Correction to "On the complex bordism group of a semi-direct product $S^1 \cdot Z_2$ "

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Correction to "On the complex bordism group of a semi-direct product $S^1 \cdot Z_2$ "

By Masayoshi KAMATA (Received April. 30, 1975)

In the preceding paper "On the complex bordism group of a semidirect product $S^1 \cdot Z_2$ " of this journal, we attacked to compute the bordism group $U_*(S^1 \cdot Z_2)$ of free $S^1 \cdot Z_2$ -actions on weakly almost complex manifolds. To accomplish the computation, we used an incorrect isomorphism $U_*(S^1 \cdot Z_2) \cong U_{*-1}(BS^1 \cdot Z_2)$. As Lazarov and Wasserman showed in [3], $U_*(S^1 \cdot Z_2)$ is not isomorphic to $U_{*-1}(BS^1 \cdot Z_2)$ because the adjoint representation of $S^1 \cdot Z_2 = O(2)$ is not trivial. In the paper [2] we computed only $U_*(BS^1 \cdot Z_2)$. In this paper we compute $U^*(S^1 \cdot Z_2)$ again.

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1. Take $gt^i \in S^1 \cdot Z_2$, where t is the generator of Z_2 . The adjoint representation of $S^1 \cdot Z_2 = O(2)$ is described by

 $Ad(gt^i)x = (-1)^i x, x \in \mathbb{R}^1.$

Let $\xi: Y \longrightarrow B$ be a principal O(2)-bundle over B. We put

$$4d(Y) = Y \times_{O(2)} \mathbb{R}^1$$
,

where $Y \times_{O(2)} \mathbb{R}^1$ is the quotient space of $Y \times \mathbb{R}^1$ under the identification $(y, x) = (yb^{-1}, Ad(b)x), b \in O(2)$, and have a line vector bundle $Ad(\xi)$: $Ad(Y) \longrightarrow B$.

Let M^n be a weakly almost complex manifold with a free O(2)-action preserving the complex structure on $\tau(M^n) \oplus \varepsilon^r$ for some r, where ε^r is the *r*-dimensional trivial bundle. We then have the complex vector bundle

(1.1)
$$(\tau(M^m) \oplus \varepsilon^r)/O(2) \simeq \pi^* \tau(M^m/O(2)) \oplus Ad(M^m) \oplus \varepsilon^r ,$$

where $\pi: M^m \longrightarrow M^m/O(2)$ is the canonical projection and $\tau()$ denotes the tangent bundle.

Denote by $U_m(O(2), Ad)$ the bordism group of pairs (W^{m-1}, γ) consisting of an (m-1)-dimensional differentiable manifold W^{m-1} and a differentiable principal O(2)-bundle γ over W^{m-1} with a complex structure on $\tau(W^{m-1}) \bigoplus$ $Ad(\gamma) \bigoplus \varepsilon^r$ for some r. Sending an element $[M^m] \in U_m(O(2))$ to an element

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 $[M^m/O(2), \gamma] \in U_m(O(2), Ad)$, where $\gamma: M^m \longrightarrow M^m/O(2)$ is the canonical principal O(2)-bundle, we can see that $U_m(O(2))$ is isomorphic to $U_m(O(2), Ad)$ [3]. For the universal principal O(2)-bundle $\gamma: EO(2) \longrightarrow BO(2)$ we consider the bundle

$$\xi = Ad(\gamma): EO(2) \times_{O(2)} \mathbb{R}^1 = Ad(EO(2)) \longrightarrow BO(2).$$

Let $D(\xi)$ and $S(\xi)$ be the total spaces of the disk bundle and the sphere bundle of ξ respectively and $T(\xi)$ the Thom complex of ξ . It follows that $U_*(D(\xi), S(\xi)) \cong \tilde{U}_*(T(\xi))$. We then have

THEOREM 1.1. $U_m(O(2), Ad)$ is isomorphic to $U_m(D(\xi), S(\xi))$.

PROOF. We first define the homomorphism $\emptyset: U_m(O(2), Ad) \longrightarrow U_m$ $(D(\xi), S(\xi))$. For $[W^{m-1}, \eta] \in U_m(O(2), Ad)$, we have a bundle map

$$Ad(\gamma) \longrightarrow \xi = Ad(\gamma)$$

and the bundle map of the disk bundles

 $f_{\eta}: D(Ad(\eta)) \longrightarrow D(\xi)$.

Here $D(Ad(\eta))$ is a weakly almost complex manifold, because

 $\tau(D(Ad(\eta)) \cong \pi^*(\tau(W^{m-1}) \bigoplus Ad(\eta))$

where the $\tau()$ are the tangent bundles and $\pi: D(Ad(\eta)) \longrightarrow W^{m-1}$ is the projection. We put

$$\varPhi[W^{m-1}, \eta] = [D(Ad(\eta)), f_{\eta}] .$$

Next we shall define $\Psi: U_m(D(\xi), S(\xi)) \longrightarrow U_m(O(2), Ad)$. Suppose that $[M^m, f] \in U_m(D(\xi), S(\xi))$ is represented by a differentiable map $f: M^m \longrightarrow D(\xi)$ which is transverse regular to BO(2). Let

$$W^{m-1} = f^{-1}(BO(2))$$

and ν be the normal bundle of W^{m-1} in M^m . Note that

$$f^*\xi \cong
u, \ \xi = Ad \ (\gamma)$$

and

$$\mathit{Ad}(ar{f}^*(\gamma))\cong
u$$
 ,

where $\bar{f}=f|W^{m-1}$. $\bar{f}^*(\gamma)$ is the principal O(2)-bundle. Since

$$\tau(D(\bar{f}^*(\xi))) \cong \pi^*(\tau(W^{m-1}) \oplus \bar{f}^*(\xi))$$

and $D(f^*(\xi))$ is diffeomorphic to the tubular neighborhood of W^{m-1} which admits a weakly almost complex structure, $\tau(W^{m-1}) \oplus Ad(\bar{f^*}(\gamma)) \oplus \varepsilon^r$ have a complex structure for some r. Therefore we can define "On the complex bordism group of a semi-direct product $S^1 \cdot Z_2$ "

$$\Psi[M^m, f] = [W^{m-1}, \overline{f}^*(\gamma)] .$$

COROLLARY 1.2. $U_m(O(2)) \simeq \widetilde{U}_m(T(\xi))$.

2. The Dold manifold D(l, k) is a quotient space under the identification $(x, z) = (-x, \overline{\lambda z}), \lambda \in S^1$. The universal principal $S^1 \cdot Z_2$ -bundle is approximated by the principal $S^1 \cdot Z_2$ -bundle $S^1 \times S^{2k+1} \longrightarrow D(l, k)$. We consider the line bundle

$$\xi_k: Ad(S^k \times S^{2k+1}) \longrightarrow D(k, k)$$
.

Making use of Theorem 1 of [1], we obtain

$$T(\xi_k) pprox D(k+1, k)/CP^k$$
 .

The direct limit space $\lim T(\xi_k)$ is $T(\xi)$. Consider the cofibration $CP^k \xrightarrow{i} D(k+1, k) \xrightarrow{j} T(\xi_k)$ and we have the exact sequence

(2.1)
$$\stackrel{\rightarrow}{\longrightarrow} H_*(BU(1)) \stackrel{i*}{\longrightarrow} H_*(BS^1 \cdot Z_2) \stackrel{i*}{\longrightarrow} \tilde{H}_*(T(\xi)) -$$

We take the cellular decomposition $\{(C_i, D_j)\}$ of $BS^1 \cdot Z_2$ given in [1].

PROPOSITION 2.2. $\tilde{H}_*(T(\xi))$ is a direct sum of the torsion group generated by

 $j_*(C_{2s}, D_{2t+1}) \quad s > 0, t \ge 0$

and

$$j_*(C_{2\lambda+1}, D_{2\mu}) \qquad \lambda \geq 0, \quad \mu \geq 0$$

whose orders are 2 and the free abelian group generated by

$$y_{4t+3}$$
 $t = 0, 1, 2, \dots$

such that $\partial y_{4t+3} = 2D_{2t+1}$.

PROOF. The proposition follows from the diagram (2.1) and Proposition 1.1 of [2].

Denote by $\{E_{s,t}^r\}$ the Atiyah-Hirzebruch spectral sequence for $\tilde{U}_*(T(\xi))$.

PROPOSITION 2.3. The element y_{4t+3} of $E_{*,0}^2$ are parmanent cycles. The 2-torsion part of $\sum_{i+j=odd} E_{i,j}^5$ is the U_* -free module generated by $\{(C_{2i+1}, D_0), i=1, 0,\}$ and the free part is generated by y_{4t+3} .

PROOF. We recall that $D(1, n)/D(0, n) \approx S^1 \wedge (CP^n)^+$, $D(0, n) = CP^n$.

The homology group $H_*(D(1, n))$ is a direct sum of the free abelian

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group generated by the (C_1, D_{2k}) and the (C_0, D_{2k}) and the torsion group generated by the (C_0, D_{2k+1}) which are of order 2. Considering the exact sequence

 $\begin{array}{c} 0 \longrightarrow H_*(D(1, n)) \longrightarrow \widetilde{H}_*(D(1, n)/CP^n) \xrightarrow{\partial'} H_{*-1}(CP^n) \longrightarrow H_{*-1}(D(1, n)) \longrightarrow 0, \\ \text{there exist the element } y'_{4k+3} \text{ such that } \partial' y'_{4k+3} = D_{2k+1}. \\ \text{We consider an inclusion } i: (D(1, n), CP^n) \longrightarrow (BS^1 \cdot Z_2, BU(1)) \\ \text{and} \end{array}$

$$\begin{array}{ccc} \widetilde{H}_*(D(1, n)/CP^n) & \xrightarrow{\partial'} H_*(CP^n) \\ & \downarrow & i_* & \xrightarrow{\partial} & \downarrow & i_* \\ \widetilde{H}_*(T(\xi)) & \longrightarrow H_*(BU(1)) & . \end{array}$$

Put $i_*y'_{4k+3} = y_{4k+3} = y_{4k+3}$ which is a desired element of the proposition. Since the homology $\tilde{H}_*(D(1, n)/CP^n)$ has no torsion, the edge homomorphism $\mu: \tilde{U}_*(D(1, n)/CP^n) \longrightarrow \tilde{H}_*(D(1, n)/CP^n)$ is epimorphic. Considering the following diagram

$$\begin{array}{ccc} \tilde{U}^*(D(1, n)/CP^n) & \stackrel{i_*}{\longrightarrow} \tilde{U}_*(T(\xi)) \\ & \downarrow \mu & & \downarrow \mu \\ H_*(D(1, n)/CP^n) & \stackrel{i_*}{\longrightarrow} \tilde{H}_*(T(\xi)) \end{array}$$

We can see that the y_{4k+3} are parmanent cycles.

Let $\{\tilde{E}_{*,*}^r\}$ be the spectral sequence with respect to $U_*(BS^1 \cdot Z_2)$. We have that $d_{*,*}^r j_* = j_* d_{*,*}^r$, where j_* is the homomorphism $\tilde{E}_{*,*}^r \longrightarrow E_{*,*}^r$ induced by the projection $j: D(k + 1, k) \longrightarrow T(\xi_k)$. From LEMMA 3.3 of [2] it follows that

$$d_{i,0}^{3}j_{*}(C_{i}, D_{j}) = \begin{cases} j_{*}(C_{i-1}, D_{j-1}) \otimes [CP^{1}] & \text{if } i \text{ is odd and } j \text{ is even } (>0) \\ 0 & \text{otherwise} \end{cases}$$

where i+j=t. Therefore we have that the $j_*(C_{2\lambda+1}, D_0)$ generate the 2-torsion part of $\sum_{i+j=odd} E_{i,j}^5$. q.e.d.

PROPOSITION 2.4. The 2-torsion part of $\tilde{U}_{odd}(T(\xi))$ is isomorphic to $\tilde{U}_{odd}(BZ_2)$.

PROOF. Consider the maps

i:
$$RP^{k+1} \longrightarrow T(\xi_k) = D(k+1, k)/CP^k$$
,
i[*x*] = [*x*, (1, 0,, 0)]

and

$$\pi: T(\xi_k) \longrightarrow RP^{k+1}$$

$$\pi\{[x, z]\} = [x].$$

Since $\pi i = \operatorname{id}$, i_* is injective. Noting that for sufficiently large k, $\tilde{U}_n(T(\xi_k)) \cong \tilde{U}_n(T(\xi))$ and $\tilde{U}_n(\mathbb{R}P^{k+1}) \cong \tilde{U}_n(\mathbb{B}Z_2)$, we have that $i_* \colon \tilde{U}_*(\mathbb{B}Z_2) \longrightarrow \tilde{U}_*(T(\xi))$ is injective. From Proposition 2.3, the order of the 2-torsion part of $\tilde{U}_{2t+1}(T(\xi)) \leq 2^\sigma$, $\sigma = \sum_{k=0}^t \pi(k)$ where $\pi(k)$ is the number of the partitions of k, and $\tilde{U}_{2t+1}(\mathbb{B}Z_2)$ has order of 2^σ . q.e.d.

Therefore we have the following

THEOREM 2.5. $U_{2t+1}(O(2))$ is isomorphic to

$$U_{2l+1}(Z_2) \oplus Z \underbrace{\oplus \cdots \oplus \oplus }_{\tau} Z$$

where

$$\tau = \begin{cases} \sum_{j=1}^{k} \pi(2j-1) & \text{if } t = 2k \\ \sum_{j=0}^{k} \pi(2j) & \text{if } t = 2k+1. \end{cases}$$

PROPOSITION 2.6. $U_{2t}(O(2))$ is order of 2^{σ} ,

$$\sigma = \begin{cases} \sum_{j=0}^{k-1} (j+1) \{ \tilde{\pi}(2(k-j)-1) + \tilde{\pi}(2(k-j)-2) \} & \text{if } t = 2k+1 \\ \sum_{j=1}^{k} j \tilde{\pi}(2(k-j)) + \sum_{j=1}^{k-1} j \tilde{\pi}(2(k-j)-1) & \text{if } t = 2k \end{cases}$$

where $\tilde{\pi}(i)$ is the number of partitions of *i* containing no 1.

PROOF. Since the elements of $\sum_{i+j=odd} E_{i,j}^{\varsigma}$ are parmanent cycles, it follows that $d_{*,0}^{r}(C_{2\lambda}, D_{2\mu+1})=0$, $r \geq 5$. Take the monomials g of the x_{i} containing no $x_{1}=[CP^{1}]$, where $U_{*}=Z[x_{1}, x_{2}, \dots]$. From Lemma 3.3 of [2] it follows that $\sum_{i+j=even} E_{i,j}^{\varsigma}$ is generated by the $(C_{2\lambda}, D_{2\mu+1}) \otimes g, \lambda > 0$. q.e.d.

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