Existence of Global (Bounded) Solutions for Some Nonlinear Evolution Equations of Second Order

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Existence of Global (Bounded) Solutions for Some Nonlinear Evolution Equations of Second Order

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Introduction

In this paper we shall consider the following evolution equations

$$\frac{\partial^2}{\partial t^2}u-\sum_{i=1}^{n}(|u_{x_i}|^{p-2}u)_{x_i}-\Delta\frac{\partial u}{\partial t}+\beta(x, u)=f(x, t) \tag{0.1}$$

in $\Omega\times(0, \infty)$ together with the initial-boundary conditions;

$$u|_{\partial\Omega}=0,\ u(x, 0)=u_0(x)\ ,\ \text{and}\ \frac{\partial u}{\partial t}(x, 0)=u_1(x)\ , \tag{0.2}$$

where Ω is a bounded domain in R^n with a smooth boundary $\partial\Omega$.

It is our purpose here to establish the existence of global (bounded) solution to the above problem (0.1)-(0.2).

In the case $\beta(x, u)=0$, problem (0.1)-(0.2) was studied by Greenberg [2] for $n=1$, and by Tsutsumi [6] for general n. On the other hand, Clement [1] and Kakita [3] have obtained the existence of weak periodic solution. (In [3] more general equations has been treated.). But in their works $\beta(x, u)=0$ was also assumed. In the case $\beta(x, u)\neq 0$, especially $\beta(x, u)$ is not monotonic in u, the methods of earier papers do not apply to our problem and here we shall consider such cases. Our method is related to Nakao [5], where semilinear hyperbolic equations have been considered.

1. Preliminaries.

We shall employ usual notations. For brevity we use the notation $\|u\|_p=\left(\int_\Omega|u|^p dx\right)^{1/p}$. The following lemma is well-known.

LEMMA 1.1. (Sobolev)

If $u\in W_0^{1,p}(\Omega)$, *then* $u\in L^q(\Omega)$ *and the inequality*

$$\|u\|_q\leq S_{q,p}(\Omega, n)\|\nabla u\|_p=S_{q,p}(\Omega, n)\left(\sum_{i=1}^{n}\left\|\frac{\partial u}{\partial x_i}\right\|_p^p\right)^{1/p} \tag{1.1}$$

holds, where q is a number satisfying

$$(1.2)\qquad 1\leq q\leq \frac{np}{n-p}\quad \textit{if}\quad n>p\quad \textit{and}\quad 1\leq q<\infty\quad \textit{if}\quad n\leq p\,.$$

Let p be a real number $\geqq 2$ and α a nonnegative real number.

Our hypotheses are:

H.1. $\beta(x, u)$ is Lipshitz continuous in u for almost all $x\in\Omega$ and satisfies

$$(1.3)\qquad |\beta(x, u)|\leq K_0|u|^{\alpha+1}\quad \text{for}\quad \forall\ u\in R,\ a.e. x\in\Omega\,,$$

where K_0 is a positive constant and

$$(1.4)\qquad 2\leq\alpha+2<\frac{np}{n-p}\quad \text{if}\quad n>p\quad \text{and}\quad 0\leq\alpha<\infty\quad \text{if}\quad 1\leq n\leq p\,.$$

H.2. $f\in C(R:\ L^2(\Omega))$ and $M=\sup_{t\in R}\left(\int_t^{t+1}\|f(s)\|_2^2 ds\right)^{1/2}<\infty\,.$

Definition 1. $u(x, t)$ is said to be a bounded solution of (0.1)-(0.2) with initial value $\{u_0(x), u_1(x)\}$ if $u\in L^\infty(R^+:\ W_0^{1,p}(\Omega))$,

$u'\in L^\infty(R^+:\ L^2(\Omega))\cap L^2_{loc}(R^+:\ W_0^{1,2}(\Omega))$ and for $\forall\varphi\in C_0^1(R^+:\ W_0^{1,p}(\Omega))$ the variational equation

$$(1.5)\qquad \int_0^\infty -(u'(t),\ \varphi'(t))dt+\int_0^\infty\left(\sum_{i=1}^n |u_{x_i}|^{p-2}u_{x_i},\ \varphi_{x_i}\right)dt+\int_0^\infty (u'_{x_i},\ \varphi_{x_i})dt$$
$$+\int_0^\infty (\beta(,u),\ \varphi)dt=\int_0^\infty (f,\ \varphi)dt+(u_1,\ \varphi(x, 0))$$

is valid, where $R^+=[0, \infty)$ and $(u, v)=\int_\Omega u(x)v(x)dx$.[1]

2. Approximate Solutions

In this section, we shall prove an existence theorem concerning approximate solutions.

For this purpose, we employ the Galerkin's approximation procedure.

Let $\{w_j\}, j=1, 2, \ldots\ldots, m, \ldots$ be the basis of $W_0^{1,p}(\Omega)$ and consider the system of ordinary differential equations:

$$(2.1)\qquad (u''_m(t),\ w_j)+<Au_m(t),\ w_j>+((u'_m(t),\ w_j))+(\beta(.,u_m),\ w_j)$$
$$=(f(t),\ w_j),\ j=1, 2, \ldots\ldots, m\,.$$

with initial values

1) For the sake of simplicity, we denote by the symbol' the differentiation with respect to t.

(2.2) $u_m(0) = \sum_{j=1}^{m} \lambda_{m,j}(0)w_j \to u_0$ strongly in $W_0^{1,p}$

(2.3) $u'_m(0) = \sum_{j=1}^{m} \lambda'_{m,j}(0)w_j \to u_1$ strongly in L^2,

where $u_m(x, t) = \sum_{j=1}^{m} \lambda_{m,j}(t)w_j(x)$, $((u, v)) = \int_\Omega \nabla u(x)\cdot\nabla v(x)\ dx$ and A is the operator from $W_0^{1,p}(\Omega)$, to $W^{-1,p/p-1}(\Omega)$ defined by

$$<Au, v> = \int_\Omega \sum_{i=1}^{n} |u_{x_i}|^{p-2}u_{x_i}v_{x_i}\ dx \quad \text{for any } u,\ v \in W_0^{1,p}(\Omega)\ .$$

The solution $\{\lambda_{m,j}(t)\}_{j=1}^{m}$, and consequently $u_m(t)$ exists on an interval, say, $[0, t_m]$ by the standard theory of ordinary differential equations. We must give a priori estimates for approximate solutions $u_m(x, t)$. First we shall consider the case where $p<\alpha+2$.

Now we introduce some functionals on $W_0^{1,p}(\Omega)$ as follows;

(2.4) $J_0(u) = (1/p)\|\nabla u\|_p^p + \int_\Omega \int_0^{u(x,t)} \beta(x, s)\ dsdx$

(2.5) $\tilde{J}_0(u) = (1/p)\|\nabla u\|_p^p - (K_0 S_{\alpha+2,p}^{\alpha+2}/\alpha+2)\|\nabla u\|_p^{\alpha+2}$

(2.6) $J_1(u) = \|\nabla u\|_p^p + \int_\Omega \beta(x, u)u\ dx$

(2.7) $\tilde{J}_1(u) = \|\nabla u\|_p^p - K_0 S_{\alpha+2,p}^{\alpha+2}\|\nabla u\|_p^{\alpha+2}$.

By the assumption (1.3) and Lemma 1. 1, we have

(2.8) $\tilde{J}_0(u) \leqq J_0(u)$ and $\tilde{J}_1(u) \leqq J_1(u)$ for $\forall\ u \in W_0^{1,p}(\Omega)$.

Also we put

(2.9) $D_1 = \operatorname{Max}_{x\geq 0} \{x^p - K_0 S_{\alpha+2,p}^{\alpha+2} x^{\alpha+2}\} = x_0^p - K_0 S_{\alpha+2,p}^{\alpha+2} x_0^{\alpha+2}$

where $x_0 = (p/K_0 S_{\alpha+2,p}^{\alpha+2}(2+\alpha))^{1/\alpha+2-p}$,

and put

(2.10) $D_0 = (1/p)x_0^p - (K_0 S_{\alpha+2,p}^{\alpha+2}/\alpha+2)x_0^{\alpha+2}$.

We define the stable set W by

(2.11) $$W \equiv \left\{(u_0, u_1) \in W_0^{1,p}\times L^2;\ \|\nabla u_0\|_p < x_0,\ \|(u_0, u_1)\|_W \equiv \frac{1}{2}\|u_1\|_2^2 + J_0(u_0) < D_0\right\}.$$

LEMMA. 2.1. *In addition to the hypotheses* H. 1 *and* H. 2, *we assume that* $(u_0, u_1) \in W$ *and* $p<\alpha+2$. *Then there exists a positive number* $M_0=M_0(\|(u_0, u_1)\|_W)$

such that if $M < M_0$, *approximate solution* $u_m(t)$ *exists on* $[0, \infty)$ *and satisfy*

$$(2.12) \qquad \|u'_m(t)\|_2^2 \leq 2D_0 \text{ and } \|\nabla u_m(t)\|_p \leq x_0 \text{ for } \forall\ t \in [0, \infty)$$

and

$$(2.13) \qquad \int_T^{T+1} \|\nabla u'_m(t)\|_2^2 dt \leq D_0 + M\sqrt{2D_0} < \infty \quad \text{for } \forall\ T > 0,$$

where m is a sufficiently large positive integer.

PROOF. We shall show that $(u_m(t), u'_m(t))$ stays in the stable set W for all time as long as they exist. Then (2.12) will follow easily from the inequality

$$(2.14) \qquad \frac{1}{2}\|u'_m(t)\|_2^2 + \tilde{J}_0(u_m(t)) < D_0 < \operatorname*{Max}_{x \geq 0} \left\{\frac{1}{p}x^p - (K_0 S_{\alpha+2,p}^{\alpha+2}/+2)x^{\alpha+2}\right\}$$

Since $(u_0, u_1) \in W$, we have by the definition of W

$$(2.15) \qquad \mathcal{E}_0 = D_0 - \left(\frac{1}{2}\|u_1\|_2^2 + J_0(u_0)\right) > 0 .$$

Also since $(u_m(0), u'_m(0)) \to (u_0, u_1)$ in $W_0^{1,p} \times L^2$, we may assume

$$(2.16) \qquad \frac{1}{2}\|u'_m(0)\|_2^2 + J_0(u_m 0)) \leq D_0 - \mathcal{E}, \quad 0 < \forall\ \mathcal{E} < \mathcal{E}_0,$$

for large m.

Suppose that our assertion was false. There would then exist the smallest time $\bar{t} \in [0, t_m]$ at which

$$(2.17) \qquad \frac{1}{2}\|u'_m(\bar{t})\|_2^2 + J_0(u_m(\bar{t})) = D_0 .$$

We shall derive a contradiction. First of all we note that

$$(2.18) \qquad \tilde{J}_0(u_m(t)) \leq J_0(u_m(t)) \leq D_0 \quad \text{for } t \in [0, \bar{t}],$$

and the continuity of $\tilde{J}_0(u_m)$, with respect to $\|\nabla u_m\|_p$, implies

$$(2.19) \qquad \|\nabla u_m(t)\|_p \leq x_0 \qquad \text{for } t \in [0, \bar{t}] .$$

Now multipling (2.1) by $\lambda'_{m,j}(t)$, summing over j from 1 to m and integrating over $[a, b] \subset [0, t_m]$, we obtain

$$(2.20) \qquad \frac{1}{2}\|u'_m(b)\|_2^2 + J_0(u_m(b)) + \int_a^b \|\nabla u'_m(t)\|_2^2 dt$$
$$= \frac{1}{2}\|u'_m(a)\|_2^2 + J_0(u_m(a)) + \int_a^b (f(t), u'_m(t)) dt .$$

In the above we take $b = \bar{t}$ and $a = 0$ to get by (2.16)

$$(2.21) \qquad \mathcal{E} + \int_0^{\bar{t}} \|\nabla u'_m(t)\|_2^2 dt \leq S_{2,2} \int_0^{\bar{t}} \|f(t)\|_2 \|\nabla u'_m(t)\|_2 dt .$$

Hence we have

$$\mathcal{E}+\int_0^{\bar t}\|\nabla u'_m(t)\|_2^2dt\leqq\int_0^{\bar t}\|\nabla u'_m(t)\|_2^2dt+\frac{S_{2,2}^2}{4}\int_0^{\bar t}\|f(t)\|_2^2dt$$

or

$$\int_0^{\bar t}\|f(t)\|_2^2dt\geqq\frac{4}{S_{2,2}^2}\mathcal{E}\ . \tag{2.22}$$

Now let us assume

$$M<M_1=\sqrt{(4/S_{2,2}^2)\,\mathcal{E}_0}\ . \tag{2.23}$$

Then from (2.22) and (2.23), it is easily seen that $\bar t>1$ for large m. Thus we can take $b=\bar t$ and $a=\bar t-1$ in (2.20), and we have

$$\int_{\bar t-1}^{\bar t}\|\nabla u'_m(t)\|_2^2dt\leqq\int_{\bar t-1}^{\bar t}|(f(t),u'_m(t))|\,dt \tag{2.24}$$

and hence

$$\int_{\bar t-1}^{\bar t}\|\nabla u'_m(t)\|_2^2dt\leqq S_{2,2}^2M^2\ ,\ \int_{\bar t-1}^{\bar t}\|u'_m(t)\|_2^2dt\leqq S_{2,2}^4M^2\ . \tag{2.25}$$

Therefore there exist numbers $\bar t_1\in[\bar t-1,\ \bar t-\frac{3}{4}]$ and $\bar t_2\in[\bar t-\frac{1}{4},\ \bar t]$ such that

$$\|u'_m(\bar t_i)\|_2\leqq 2S_{2,2}^2\ M\ ,\quad i=1,2\ . \tag{2.26}$$

Next, multipling (2.1) by $\lambda_{m,j}(t)$, summing over j and integrating over $[\bar t_1,\bar t_2]$, we obtain

$$\int_{\bar t_1}^{\bar t_2}J_1(u_m(t))dt\leqq|(u'_m(t_1),\ u_m(t_1))|+|(u'_m(t_2),\ u_m(t_2))|+$$
$$+\int_{\bar t_1}^{\bar t_2}\{\|u'_m(t)\|_2^2+|(\nabla u'_m(t),\ \nabla u_m(t))|+|(f(t),\ u_m(t)|\}\ dt$$

and by virtue of (2.25), (2.26) and (2.19),

$$\leqq 4S_{2,2}^2M\ \underset{[t-1,t]}{\mathrm{Max}}\ \|u_m(t)\|^2+S_{2,2}^4M^2+S_{2,2}M(\mathrm{mes}\ \Omega)^{p-2/2p}x_0+M\ \underset{[t-1,t]}{\mathrm{Max}}\ \|u_m(t)\|_2$$
$$\leqq S_{2,2}^4M^2+x_0\{4S_{2,2}^2MS_{2,p}+S_{2,2}M(\mathrm{mes}\ \Omega)^{p-2/2p}+MS_{2,p}\}$$
$$\equiv C_1(M)+x_0C_2(M)\ . \tag{2.27}$$

On the other hand,

$$\int_{\bar t_1}^{\bar t_2}J_1(u_m(t))dt\geqq\int_{\bar t_1}^{\bar t_2}\tilde J_1(u_m(t))dt$$
$$=\int_{\bar t_1}^{\bar t_2}\|\nabla u_m(t)\|_p^p(1-K_0S_{\alpha+2,p}^{\alpha+2}\|\nabla u_m(t)\|_p^{\alpha+2-p})dt$$

by (2.19) and the definition of x_0

$$\geqq (1-(p/\alpha+2))\int_{\bar{t}_1}^{\bar{t}_2} \|\nabla u_m(t)\|_p^p dt .$$

From (2.27) and (2.28), we have

$$\int_{\bar{t}_1}^{\bar{t}_2} \|\nabla u_m(t)\|_p^p dt \leqq \frac{\alpha+2}{\alpha+2-p}\int_{\bar{t}_1}^{\bar{t}_2} J_1(u_m(t))dt$$

$$\leqq \frac{\alpha+2}{\alpha+2-p}\{C_1(M)+x_0C_2(M)\} . \tag{2.29}$$

The inequality (2.29) together with (2.25) implies that there exists a point $t^* \in [\bar{t}_1, \bar{t}_2]$ such that

$$\|u'_m(t^*)\|_2^2+\|\nabla u_m(t^*)\|_p^p \leqq 2C_3(M) , \tag{2.30}$$

where $C_3(M) \equiv S_{2,2}^4 M^2+(\alpha+2/\alpha+2-p)(C_1(M)+x_0C_2(M))$.

In (2.20) we take $b=\bar{t}$ and $a=t^*$ to get by (2.30)

$$\frac{1}{2}\|u'_m(\bar{t})\|_2^2+J_0(u_m(\bar{t}))$$

$$\leqq \frac{1}{2}\|u_m(t^*)\|_2^2+J_0(u_m(t^*))+\int_{\bar{t}-1}^{\bar{t}} |(f(t), u'_m(t))|dt$$

$$\leqq C_4(M) \equiv C_3(M)+\frac{K_0}{\alpha+2}S_{\alpha+2,p}^{\alpha+2}(2C_3(M))^{2+\alpha/p}+S_{2,2}^2M^2$$

This contradicts (2.17) if we choose $M<M_2$, M_2 being the smallest number satisfying $C_4(M)=D_0$.

Thus if we take $M_0=\mathrm{Min}(M_1, M_2)$, the proof of (2.12) is completed. Furthermore if we take $b=T+1$ and $a=T$ in (2.20), then we have

$$\int_T^{T+1} \|\nabla u'_m(t)\|_2^2 dt \leqq \frac{1}{2}\|u'_m(T)\|_2^2+J_0(u_m(T))+\int_T^{T+1} |(f(t), u'_m(t))|dt$$

$$\leqq D_0+M\left(\int_T^{T+1}\|u'_m(t)\|_2^2 dt\right)^{1/2} ,$$

and hence $\int_T^{T+1} \|\nabla u'_m(t)\|_2^2 dt \leqq D_0+M\sqrt{2D_0}$,

where T is any positive number.

Next we shall consider the case where $p>\alpha+2$.

LEMMA 2.2. *In addition to hypotheses* H.1 *and* H.2., *we assume that* $p>\alpha+2$. *Then approximate solutions of the problem* (2.1)-(2.3) *exist on* $[0, \infty)$ *and satisfy*

(2.12)′ $\|u'_m(t)\|_2^2+\|\nabla u_m(t)\|_p^p \leqq C(u_0, u_1, M)$ *for* $\forall$ $t \in [0, \infty)$,

(2.13)′ $\int_T^{T+1} \|\nabla u'_m(t)\|_2^2 dt \leqq C(u_0, u_1, M, T)$ *for* $\forall$ $T>0$ and $\forall m$.

PROOF. First we can obtain the estimate

$$\frac{1}{2}\|u'_m(t)\|_2^2+J_0(u_m(t))\leqq C(u_0, u_1,M) \quad \text{for } t\in[0,\infty] \text{ and large } m.$$

This is verified by the similar way of the Lemma 2.1.
Next, using Lemma 1.1 and Young's inequality, we have

$$\frac{1}{2}\left(\|u'_m(t)\|_2^2+\frac{1}{p}\|\nabla u_m(t)\|_p^p\right)\leqq C(u_0, u_1, M),$$

which implies (2.12)′.

3. Passage to the Limit

By the passage to the limit of approximate solutions, we shall obtain;

THEOREM 3.1. *In addition to the hypotheses* H. 1 *and* H. 2, *we assume that* $\alpha+2>p$, $(u_0, u_1)\in W$ *and* $M<M_0$. *Then the problem* (0.1)-(0.2) *admits a bounded solution* u *with initial data* (u_0, u_1) *in the sense of Definition* 1. *Moreover the estimates* (2.12) *and* (2.13) *hold for* $u_m=u$.

THEOREM 3.2. *In addition to the hypotheses* H.1 *and* H.2, *we assume that* $\alpha+2<p$ *and* $(u_0, u_1)\in W_0^{1,p}\times L^2$. *Then we have the same coclusion as Theorem* 3.1 *without any restrictions on* M *and* (u_0, u_1).

We shall verify Theorem 3.1 only, because Theorem 3.2 is verified in a similar way.

PROOF *of Theorem* 3.1. We construct the approximate solutions $\{u_m(t)\}$ by (2.1), (2.2) and (2.3). By the Lemma 2.1 we have

(3.1) $\|Au_m(t)\|_{W^{-1,p/p-1}}\leqq x_0^{p-1}$ for $\forall\, t\in[0,\infty)$.

Let $\{T_n\}$ be any positive sequence tending to ∞ as $n\to\infty$. Then by (2.12), (2.13) and (3.1), we can use the standard compactness arguments to extract a subsequence from $u_m(t)$, which will be denoted also by $u_m(t)$, satisfying;

(3.2) $u_m(t)\longrightarrow u(t)$ weakly star in $L^\infty(R^+: W_0^{1,p}(\Omega))$ and *a.e.* in $\Omega\times R^+$,

(3.3) $\beta\cdot,u_m(t))\longrightarrow\beta(\cdot,u(t))$ weakly star in $L^\infty(R^+: L^{\frac{\alpha+2}{\alpha+1}}(\Omega))$

(3.4) $u'_m(t)\longrightarrow u'(t)$ weakly star in $L^\infty(R^+: L^2(\Omega))$ and *a.e.* in $\Omega\times R^+$,

(3.5) $u'_m(t)\longrightarrow u'(t)$ weakly in $L^2_{\mathrm{loc}}(R^+: W_0^{1,2}(\Omega))$

(3.6) $Au_m\longrightarrow\chi$ weakly star in $L^\infty(R^+: W^{-1,p/p-1})$,

(3.7) $u_m(T_n)\longrightarrow u(T_n)$ in L^2 strongly

(3.8) $u'_m(T_n) \longrightarrow u'(T_n)$ in L^2 weakly

(3.9) $u_m(T_n) \longrightarrow u(T_n)$ in $W_0^{1,2}$ weakly .

To show that the function $u(t)$ is a solution, it is sufficient to prove that $\chi = Au$.

From (2.1), for every T_n, we have the identity

$$-\int_0^{T_n} (u'_m(t), \psi'(t))dt + \int_0^{T_n} <Au_m(t), \psi(t)> dt + \int_0^{T_n} ((u'_m(t), \psi(t)))dt +$$
$$+\int_0^{T_n} (\beta(\cdot, u_m), \psi(t))dt = \int_0^{T_n} (f(t), \psi(t))dt + (u'_m(0), \psi(0)) - (u'_m(T_n), \psi(T_n))$$

for all functions of the form $\psi = \sum_{k=1}^{N} d_k(t)w_k$, where $d_k(t)$ are smooth function on $[0, T_n]$.

Taking the limit as $m \to \infty$ in the above equation, we get

$$-\int_0^{T_n} (u'(t), \psi'(t))dt + \int_0^{T_n} <\chi, \psi(t)> dt + \int_0^{T_n} ((u'(t), \psi(t)))dt +$$
$$+\int_0^{T_n} (\beta(\cdot, u), \psi(t))dt = \int_0^{T_n} (f(t), \psi(t))dt + (u_1, \psi(0)) - (u'(T), \psi(T_n)) .$$

Thus we can replace ψ by u in the above equation to obtain

$$(3.10) \qquad -\int_0^{T_n} \|u'\|_2^2 dt + \int_0^{T_n} <\chi, u> dt + \frac{1}{2}\|\nabla u(T_n)\|_2^2 - \frac{1}{2}\|\nabla u_0\|_2^2 +$$
$$+\int_0^{T_n} (\beta(\cdot, u), u)dt = \int_0^{T_n} (f(t), u(t))\, dt + (u_0, u_1) - (u'(T_n), u(T_n)) .$$

Also we have

$$X_m = \int_0^{T_n} <A(u_m) - A(v), u_m - v> dt \geqq 0, \qquad v \in L^2((0, T_n): W_0^{1,p}(\Omega)) ,$$

and, replacing w_j by u_m in (2.1),

$$X_m = \int_0^{T_n} \|u'_m(t)\|_2^2 dt - \frac{1}{2}\|\nabla u_m(T_n)\|_2^2 + \frac{1}{2}\|\nabla u_m(0)\|_2^2 + \int_0^{T_n} (\beta(\cdot, u_m), u_m)dt$$
$$+\int_0^{T_n} (f(t), u_m(t))dt + (u'_m(0), u_m(0)) - (u'_m(T_n), u_m(T_n)) -$$
$$-\int_0^{T_n} <Av, u_m - v> dt - \int_0^{T_n} <A(u_m), v> dt .$$

Taking the limit as $m \to \infty$ in the above equation, we get

$$(3.11) \qquad \varliminf_{m\to\infty} X_m \leqq \int_0^{T_n} \|u'(t)\|_2^2\, dt - \frac{1}{2}\|\nabla u(T_n)\|_2^2 + \frac{1}{2}\|\nabla u_0\|_2^2 +$$

$$+\int_0^{T_n}(\beta(\cdot,u),u)dt+\int_0^{T_n}(f(t),u(t))dt=(u_1,u_0)-$$

$$-(u'(T_n),u(T_n))-\int_0^{T_n}<Av,u-v>dt-\int_0^{T_n}<\chi,v>dt.$$

From (3.10) and (3.11), we have

$$\int_0^{T_n}<\chi-Av,u-v>dt\geqq 0 \qquad \text{for } v\in L^2((0,T_n):W_0^{1,p})$$

which gives $\chi=Au$.

The limit function $u(x,t)$ satisfies evidently the equality (1.5), and the estimates (2.12), (2.13) and (3.1) hold for u.

Remark 1. We note that $\beta(x,u)$ is a monotonic in u, then the Theorem 3.1 holds for arbitrary f satisfying only the hypothesis H. 2.

Remark 2. If we choose the sequence w_j as the basis of $W_0^{s,2}(\Omega)$, $\left(s>n\left(\frac{1}{2}-\frac{1}{p}\right)+1\right)$, we can easily obtain a priori estimate of u_m'' in $||u_m''||$ $L^2_{loc}(R: W^{-s,2})$ and consequently we have slightly better soltuions. (see [6]).

Remark 3. Under additional appropriate condition, we can obtain the Theorems 3.1 for more general equation;

$$u_{tt}-\sum_{i=1}^n\{a_i(x,u_{xi})\}_{xi}-\Delta u_t+\beta(x,u,u_t)=f. \quad ([1],[3]).$$

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