Existence of Global (Bounded) Solutions for Some Nonlinear Evolution Equations of Second Order

Nakao, Mitsuhiro Department of Mathematics, Faculty of Science, Kyushu University

Nanbu, Tokumori Department of Mathematics, College of General Education, Kyushu University

https://doi.org/10.15017/1448981

出版情報:九州大学教養部数学雑誌. 10(1), pp.67-75, 1975-08. College of General Education, Kyushu University バージョン: 権利関係:

Existence of Global (Bounded) Solutions for Some Nonlinear Evolution Equations of Second Order

By Mitsuhiro NAKAO and Tokumori NANBU (Received May 6, 1975)

Introduction

In this paper we shall consider the following evolution equations

$$(0.1) \qquad \frac{\partial^2}{\partial t^2} u - \sum_{i=1}^n \left(|u_{x_i}|^{p-2} u \right)_{x_i} - \Delta \frac{\partial u}{\partial t} + \beta(x, u) = f(x, t)$$

in $\Omega \times (0, \infty)$ together with the initial-boundary conditions;

(0.2)
$$u|_{\partial a} = 0, \ u(x, 0) = u_0(x), \ \text{and} \ \frac{\partial u}{\partial t}(x, 0) = u_1(x),$$

where \mathcal{Q} is a bounded domain in \mathbb{R}^n with a smooth boundary $\partial \mathcal{Q}$.

It is our purpose here to establish the existence of global (bounded) solution to the above problem (0.1)-(0.2).

In the case $\beta(x, u) = 0$, problem (0.1)-(0.2) was studied by Greenberg [2] for n=1, and by Tsutsumi [6] for general n. On the other hand, Clement [1] and Kakita [3] have obtained the existence of weak periodic solution. (In [3] more general equations has been treated.). But in their works $\beta(x, u) = 0$ was also assumed. In the case $\beta(x, u) \ge 0$, especially $\beta(x, u)$ is not monotonic in u, the methods of earier papers do not apply to our problem and here we shall consider such cases. Our method is related to Nakao [5], where semilinear hyperbolic equations have been considered.

1. Preliminaries.

We shall employ usual notations. For brevity we use the notation $||u||_{p} = \left(\int_{a} |u|^{p} dx\right)^{1/p}$. The following lemma is well-known. LEMMA 1.1. (Sobolev)

If $u \in W_0^{1,p}(\Omega)$, then $u \in L^q(\Omega)$ and the inequality

(1.1)
$$||u||_q \leq S_{q,p}(\Omega, n) ||\nabla u||_p = S_{q,p}(\Omega, n) \left(\sum_{i=1}^n \left\| \frac{\partial u}{\partial x_i} \right\|_p^p \right)^{1/p}$$

Mitsuhiro NAKAO and Tokumori NANBU

holds, where q is a number satisfying

(1.2)
$$1 \leq q \leq \frac{np}{n-p}$$
 if $n > p$ and $1 \leq q < \infty$ if $n \leq p$.

Let p be a real number ≥ 2 and α a nonnegative real number. Our hypotheses are:

H.1. $\beta(x, u)$ is Lipshitz continuous in u for almost all $x \in Q$ and satisfies

(1.3)
$$|\beta(x, u)| \leq K_0 |u|^{\alpha+1}$$
 for $\forall u \in \mathbb{R}, a.e.x \in \mathcal{Q}$,

where K_0 is a positive constant and

(1.4)
$$2 \leq \alpha + 2 < \frac{np}{n-p}$$
 if $n > p$ and $0 \leq \alpha < \infty$ if $1 \leq n \leq p$.

H.2.
$$f \in C(\mathbf{R}: L^2(\mathcal{Q}))$$
 and $M = \sup_{t \in \mathcal{R}} \left(\int_t^{t+1} ||f(s)||_2^2 ds \right)^{1/2} < \infty$.

Definition 1. u(x, t) is said to be a bounded solution of (0.1)-(0.2) with initial value $\{u_0(x), u_1(x)\}$ if $u \in L^{\infty}(\mathbb{R}^+; W^{1,p}_0(\Omega))$,

 $u' \in L^{\infty}(\mathbb{R}^+: L^2(\mathcal{Q})) \cap L^2_{loc}(\mathbb{R}^+: W^{1,2}_0(\mathcal{Q}))$ and for $V\varphi \in C^1_0(\mathbb{R}^+: W^{1,p}_0(\mathcal{Q}))$ the variational equation

(1.5)
$$\int_{0}^{\infty} -(u'(t), \varphi'(t))dt + \int_{0}^{\infty} \left(\sum_{i=1}^{n} |u_{x_{i}}|^{p-2}u_{x_{i}}, \varphi_{x_{i}}\right)dt + \int_{0}^{\infty} (u'_{x_{i}}, \varphi_{x_{i}})dt + \int_{0}^{\infty} (\beta(u), \varphi)dt = \int_{0}^{\infty} (f, \varphi)dt + (u_{1}, \varphi(x, 0))$$

is valid, where $R^+ = [0, \infty)$ and $(u, v) = \int_{g} u(x)v(x)dx$.¹⁾

2. Approximate Solutions

In this section, we shall prove an existence theorem concerning approximate solutions.

For this purpose, we employ the Galerkin's approximation procedure.

Let $\{w_j\}, j = 1, 2, \dots, m, \dots$ be the basis of $W_0^{1,p}(\mathcal{Q})$ and consider the system of ordinary differential equations:

(2.1)
$$(u''_m(t), w_j) + \langle Au_m(t), w_j \rangle + ((u'_m(t), w_j)) + (\beta(., u_m), w_j)$$

= $(f(t), w_j), j = 1, 2, \dots, m.$

with initial values

¹⁾ For the sake of simplicity, we denote by the symbol' the differentiation with respect to t.

Solutions for Some Nonlinear Evolution Equations of Second Order

(2.2)
$$u_m(0) = \sum_{j=1}^m \lambda_{m,j}(0) w_j \to u_0 \text{ strongly in } W_0^1.$$

(2.3)
$$u'_{m}(0) = \sum_{j=1}^{m} \lambda'_{m,j}(0) w_{j} \rightarrow u_{1} \text{ strongly in } L^{2}$$

where $u_m(x, t) = \sum_{j=1}^m \lambda_{m,j}(t) w_j(x)$, $((u, v)) = \int_g \nabla u(x) \cdot \nabla v(x) dx$ and A is the operator from $W_0^{1,p}(\Omega)$, to $W^{-1,p/p-1}(\Omega)$ defined by

$$\langle Au, v \rangle = \int_{\mathcal{G}} \sum_{i=1}^{n} |u_{x_i}|^{p-2} u_{x_i} v_{x_i} dx$$
 for any $u, v \in W_0^{1,p}(\mathcal{Q})$.

The solution $\{\lambda_{m,j}(t)\}_{j=1}^{m}$, and consequently $u_m(t)$ exists on an interval, say, $[0, t_m]$ by the standard theory of ordinary differential equations. We must give a priori estimates for approximate solutions $u_m(x, t)$. First we shall consider the case where $p < \alpha + 2$.

Now we introduce some functionals on $W_0^{1,p}(\mathcal{Q})$ as follows;

(2.4)
$$J_0(u) = (1/p) || \nabla u ||_p^p + \int_g \int_0^{u(x,t)} \beta(x, s) \, ds dx$$

$$(2.5) J_0(\tilde{u}) = (1/p) || \nabla u ||_p^p - (K_0 S_{\alpha+2,p}^{\alpha+2}/\alpha + 2) || \nabla u ||_p^{\alpha+2}$$

(2.6)
$$J_1(u) = || \nabla u ||_{\rho}^{\rho} + \int_{g} \beta(x, u) u \, dx$$

(2.7)
$$\widetilde{J}_{1}(u) = || \nabla u ||_{p}^{p} - K_{0} S_{\alpha^{+2},p}^{\alpha^{+2}} || \nabla u ||_{p}^{\alpha^{+2}}.$$

By the assumption (1.3) and Lemma 1. 1, we have

(2.8)
$$J_0(u) \leq J_0(u)$$
 and $J_1(u) \leq J_1(u)$ for V $u \in W_0^{1,p}(\Omega)$.

Also we put

$$(2.9) D_1 = \max_{x\geq 0} \{x^p - K_0 S_{\alpha+2,p}^{\alpha+2} x^{\alpha+2}\} = x_0^p - K_0 S_{\alpha+2,p}^{\alpha+2} x_0^{\alpha+2}$$

where

$$x_{\scriptscriptstyle 0} = (p/K_{\scriptscriptstyle 0}S^{lpha+2}_{lpha+2,
u}(2\!+\!lpha))^{1/lpha+2-
u}$$
 ,

and put

$$(2.10) \qquad D_0 = (1/p) x_0^p - (K_0 S_{\alpha+2,p}^{\alpha+2} / \alpha + 2) x_0^{\alpha+2}$$

We define the stable set W by

(2.11)
$$W \equiv \left\{ (u_0, u_1) \in W_0^{1,p} \times L^2; || \nabla u_0 ||_p < x_0, || (u_0, u_1) ||_w \\ \equiv \frac{1}{2} || u_1 ||_2^2 + J_0(u_0) < D_0 \right\}.$$

LEMMA. 2.1. In addition to the hypotheses H.1 and H.2, we assume that $(u_0, u_1) \in W$ and $p < \alpha + 2$. Then there exists a positive number $M_0 = M_0(||(u_0, u_1)||_W)$

69

such that if $M < M_0$, approximate solution $u_m(t)$ exists on $[0, \infty)$ and satisfy

$$(2.12) \qquad ||u'_{m}(t)||_{2}^{2} \leq 2D_{0} \text{ and } ||\nabla u_{m}(t)||_{p} \leq x_{0} \text{ for } \forall t \in [0, \infty)$$

and

(2.13)
$$\int_{T}^{T+1} || \nabla u'_{\pi}(t) ||_{2}^{2} dt \leq D_{0} + M \sqrt{2D_{0}} < \infty \quad \text{for } \forall T > 0,$$

where m is a sufficiently large positive integer.

PROOF. We shall show that $(u_m(t), u'_m(t))$ stays in the stable set W for all time as long as they exist. Then (2.12) will follow easily from the inequality

$$(2.14) \qquad \frac{1}{2} ||u'_{m}(t)||_{2}^{2} + \tilde{J}_{0}(u_{m}(t)) < D_{0} < \max_{x \geq 0} \left\{ \frac{1}{p} x^{p} - (K_{0} S_{\alpha+2,p}^{\alpha+2} / + 2) x^{\alpha+2} \right\}$$

Since $(u_0, u_1) \in W$, we have by the definition of W

$$(2.15) \qquad \mathcal{E}_{0} = D_{0} - \left(\frac{1}{2} ||u_{1}||_{2}^{2} + J_{0}(u_{0})\right) > 0 \; .$$

Also since $(u_m(0), u'_m(0)) \rightarrow (u_0, u_1)$ in $W_0^{1,p} \times L^2$, we may assume

(2.16)
$$\frac{1}{2} ||u'_{m}(0)||_{2}^{2} + J_{0}(u_{m}(0)) \leq D_{0} - \mathcal{E}, \ 0 < V \ \mathcal{E} < \mathcal{E}_{0},$$

for large m.

Suppose that our assertion was false. There would then exist the smallest time $\bar{t} \in [0, t_m]$ at which

(2.17)
$$\frac{1}{2} ||u'_{m}(\bar{t})||_{2}^{2} + J_{0}(u_{m}(\bar{t})) = D_{0}$$
.

We shall derive a contradiction. First of all we note that

$$(2.18) \qquad \widetilde{J}_0(u_m(t)) \leq J_0(u_m(t)) \leq D_0 \quad \text{for} \quad t \in [0, \, \overline{t}],$$

and the continuity of $\tilde{J}_0(u_m)$, with respect to $\|\nabla u_m\|_p$, implies

(2.19) $|| \nabla u_m(t) ||_p \leq x_0$ for $t \in [0, \bar{t}]$.

Now multipling (2.1) by $\lambda'_{m,j}(t)$, summing over j from 1 to m and integrating over $[a, b] \subset [0, t_m]$, we obtain

$$(2.20) \qquad \frac{1}{2} ||u'_{m}(b)||_{2}^{2} + J_{0}(u_{m}(b)) + \int_{a}^{b} ||\nabla u'_{m}(t)||_{2}^{2} dt$$
$$= \frac{1}{2} ||u'_{m}(a)||_{2}^{2} + J_{0}(u_{m}(a)) + \int_{a}^{b} (f(t), u'_{m}(t)) dt$$

In the above we take $b=\bar{t}$ and a=0 to get by (2.16)

(2.21)
$$\mathscr{E} + \int_{0}^{\overline{t}} || \nabla u'_{m}(t) ||_{2}^{2} dt \leq S_{2,2} \int_{0}^{\overline{t}} || f(t) ||_{2} || \nabla u'_{m}(t) ||_{2} dt$$
.

Solutions for Some Nonlinear Evolution Equations of Second Order

Hence we have

$$\mathcal{E} + \int_{0}^{\overline{t}} || \nabla u'_{m}(t) ||_{2}^{2} dt \leq \int_{0}^{\overline{t}} || \nabla u'_{m}(t) ||_{2}^{2} dt + \frac{S_{2,2}^{2}}{4} \int_{0}^{\overline{t}} || f(t) ||_{2}^{2} dt$$

or

(2.22)
$$\int_{0}^{\bar{t}} ||f(t)||_{2}^{2} dt \ge \frac{4}{S_{2,2}^{2}} \mathcal{E}$$

Now let us assume

$$(2.23) M < M_1 = \sqrt{(4/S_{2,2}^2) \mathcal{E}_0} .$$

Then from (2.22) and (2.23), it is easily seen that t > 1 for large *m*. Thus we can take b=t and a=t-1 in (2.20), and we have

(2.24)
$$\int_{\bar{t}-1}^{\bar{t}} \| \mathcal{F}u'_{m}(t) \|_{2}^{2} dt \leq \int_{\bar{t}-1}^{\bar{t}} |(f(t), u'_{m}(t))| dt$$

and hence

(2.25)
$$\int_{\overline{t}-1}^{\overline{t}} || \nabla u'_{m}(t) ||_{2}^{2} dt \leq S_{2,2}^{2} M^{2}, \int_{\overline{t}-1}^{\overline{t}} || u'_{m}(t) ||_{2}^{2} dt \leq S_{2,2}^{4} M^{2}.$$

Therefore there exist numbers $\bar{t}_1 \in [\bar{t}-1, \bar{t}-\frac{3}{4}]$ and $\bar{t}_2 \in [\bar{t}-\frac{1}{4}, \bar{t}]$ such that

 $(2.26) \qquad ||u'_{m}(\bar{t_{i}})||_{2} \leq 2S_{2,2}^{2} M, \quad i=1,2.$

Next, multipling (2.1) by $\lambda_{m,i}(t)$, summing over *j* and integrating over $[\bar{t}_1, \bar{t}_2]$, we obtain

$$\int_{\overline{t_1}}^{\overline{t_2}} J_1(u_m(t)) dt \leq |(u'_m(t_1), u_m(t_1))| + |(u'_m(t_2), u_m(t_2))| + \\ + \int_{\overline{t_1}}^{\overline{t_2}} (||u'_m(t)||_2^2 + |(\nabla u'_m(t), \nabla u_m(t))| + |(f(t), u_m(t)|) dt$$

and by virtue of (2.25), (2.26) and (2.19),

$$\leq 4S_{2,2}^2 M \max_{[t-1,t]} ||u_m(t)||^2 + S_{2,2}^4 M^2 + S_{2,2} M(\operatorname{mes} \mathcal{Q})^{p-2/2p} x_0 + M \max_{[t-1,t]} ||u_m(t)||_2$$

$$\leq S_{2,2}^4 M^2 + x_0 \{4S_{2,2}^2 M S_{2,p} + S_{2,2} M(\operatorname{mes} \mathcal{Q})^{p-2/2p} + M S_{2,p}\}$$

(2. 27) $\equiv C_1(M) + x_0 C_2(M)$.

On the other hand,

$$\int_{\tilde{t}_{1}}^{\tilde{t}_{2}} J_{1}(u_{m}(t)) dt \geq \int_{\tilde{t}_{1}}^{\tilde{t}_{2}} \widetilde{J}_{1}(u_{m}(t)) dt$$
$$= \int_{\tilde{t}_{1}}^{\tilde{t}_{2}} || \mathcal{V}u_{m}(t) ||_{p}^{p} (1 - K_{0} S_{\alpha+2,p}^{\alpha+2}) || \mathcal{V}u_{m}(t) ||_{p}^{\alpha+2-p}) dt$$

by (2.19) and the definition of x_0

71

$$\geq (1 - (p/\alpha + 2)) \int_{\bar{t}_1}^{\bar{t}_2} || \mathbf{r} u_m(t) ||_p^p dt .$$

From (2.27) and (2.28), we have

(2.29)
$$\int_{\bar{t}_{1}}^{\bar{t}_{2}} || \nabla u_{m}(t) ||_{p}^{p} dt \leq \frac{\alpha+2}{\alpha+2-p} \int_{\bar{t}_{1}}^{\bar{t}_{2}} J_{1}(u_{m}(t)) dt \leq \frac{\alpha+2}{\alpha+2-p} \{C_{1}(M) + x_{0}C_{2}(M)\}$$

The inequality (2.29) together with (2.25) implies that there exists a point $t^* \in [\bar{t_1}, \bar{t_2}]$ such that

(2.30)
$$||u'_{m}(t^{*})||_{2}^{2} + ||\nabla u_{m}(t^{*})||_{p}^{b} \leq 2C_{3}(M)$$
,
where $C_{3}(M) \equiv S_{2,2}^{4}M^{2} + (\alpha + 2/\alpha + 2 - p)(C_{1}(M) + x_{0}C_{2}(M))$

In (2.20) we take b=t and $a=t^*$ to get by (2.30)

$$\frac{1}{2} ||u'_{m}(\bar{t})||_{2}^{2} + J_{0}(u_{m}(\bar{t}))$$

$$\leq \frac{1}{2} ||u_{m}(t^{*})||_{2}^{2} + J_{0}(u_{m}(t^{*})) + \int_{\bar{t}^{-1}}^{\bar{t}} |(f(t), u'_{m}(t))| dt$$

$$\leq C_{4}(M) \equiv C_{3}(M) + \frac{K_{0}}{\alpha + 2} S_{\alpha + 2, \rho}^{\alpha + 2} (2C_{3}(M))^{2 + \alpha/\rho} + S_{2, 2}^{2} M^{2}$$

This contradicts (2.17) if we choose $M < M_2$, M_2 being the smallest number satisfying $C_4(M) = D_0$.

Thus if we take $M_0 = Min(M_1, M_2)$, the proof of (2.12) is completed. Furthermore if we take b=T+1 and a=T in (2.20), then we have

$$\int_{T}^{T+1} || \mathcal{F} u'_{m}(t) ||_{2}^{2} dt \leq \frac{1}{2} || u'_{m}(T) ||_{2}^{2} + J_{0}(u_{m}(T)) + \int_{T}^{T+1} |(f(t), u'_{m}(t))| dt$$
$$\leq D_{0} + M \left(\int_{T}^{T+1} || u'_{m}(t) ||_{2}^{2} dt \right)^{1/2},$$

and hence $\int_{T}^{T+1} || \nabla u'_{m}(t) ||_{2}^{2} dt \leq D_{0} + M \sqrt{2D_{0}}$,

where T is any positive number.

Next we shall consider the case where $p > \alpha + 2$.

LEMMA 2.2. In addition to hypotheses H.1 and H.2., we assume that $p > \alpha + 2$. Then approximate solutions of the problem (2.1)-(2.3) exist on $[0, \infty)$ and satisfy

$$(2.12)' \qquad ||u'_{m}(t)||_{2}^{2} + ||\nabla u_{m}(t)||_{p}^{p} \leq C(u_{0}, u_{1}, M) \text{ for } \forall t \in [0, \infty),$$

$$(2.13)' \int_{T}^{T} || \nabla u'_{m}(t) ||_{2}^{2} dt \leq C(u_{0}, u_{1}, M, T) \text{ for } V T > 0 \text{ and } Vm.$$

PROOF. First we can obtain the estimate

$$\frac{1}{2} \|u'_{m}(t)\|_{2}^{2} + J_{0}(u_{m}(t)) \leq C(u_{0}, u_{1}, M) \text{ for } t \in [0, \infty] \text{ and large } m.$$

73

This is verified by the similar way of the Lemma 2.1. Next, using Lemma 1.1 and Young's inequality, we have

$$\frac{1}{2}\Big(||u'_{m}(t)||_{2}^{2}+\frac{1}{p}||\nabla u_{m}(t)||_{p}^{p}\Big) \leq C(u_{0}, u_{1}, M) ,$$

which implies (2.12)'.

3. Passage to the Limit

By the passage to the limit of approximate solutions, we shall obtain;

THEOREM 3.1. In addition to the hypotheses H. 1 and H. 2, we assume that $\alpha + 2 > p$, $(u_0, u_1) \in W$ and $M < M_0$. Then the problem (0.1)-(0.2) admits a bounded solution u with initial data (u_0, u_1) in the sense of Definition 1. Moreover the estimates (2.12) and (2.13) hold for $u_m = u$.

THEOREM 3.2. In addition to the hypotheses H.1 and H.2, we assume that $\alpha + 2 < p$ and $(u_0, u_1) \in W_0^{1,p} \times L^2$. Then we have the same coclusion as Theorem 3.1 without any restrictions on M and (u_0, u_1) .

We shall verify Theorem 3.1 only, because Theorem 3.2 is verified in a similar way.

PROOF of Theorem 3.1. We construct the approximate solutions $\{u_m(t)\}$ by (2.1), (2.2) and (2.3). By the Lemma 2.1 we have

 $(3.1) || Au_m(t) ||_{\Psi^{-1,p/p-1}} \leq x_0^{p-1} ext{ for } \Psi \ t \in [0,\infty) \ .$

Let (T_n) be any positive sequence tending to ∞ as $n \to \infty$. Then by (2.12), (2.13) and (3.1), we can use the standard compactness arguments to extract a subsequence from $u_m(t)$, which will be denoted also by $u_m(t)$, satisfying;

(3.2)
$$u_m(t) \longrightarrow u(t)$$
 weakly star in $L^{\infty}(\mathbb{R}^+: W^{1,p}_0(\mathcal{Q}))$ and a.e. in $\mathcal{Q} \times \mathbb{R}^+$,

(3.3) $\beta \cdot , u_m(t) \longrightarrow \beta(\cdot, u(t))$ weakly star in $L^{\infty}(\mathbb{R}^+ : L^{\frac{\alpha+2}{\alpha+1}}(\Omega))$

 $(3.4) \qquad u'_{\mathfrak{m}}(t) \longrightarrow u'(t) \text{ weakly star in } L^{\infty}(\mathbb{R}^+: L^2(\mathcal{Q})) \text{ and } a.e. \text{ in } \mathcal{Q} \times \mathbb{R}^+,$

 $(3.5) \qquad u'_{\mathfrak{m}}(t) \longrightarrow u'(t) \text{ weakly in } L^2_{\text{loc}}(R^+: W^{1,2}_0(\Omega))$

- (3.6) $Au_m \longrightarrow \chi$ weakly star in $L^{\infty}(\mathbb{R}^+: W^{-1,p/p-1})$,
- $(3.7) u_m(T_n) \longrightarrow u(T_n) \text{ in } L^2 \text{ strongly}$

$$(3.8) \qquad u'_{m}(T_{n}) \longrightarrow u'(T_{n}) \text{ in } L^{2} \text{ weakly}$$

(3.9) $u_m(T_n) \longrightarrow u(T_n)$ in $W_0^{1,2}$ weakly.

To show that the function u(t) is a solution, it is sufficient to prove that $\chi = Au$.

From (2.1), for every T_n , we have the identity

$$-\int_{0}^{T_{n}} (u'_{m}(t), \psi'(t))dt + \int_{0}^{T_{n}} \langle Au_{m}(t), \psi(t) \rangle dt + \int_{0}^{T_{n}} ((u'_{m}(t), \psi(t)))dt + \int_{0}^{T_{n}} (\beta(\cdot, u_{m}), \psi(t))dt = \int_{0}^{T_{n}} (f(t), \psi(t))dt + (u'_{m}(0), \psi(0)) - (u'_{m}(T_{n}), \psi(T_{n}))dt$$

for all functions of the form $\psi = \sum_{k=1}^{N} d_k(t) w_k$, where $d_k(t)$ are smooth function on $[0, T_n]$.

Taking the limit as $m \to \infty$ in the above equation, we get

$$-\int_{0}^{T_{n}} (u'(t), \psi'(t)) dt + \int_{0}^{T_{n}} <\chi, \ \psi(t) > dt + \int_{0}^{T_{n}} ((u'(t), \ \psi(t))) dt + \\ + \int_{0}^{T_{n}} (\beta(\cdot, u), \ \psi(t)) dt = \int_{0}^{T_{n}} (f(t), \ \psi(t)) dt + (u_{1}, \ \psi(0)) - (u'(T), \psi(T_{n})) .$$

Thus we can replace ψ by u in the above equation to obtain

$$(3.10) \qquad -\int_{0}^{T_{n}} ||u'||_{2}^{2} dt + \int_{0}^{T_{n}} \langle \chi, u \rangle dt + \frac{1}{2} ||\nabla u(T_{n})||_{2}^{2} - \frac{1}{2} ||\nabla u_{0}||_{2}^{2} + \int_{0}^{T_{n}} (\beta(\cdot, u), u) dt = \int_{0}^{T_{n}} (f(t), u(t)) dt + (u_{0}, u_{1}) - (u'(T_{n}), u(T_{n})) .$$

Also we have

$$X_{m} = \int_{0}^{T_{m}} \langle A(u_{m}) - A(v), u_{m} - v \rangle dt \geq 0, \quad v \in L^{2}((0, T_{n}): W_{0}^{1,p}(\Omega)),$$

and, replacing w_i by u_m in (2.1),

$$X_{m} = \int_{0}^{T_{n}} ||u'_{m}(t)||_{2}^{2} dt - \frac{1}{2} ||\nabla u_{m}(T_{n})||_{2}^{2} + \frac{1}{2} ||\nabla u_{m}(0)||_{2}^{2} + \int_{0}^{T_{n}} (\beta(\cdot, u_{m}), u_{m}) dt + \int_{0}^{T_{n}} (f(t), u_{m}(t)) dt + (u'_{m}(0), u_{m}(0)) - (u'_{m}(T_{n}), u_{m}(T_{n})) - \int_{0}^{T_{n}} \langle Av, u_{m} - v \rangle dt - \int_{0}^{T_{n}} \langle A(u_{m}), v \rangle dt .$$

Taking the limit as $m \to \infty$ in the above equation, we get

(3.11)
$$\lim_{m \to \infty} X_m \leq \int_0^{T_m} ||u'(t)||_2^2 dt - \frac{1}{2} ||\nabla u(T_n)||_2^2 + \frac{1}{2} ||\nabla u_0||_2^2 + \frac{1}{2} ||\nabla u_0||_2^2$$

Solutions for some Nonlinear Evolution Equations of Second Order

$$+ \int_{0}^{T_{n}} (\beta(\cdot, u), u) dt + \int_{0}^{T_{n}} (f(t), u(t)) dt = (u_{1}, u_{0}) - (u'(T_{n}), u(T_{n})) - \int_{0}^{T_{n}} \langle Av, u - v \rangle dt - \int_{0}^{T_{n}} \langle \chi, v \rangle dt$$

From (3.10) and (3.11), we have

$$\int_{0}^{T_n} \langle \chi - Av, u - v \rangle dt \geq 0 \quad \text{for } v \in L^2((0, T_n) \colon W_0^{1, p})$$

which gives $\chi = Au$.

The limit function u(x, t) satisfies evidently the equality (1.5), and the estimates (2.12), (2.13) and (3.1) hold for u.

Remark 1. We note that $\beta(x, u)$ is a monotonic in u, then the Theorem 3.1 holds for arbitrary f satisfying only the hypothesis H.2.

Remark 2. If we choose the sequence w_i as the basis of $W_{5}^{s,2}(\Omega)$, $\left(s > n\left(\frac{1}{2} - \frac{1}{p}\right) + 1\right)$, we can easily obtain a priori estimate of u''_m in $||u''_m|| L_{1oc}^2(R; W^{-s,2})$ and consequently we have slightly better soltuions. (see [6]). **Remark 3.** Under additional appropriate condition, we can obtain the Theorems 3.1 for more general equation;

$$u_{ii} - \sum_{i=1}^{n} \{a_i(x, u_{xi})\}_{xi} - \Delta u_i + \beta(x, u, u_i) = f. \quad ([1], [3]).$$

References

- J. CLEMENT; Existence Theorems for a quasilinear evolution equation, SIAM J. Appl. Math. 26 (1974), pp. 745-752.
- [2] J. M. GREENBERG; On the existence, uniqueness and stability of solutions of the equations $X_{tt} = E(X_x)X_{xx} + X_{xxt}$, J. Math. Anal. Appl. 25 (1969), pp. 575-591.
- [3] T. KAKITA; Time periodic solutions of some nonlinear evolution equations, Publ. RIMS. Kyoto Univ. 9 (1974) pp. 477-492.
- [4] J. L. LIONS and W. A. STRAUSS; Some nonlinear evolution equations, Bull, Soc. Math. France 93(1965), pp. 43-96.
- [5] M. NAKAO; Bounded, Periodic or Almost-Periodic Solutions of Nonlinear Hyperbolic Partial Differential Equations (to appear).
- [6] M. TSUTSUMI; Some Evolution Equations of Second order, Proc. Jap. Acad. 47 (1970), pp. 950-955.

Department of Mathematics, Faculity of Science, Kyushu University and Department of Mathematics, College of General Education, Kyushu University. **7**5