Correction to "On the complex bordism group of a semi-direct product S^1 ・Z_2"

Kamata, Masayoshi
Department of Mathematics, College of General Education, Kyushu University

https://doi.org/10.15017/1448983

出版情報：九州大学教養部数学雑誌. 10 (1), pp.85-89, 1975-08. College of General Education, Kyushu University

Math. Rep.
X-1, 1975.

Correction to "On the complex bordism group of a semi-direct product $S^1 \cdot Z_2$"

By Masayoshi KAMATA

(Received April. 30, 1975)

In the preceding paper "On the complex bordism group of a semi-direct product $S^1 \cdot Z_2$" of this journal, we attacked to compute the bordism group $U_*(S^1 \cdot Z_2)$ of free $S^1 \cdot Z_2$-actions on weakly almost complex manifolds. To accomplish the computation, we used an incorrect isomorphism $U_*(S^1 \cdot Z_2) \cong U_{*-1}(BS^1 \cdot Z_2)$. As Lazarov and Wasserman showed in [3], $U_*(S^1 \cdot Z_2)$ is not isomorphic to $U_{*-1}(BS^1 \cdot Z_2)$ because the adjoint representation of $S^1 \cdot Z_2 = O(2)$ is not trivial. In the paper [2] we computed only $U_*(BS^1 \cdot Z_2)$. In this paper we compute $U^*(S^1 \cdot Z_2)$ again.

I am grateful to Professor R. E. Stong and Doctor K. Shibata for catching the above embarrassing errors and their useful suggestions.

1. Take $gt^i \in S^1 \cdot Z_2$, where t is the generator of Z_2. The adjoint representation of $S^1 \cdot Z_2 = O(2)$ is described by

$$Ad(gt^i)x = (-1)^i x, \; x \in R^1 .$$

Let $\xi: Y \longrightarrow B$ be a principal $O(2)$-bundle over B. We put

$$Ad(Y) = Y \times_{O(2)} R^1 ,$$

where $Y \times_{O(2)} R^1$ is the quotient space of $Y \times R^1$ under the identification $(y, x) = (yb^{-1}, Ad(b)x)$, $b \in O(2)$, and have a line vector bundle $Ad(\xi)$: $Ad(Y) \longrightarrow B$.

Let M^m be a weakly almost complex manifold with a free $O(2)$-action preserving the complex structure on $\tau(M^m) \oplus \varepsilon^r$ for some r, where ε^r is the r-dimensional trivial bundle. We then have the complex vector bundle

$$(1.1) \qquad (\tau(M^m) \oplus \varepsilon^r)/O(2) \cong \pi^*\tau(M^m/O(2)) \oplus Ad(M^m) \oplus \varepsilon^r ,$$

where $\pi: M^m \longrightarrow M^m/O(2)$ is the canonical projection and $\tau(\quad)$ denotes the tangent bundle.

Denote by $U_m(O(2), Ad)$ the bordism group of pairs (W^{m-1}, η) consisting of an $(m-1)$-dimensional differentiable manifold W^{m-1} and a differentiable principal $O(2)$-bundle η over W^{m-1} with a complex structure on $\tau(W^{m-1}) \oplus Ad(\eta) \oplus \varepsilon^r$ for some r. Sending an element $[M^m] \in U_m(O(2))$ to an element

$[M^m/O(2), \eta] \in U_m(O(2), Ad)$, where $\eta: M^m \longrightarrow M^m/O(2)$ is the canonical principal $O(2)$-bundle, we can see that $U_m(O(2))$ is isomorphic to $U_m(O(2), Ad)$ [3]. For the universal principal $O(2)$-bundle $\gamma: EO(2) \longrightarrow BO(2)$ we consider the bundle

$$\xi = Ad(\gamma): EO(2) \times_{O(2)} R^1 = Ad(EO(2)) \longrightarrow BO(2) .$$

Let $D(\xi)$ and $S(\xi)$ be the total spaces of the disk bundle and the sphere bundle of ξ respectively and $T(\xi)$ the Thom complex of ξ. It follows that $U_*(D(\xi), S(\xi)) \cong \tilde{U}_*(T(\xi))$. We then have

THEOREM 1.1. *$U_m(O(2), Ad)$ is isomorphic to $U_m(D(\xi), S(\xi))$.*

PROOF. We first define the homomorphism $\Phi: U_m(O(2), Ad) \longrightarrow U_m(D(\xi), S(\xi))$. For $[W^{m-1}, \eta] \in U_m(O(2), Ad)$, we have a bundle map

$$Ad(\eta) \longrightarrow \xi = Ad(\gamma)$$

and the bundle map of the disk bundles

$$f_\eta: D(Ad(\eta)) \longrightarrow D(\xi) .$$

Here $D(Ad(\eta))$ is a weakly almost complex manifold, because

$$\tau(D(Ad(\eta)) \cong \pi^*(\tau(W^{m-1}) \oplus Ad(\eta))$$

where the $\tau(\quad)$ are the tangent bundles and $\pi: D(Ad(\eta)) \longrightarrow W^{m-1}$ is the projection. We put

$$\Phi[W^{m-1}, \eta] = [D(Ad(\eta)), f_\eta] .$$

Next we shall define $\Psi: U_m(D(\xi), S(\xi)) \longrightarrow U_m(O(2), Ad)$. Suppose that $[M^m, f] \in U_m(D(\xi), S(\xi))$ is represented by a differentiable map $f: M^m \longrightarrow D(\xi)$ which is transverse regular to $BO(2)$.
Let

$$W^{m-1} = f^{-1}(BO(2))$$

and ν be the normal bundle of W^{m-1} in M^m. Note that

$$\bar{f}^*\xi \cong \nu, \ \xi = Ad(\gamma)$$

and

$$Ad(\bar{f}^*(\gamma)) \cong \nu ,$$

where $\bar{f} = f | W^{m-1}$. $\bar{f}^*(\gamma)$ is the principal $O(2)$-bundle. Since

$$\tau(D(\bar{f}^*(\xi))) \cong \pi^*(\tau(W^{m-1}) \oplus \bar{f}^*(\xi))$$

and $D(f^*(\xi))$ is diffeomorphic to the tubular neighborhood of W^{m-1} which admits a weakly almost complex structure, $\tau(W^{m-1}) \oplus Ad(\bar{f}^*(\gamma)) \oplus \varepsilon^r$ have a complex structure for some r. Therefore we can define

$$\Psi[M^m, f] = [W^{m-1}, \bar{f}^*(\gamma)] .$$

It is easy to see that $\Phi\Psi = id$ and $\Psi\Phi = id$. q.e.d.

COROLLARY 1.2. $U_m(O(2)) \cong \tilde{U}_m(T(\xi))$.

2. The Dold manifold $D(l, k)$ is a quotient space under the identification $(x, z) = (-x, \overline{\lambda z})$, $\lambda \in S^1$. The universal principal $S^1\cdot Z_2$-bundle is approximated by the principal $S^1\cdot Z_2$-bundle $S^l\times S^{2k+1} \longrightarrow D(l, k)$. We consider the line bundle

$$\xi_k\colon Ad(S^k\times S^{2k+1}) \longrightarrow D(k, k) .$$

Making use of Theorem 1 of [1], we obtain

$$T(\xi_k) \approx D(k+1, k)/CP^k .$$

The direct limit space lim $T(\xi_k)$ is $T(\xi)$. Consider the cofibration $CP^k \xrightarrow{i} D(k+1, k) \xrightarrow{j} T(\xi_k)$ and we have the exact sequence

$$(2.1) \qquad \to H_*(BU(1)) \xrightarrow{i*} H_*(BS^1\cdot Z_2) \xrightarrow{j*} \tilde{H}_*(T(\xi)) \xrightarrow{\partial} \cdots$$

We take the cellular decomposition $\{(C_i, D_j)\}$ of $BS^1\cdot Z_2$ given in [1].

PROPOSITION 2.2. *$\tilde{H}_*(T(\xi))$ is a direct sum of the torsion group generated by*

$$j_*(C_{2s}, D_{2t+1}) \qquad s>0,\ t\geqq 0$$

and

$$j_*(C_{2\lambda+1}, D_{2\mu}) \qquad \lambda\geqq 0,\ \mu\geqq 0$$

whose orders are 2 *and the free abelian group generated by*

$$y_{4t+3} \qquad t = 0, 1, 2, \ldots\ldots$$

such that $\partial y_{4t+3} = 2D_{2t+1}$.

PROOF. The proposition follows from the diagram(2.1) and Proposition 1.1 of [2].

Denote by $\{E^r_{s,t}\}$ the Atiyah-Hirzebruch spectral sequence for $\tilde{U}_*(T(\xi))$.

PROPOSITION 2.3. *The element y_{4t+3} of $E^2_{*,0}$ are parmanent cycles. The 2-torsion part of $\sum_{i+j=odd} E^5_{i,j}$ is the U_*-free module generated by $\{(C_{2i+1}, D_0)$, $i=1, 0, \ldots\ldots\}$ and the free part is generated by y_{4t+3}.*

PROOF. We recall that $D(1, n)/D(0, n) \approx S^1\wedge(CP^n)^+$, $D(0, n) = CP^n$. The homology group $H_*(D(1, n))$ is a direct sum of the free abelian group generated by the (C_1, D_{2k}) and the (C_0, D_{2k}) and the torsion group generated by the (C_0, D_{2k+1}) which are of order 2. Considering the exact sequence

$$0 \longrightarrow H_*(D(1, n)) \longrightarrow \tilde{H}_*(D(1, n)/CP^n) \xrightarrow{\partial'} H_{*-1}(CP^n) \longrightarrow H_{*-1}(D(1, n)) \longrightarrow 0,$$

there exist the element y'_{4k+3} such that $\partial' y'_{4k+3} = D_{2k+1}$.

We consider an inclusion i: $(D(1, n), CP^n) \longrightarrow (BS^1 \cdot Z_2, BU(1))$ and

$$\begin{array}{ccc} \tilde{H}_*(D(1, n)/CP^n) & \xrightarrow{\partial'} & H_*(CP^n) \\ \downarrow i_* & & \downarrow i_* \\ \tilde{H}_*(T(\xi)) & \xrightarrow{\partial} & H_*(BU(1)) \end{array} .$$

Put $i_* y'_{4k+3} = y_{4k+3}$ which is a desired element of the proposition. Since the homology $\tilde{H}_*(D(1, n)/CP^n)$ has no torsion, the edge homomorphism μ: $\tilde{U}_*(D(1, n)/CP^n) \longrightarrow \tilde{H}_*(D(1, n)/CP^n)$ is epimorphic. Considering the following diagram

$$\begin{array}{ccc} \tilde{U}^*(D(1, n)/CP^n) & \xrightarrow{i_*} & \tilde{U}_*(T(\xi)) \\ \downarrow \mu & & \downarrow \mu \\ H_*(D(1, n)/CP^n) & \xrightarrow{i_*} & \tilde{H}_*(T(\xi)) \end{array} ,$$

We can see that the y_{4k+3} are parmanent cycles.

Let $\{\tilde{E}^r_{*,*}\}$ be the spectral sequence with respect to $U_*(BS^1 \cdot Z_2)$. We have that $d^r_{*,*} j_* = j_* d^r_{*,*}$, where j_* is the homomorphism $\tilde{E}^r_{*,*} \longrightarrow E^r_{*,*}$ induced by the projection j: $D(k+1, k) \longrightarrow T(\xi_k)$. From LEMMA 3.3 of [2] it follows that

$$d^3_{t,0} j_*(C_i, D_j) = \begin{cases} j_*(C_{i-1}, D_{j-1}) \otimes [CP^1] & \text{if } i \text{ is odd and } j \text{ is even } (>0) . \\ 0 & \text{otherwise} \end{cases}$$

where $i+j=t$. Therefore we have that the $j_*(C_{2\lambda+1}, D_0)$ generate the 2-torsion part of $\sum_{i+j=odd} E^5_{i,j}$. q.e.d.

PROPOSITION 2.4. *The 2-torsion part of* $\tilde{U}_{odd}(T(\xi))$ *is isomorphic to* $\tilde{U}_{odd}(BZ_2)$.

PROOF. Consider the maps

$$i: RP^{k+1} \longrightarrow T(\xi_k) = D(k+1, k)/CP^k ,$$
$$i[x] = [x, (1, 0, \ldots\ldots, 0)]$$

and

$$\pi: T(\xi_k) \longrightarrow RP^{k+1}$$
$$\pi\{[x, z]\} = [x] .$$

Since $\pi i = \mathrm{id}$, i_* is injective. Noting that for sufficiently large k, $\tilde{U}_n(T(\xi_k)) \cong \tilde{U}_n(T(\xi))$ and $\tilde{U}_n(RP^{k+1}) \cong \tilde{U}_n(BZ_2)$, we have that i_*: $\tilde{U}_*(BZ_2) \longrightarrow \tilde{U}_*(T(\xi))$ is injective. From Proposition 2.3, the order of the 2-torsion part of $\tilde{U}_{2t+1}(T(\xi)) \leqq 2^\sigma$, $\sigma = \sum_{k=0}^{t} \pi(k)$ where $\pi(k)$ is the number of the partitions of k, and $\tilde{U}_{2t+1}(BZ_2)$ has order of 2^σ. q.e.d.

Therefore we have the following

THEOREM 2.5. $U_{2t+1}(O(2))$ is isomorphic to

$$U_{2t+1}(Z_2) \oplus \underbrace{Z \oplus \cdots\cdots \oplus Z}_{\tau}$$

where

$$\tau = \begin{cases} \sum_{j=1}^{k} \pi(2j-1) & \text{if } t = 2k \\ \sum_{j=0}^{k} \pi(2j) & \text{if } t = 2k+1 . \end{cases}$$

PROPOSITION 2.6. *$U_{2t}(O(2))$ is order of 2^σ,*

$$\sigma = \begin{cases} \sum_{j=0}^{k-1} (j+1)\{\tilde{\pi}(2(k-j)-1) + \tilde{\pi}(2(k-j)-2)\} & \text{if } t = 2k+1 \\ \sum_{j=1}^{k} j\tilde{\pi}(2(k-j)) + \sum_{j=1}^{k-1} j\tilde{\pi}(2(k-j)-1) & \text{if } t = 2k \end{cases}$$

where $\tilde{\pi}(i)$ is the number of partitions of i containing no 1.

PROOF. Since the elements of $\sum_{i+j=odd} E_{i,j}^5$ are parmanent cycles, it follows that $d_{*,0}^r(C_{2\lambda}, D_{2\mu+1}) = 0$, $r \geqq 5$. Take the monomials g of the x_i containing no $x_1 = [CP^1]$, where $U_* = Z[x_1, x_2, \cdots\cdots]$. From Lemma 3.3 of [2] it follows that $\sum_{i+j=even} E_{i,j}^5$ is generated by the $(C_{2\lambda}, D_{2\mu+1}) \otimes g$, $\lambda > 0$. q.e.d.

References

[1] M. Fujii: *Ring structure of K_U-cohomologies of Dold manifolds*, Osaka J. Math. **6** (1969), 107-115.

[2] M. Kamata: *On the complex bordism group of a semi-direct product $S^1 \cdot Z_2$*, Math. Rep. of Colledge of General Education Kyushu Uni. vol. **VIII**, No. 2. (1972) 21-27.

[3] Connor Lazarov and Arthur Wasserman: Complex Actions of Lie Groups, Memoirs of Amer. Math. Soc. No. **137** (1973).

Department of Mathematics,
College of General Education
Kyushu University.