \psi_k'(t) \leqq c_{10}\{-\psi_k(t)+c_{11}\varphi_k^r(t)(1+\varphi_k^{r'}(t))+c_{12}\psi_k^s(t)\}\end{cases}$$

for some positive numbers $c_8 \sim c_{12}$ and p, q, r, s, p', r'.
Here we note these numbers are independent of k. Thus from the analogous way of Lemma 2 (taking no account of the difference of coefficients of equations), there exists $\delta > 0$ such that,
if,

$$\varphi_k(0)+\psi_k(0) < \delta$$

then, $\varphi_k(t)$ is decreasing and $\psi_k(t) < K_0(\delta)$
that is,

$$\varphi_k(t)+\psi_k(t) < K(\delta)$$

for some positive number $K(\delta)$. This shows the existence interval $[0, \varepsilon_k)$ can be extended as far as desired.
Here, if we set for the number δ such that

$$|u_0|_{m+1}^2+|u_1|_m^2+|v_0|_m < \delta$$

then from the continuity of the functionals $|\cdot|_m$, $|\cdot|_{m+1}$ and the conditions (2. 5), we obtain for $k \geqq k_0$ that

$$\begin{aligned}&|u_{0,k}|_{m+1}^2+|u_{1,k}|_m^2+|v_{0,k}|_m \\ &=\varphi_k(0)+\psi_k(0) < \delta.\end{aligned}$$

Therefore we can conclude that $u_k(t)$, $v_k(t)$ $(k \geqq k_0)$ exist globally and satisfy

$$\sup_{k \geq k0} \sup_{t \in [0,T)} \{|u_k(t)|_{m+1}+|u_k'(t)|_m+|v_k(t)|_m\} \leqq c(\delta, \mathrm{T}).$$

This completes the proof. (q.e.d)

COR. *If,* $m=\left[\frac{n}{2}\right]+1+m_0$, *then the solution* $u(x, t)$, $v(x, t)$ *of the theorem belong to*

$$\mathcal{E}^0_{[0,\infty)}[c^{m_0+2}(\overline{\Omega})] \cap \mathcal{E}^1_{[0,\infty)}[c^{m_0+1}(\overline{\Omega})] \cap \mathcal{E}^2_{[0,\infty)}[c^{m_0}(\overline{\Omega})],$$

$$\mathcal{E}^0_{[0,\infty)}[c^{m_0+2}(\overline{\Omega})] \cap \mathcal{E}^1_{[0,\infty)}[c^{m_0}(\overline{\Omega})]$$

respectively.

REMARK. *Though it seems that we can not hope a global solution of the Initial Value Problem for* (1) *for any initial values, if* g_1, g_2 *are non-zero constants then we have a global solution by the method introduced here for the equations of the form:*

$$\begin{cases} u''-\triangle u+g_1^2u+u'(\gamma+(u)^{p_1}+(u')^{q_1}+(v)^{r_1})=0 \\ v'-\triangle v+g_2^2v+(u)^{p_2}+(u')^{q_2}+(v)^{r_2}=0 \end{cases}$$

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