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FINITE AND SIGMA-FINITE INVARIANT MEASURES OF NON-SINGULAR TRANSFORMATIONS*

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§ 1. Introduction.

In the theory of classical dynamical system we know the Liouville's theorem that Hamiltonian systems have an invariant measure. The main part of recent ergodic theory is to investigate the metrical and the spectral properties of dynamical systems with a finite invariant measure. In 1932 E. Hopf [13] formulated the following problem; Let T be an invertible null-measure preserving (we say, nonsingular) transformation of a σ-finite measure space $(\Omega, \mathscr{F}, P)$. When does there exist a finite invariant measure which is equivalent to a given measure P? Necessary and sufficient conditions for the existence of a finite invariant equivalent measure have been given by many authors [2], [5], [6], [8], [9], [13], [16], [20]. One due to E. Hopf [13] is that the space Ω is T-bounded and another one due to A. Hajian and S. Kakutani [9] is that there does not exist a weakly wandering set. The meaning of the assumption that a non-singular transformation preserves a finite measure was made clear by their works. It means, roughly speeking, the strong recurrence and the ergodic convergence of the transformation. On the other hand necessary and sufficient conditions for the existence of a σ-finite invariant equivalent measure were given by P. Halmos [10] and L. Arnold [1]. In 1960 D. Ornstein [21] gave an example of an ergodic non-singular transformation without a σ-finite invariant equivalent measure.

We see that among ergodic non-singular transformations there are three classes; The first is the class of ergodic transformations with a finite invariant measure. The second is the class of ergodic transformations with a σ-finite, infinite invariant measure. The third is the class of ergodic non-singular transformations without a σ-finite invariant equivalent measure. Recently more detailed classifications of the third class have been given by W. Krieger [15] and Hamachi-Oka-Osikawa [12]. They are closely related to the classification of factors in the theory of

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von Neumann [3], [23].

In this thesis we discuss the existence problem of finite and σ-finite invariant equivalent measures of non-singular transformations from a new point of view of classification of non-singular transformations. For this purpose we introduce the notion of T-equivalence among measurable sets and a one-parameter flow associated with a non-singular transformation. They play important roles in our discussion.

In section 2 we introduce fundamental terminologies about non-singular transformations and non-singular flows. In section 3 we construct a finite invariant equivalent measure under the Hopf's condition that Ω is T-bounded using the T-equivalence relation among measurable sets. Our method of construction of a finite invariant equivalent measure is more natural and simple than the Hopf's one In section 4 we prove directly the equivalence of the Hopf's condition and the Hajian-Kakutani's condition (non-existence of weakly wandering set). It was an open problem. In section 5 we define a flow associated with a non-singulai transformation. The associated flow is a nice invariant of non-singular transformations under weak equivalence and gives many informations about them. The weakly equivalent types of non-singular transformations are determined by the metrical types of their associated flows. We show that a non-singular transformation has a σ-finite invariant equivalent measure if and only if its associated flow is of translation type. As its corollary we obtain the L. Arnold's condition [1] which has been most useful for the existence of σ-finite invariant equivalent measure. In the last section we show that a non-singular flow whose point spectrum set is the set of all real numbers is of translation type. And we apply it to obtain a new existence condition of a σ-finite invariant equivalent measure. This method of the associated flow is related to the treatment of unbounded vectors in the Tomita's theory of generalized Hilbert algebra and makes a beginning of classification of ergodic transformations with a σ-finite infinite invariant measure.

§ 2. Non-singular transformations and non-singular flows

By a measure space we mean a triple $(\Omega, \mathscr{F}, P)$ where $\mathscr{F}$ is a σ-field of subsets of an abstract set Ω and P is a countably additive non-negative set function defined on $\mathscr{F}$. A set A is called measurable if $A \varepsilon \mathscr{F}$ and a real valued function $f(\omega)$ defined on Ω is called measurable if for any real number a, $\{\omega | f(\omega) < a\} \varepsilon \mathscr{F}$ and a complex valued function is called measurable if its real part and imaginary part are measurable. We use the notations $A=B$ *a.e.* when $P((A\Delta B))=0$ and $f(\omega)=g(\omega)$ *a.e.* ω when $P(\omega|f(\omega)\neq g(\omega))=0$. A measure P is called finite if $P(\Omega)<+\infty$ and is called σ-finite if there exist countable subsets Ω_n such that $P(\Omega_n)<+\infty$, $n=1,2,\ldots\ldots$ and $\Omega=\bigcup_{n=1}^{\infty}\Omega_n$ *a.e.*.... If μ is a second measure defined on $\mathscr{F}$, then μ is said to be mutually equivalent to P, $\mu\sim P$, when $\mu(A)=0$ if and only if $P(A)=0$. We denote by $\frac{d\mu}{dP}(\omega)$ a Radon-Nikodym derivative of μ with respect to P.

Let $(\Omega, \mathscr{F}, P)$ be a σ-finite measure space. A 1-1 transformation T of Ω onto itself is said to be bi-measurable if $T\mathscr{F}=\mathscr{F}$ *i.e.* $A\in\mathscr{F}$ implies $TA\in\mathscr{F}$ and $T^{-1}A\in\mathscr{F}$. We denote by PT a measure such that $PT(A)=P(TA)$, $A\in\mathscr{F}$. A bi-measurable 1-1 transformation T is said to be non-singular if $PT\sim P$ *i.e.* $P(A)=0$ implies $P(TA)=P(T^{-1}A)=0$. A measure μ is said to be T-invariant if $\mu(TA)=\mu(A)$ for each $A\in\mathscr{F}$. A measurable subset C is T-invariant if $TC=C$ *a.e.* and a measurable function $f(\omega)$ is T-invariant ir $f(T\omega)=f(\omega)$ *a.e.* ω. A non-singular transformation T is said to be ergodic if every T-invariant function is a constant *a.e.* ω. T is said to be conservative if for any measurable set A, $P\left(A-\bigcup_{n=1}^{\infty}T^{-n}A\right)=0$.

Let T be a conservative non-singular transformation and A be a measurable set such that $P(A)>0$. Putting $A_1=A\cup T^{-1}A$ and $A_n=A\cap T^{-n}A-\bigcup_{i=1}^{n-1}T^{-i}A$, $n=2,3,\ldots\ldots$, we have $A=\sum_{n=1}^{\infty}A_n$ *a.e.*. A transformation T_A defined by $T_A\omega=T^n\omega$, $\omega\in A_n$, $n=1,2,\ldots\ldots$ is non-singular and conservative, and is called the induced transformation of T on A.

E. Hopf gave the follwing problems.

Let T be a non-singular transformation of a measure space $(\Omega, \mathscr{F}, P)$.

(I) Find necessary and sufficient conditions for the existence of a finite T-ınvariant measure which is equivalent to P.

(II) Find necessary and sufficient conditions for the existence of a σ-finite T-invariant measure which is equivalent to P.

If T is ergodic there exists a unique T-invariant measure equivalent to P except a constant multiple. Indeed, if μ and μ' are T-invariant measure equivalent to P the Radon- Nikodym derivative $\frac{d\mu}{d\mu'}(\omega)$ is a T-invariant function. Hence, there exists a constant c such as $\mu=c\mu'$ if T is ergodic.

We denote by (T) the set of all mappings S such that the domain $D(S)$ and the range $R(S)$ of S are measurable subsets of Ω with $P(D(S))>0$ and $P(R(S))>0$ and there exist a countable partition $\{A_i|i=1,2,\ldots\ldots\}$ of $D(S)$ and a sequence $\{n_i|i=1,2,\ldots\ldots\}$ of integers with $S\omega=T^{n_i}\omega$ for $\omega\in A_i$, $i=1,2,\ldots\ldots$. For $S\in(T)$ and $S'\in(T)$ with $P(R(S)\cap D(S'))>0$, the composed mapping $S'\circ S$ ($S'\circ S\omega = S'(S\omega)$) belongs to (T) with $D(S'\circ S) = S^{-1}(R(S) \cap D(S'))$ and $R(S'\circ S) = S'(R(S) \cup D(S'))$. And for $S \varepsilon (T)$ the inverse mapping S^{-1} belongs to (T) with $D(S^{-1}) = R(S)$ and $R(S^{-1}) = D(S)$.

We denote by $[T]$ the set of all $S\varepsilon(T)$ such that $D(S) = R(S) = \Omega$. $[T]$ is a group of non-singular transformations of $(\Omega, \mathscr{F}, P)$ and is called the full group of T.

Two non-singular transformations T and T' of measure spaces $(\Omega, \mathscr{F}, P)$ and $(\Omega', \mathscr{F}', P')$, respectively, are mutually weakly equivalent if there exists a 1-1 mapping ψ from Ω onto Ω' such that $\psi\mathscr{F} = \mathscr{F}'$, $P \sim P'\psi$ and $\psi[T] = [T']\psi$ where $P'\psi(A) = P'(\psi A)$, $A \varepsilon \mathscr{F}$. It is easy to see that a measure μ is T-invariant if and only if it is S-invariant for any $S \varepsilon (T)$. If T and T' are mutually weakly equivalent there exists a finite (or σ-finite) T-invariant measure equivalent to P according as there exists a finite (or σ-finite) T'-invariant measure equivalent to P'.

A one-parameter group $\{\phi_s\}_{-\infty<s<+\infty}$ of non-singular transformations of a measure space $(X, \mathscr{B}, m)$ is called a non-singular flow. A non-singular flow $\{\phi_s\}_{-\infty<s<+\infty}$ of $(X, \mathscr{B}, m)$ is said to be measurable if the mapping $X \times R \ni (x, s) \to \phi_s x \varepsilon X$ is measurable. Two non-singular flows $\{\phi_s\}_{-\infty<s<+\infty}$ and $\{\phi'_s\}_{-\infty<s<+\infty}$ of measure spaces $(X, \mathscr{B}, m)$ and $(X', \mathscr{B}', m')$, respectively, are mutually strongly equivalent if there exists a 1-1 mapping ψ from X onto X' such that $\psi\mathscr{B} = \mathscr{B}'$, $m \sim m'\psi$ and for $-\infty < s < +\infty$, $\psi\phi_s x = \phi'_s \psi x$, *a.e.* $x \in X$. A non-singular flow $\{\phi_s\}_{-\infty<s\wedge+\infty}$ is said to be ergodic if every function which is ϕ_s-invariant for all $-\infty < s < +\infty$ is a constant *a.e.* A real number t is said to be a point spectrum of $\{\phi_s\}_{-\infty<s<+\infty}$ if there exists a measurablefunction $\exp i\xi(x)$ such that for all $-\infty < s < +\infty$, $\exp i\xi(\phi_s x) = \exp its \cdot \exp i\xi(x)$ *a.e.* $x \varepsilon X$ and we denote by $\sigma(\{\phi_s\})$ the set of all point spectra t of $\{\phi_s\}_{-\infty<s<+\infty}$.

DEFINITION 1. Let $(X, \mathscr{B}, m)$ be a measure space. A flow $\{\phi_s\}_{-\infty<s<+\infty}$ of the product space $(X \times R, \mathscr{B} \times \mathscr{B}(R), m \times du)$ defined by $\phi_s(x, u) = (x, u+s)$, is called a flow of translation type.

§ 3. A construction of the finite invariant measure.

Let T be a non-singular transformation of a measure space $(\Omega, \mathscr{F}, P)$. A measurable set A is said to be countably T-equivalent to a measurable set B, $A \sim B$, if there exists a mapping $S \varepsilon (T)$ such that $D(S) = A$ and $R(S) = B$. We have the following properties.

(1) $A \sim A$.

(2) $A \sim B$ implies $B \sim A$.

(3) $A \sim B$ and $B \sim C$ imply $A \sim C$.

(4) For any sets A and B which are mutually T-equivalent, and for any subset A' of A, there exists a subset B' of B such that $A' \sim B'$ and $A - A' \sim B - B'$.

(5) $A = \sum_{i=1}^{\infty} A_i$, $B = \sum_{i=1}^{\infty} B_i$ and $A_i \sim B_i$ for $i=1, 2, \ldots\ldots$, imply $A \sim B$.

(6) $A \sim B$ and $TC = C$ imply $A \cap C \sim B \cap C$.

A set A is said to be T-unbounded if there exists a subset A' of A such that $A' \sim A$ and $A' \neq A$. A set is said to be T-bounded if it is not T-unbounded. For a pair of sets A, B we use the notation $A < B$ if there exists a subset B' of B such that $A \sim B'$. We have the following properties.

(7) $A < A$.

(8) $A < B$ and $B < A$ imply $A \sim B$.

(9) $A < B$ and $B < C$ imply $A < C$.

(10) $A \subset B$ implies $A < B$.

(11) If a set A is T-unbounded and $A < B$ then B is T-unbounded.

A proof of property (8) is similar to the Bernstein's method in general theory of sets. Property (11) means that the equivalent relation $\sim$ preserves T-boundedness of measurable sets. The ordered relation $<$ is not totally ordered but we have the following lemma.

LEMMA 1. For a pair of T-bounded sets $\{A, B\}$ there uniquely exists a pair of subsets $\{C, D\}$ of $\bigcup_{n=-\infty}^{\infty} T^n A$ with the following properties.

1) Sets C, D are mutually disjoint and T-invariant.

2) $C \cup D = \bigcup_{n=-\infty}^{\infty} T^n A$.

3) For any T-invariant subset C' of C, $B \cap C' < A \cap C'$ and $B \cap C'$ is not T-equivalent to $A \cap C'$.

4) For any T-invariant subset D' of D, $A \cap D' < B \cap D'$.

PROOF. We inductively define sets as follows.

$$A^0 = A, \quad B^0 = B, \quad A_0 = A \cap B, \quad B_0 = A \cap B,$$

$$A^i = A^{i-1} - A_{i-1}, \qquad B^i = B^{i-1} - B_{i-1},$$

$$A_i = A^i \cap T^i B^i, \qquad B_i = T^{-i} A^i \cap B^i, \qquad i = 1, 2, \ldots\ldots,$$

$$A^{-1} = \bigcap_{i=0}^{\infty} A^i = A - \sum_{i=0}^{\infty} A_i, \quad B^{-1} = \bigcap_{i=0}^{\infty} B^i = B - \sum_{i=0}^{\infty} B_i,$$

$$A_{-1} = A^{-1} \cap T^{-1} B^{-1}, \quad B_{-1} = T A^{-1} \cap B^{-1},$$

$$A^{-i} = A^{-i+1} - A_{-i+1}, \quad B^{-i} = B^{-i+1} - B_{-i+1},$$

$$A_{-i} = A^{-i} \cap T^{-i} B^{-i}, \quad B_{-i} = T^i A^{-i} \cap B^{-i}, \qquad i = 1, 2, \ldots\ldots,$$

$$A^* = A - \sum_{i=-\infty}^{\infty} A_i, \quad B^* = B - \sum_{i=-\infty}^{\infty} B_i,$$

$$C = \bigcup_{n=-\infty}^{\infty} T^i A^*, \qquad D = \bigcup_{n=-\infty}^{\infty} T^i A - C.$$

Properties 1) and 2) are evident. For any non-vacant T-invariant subset C' of C let us assume $B^* \cap C' \neq \phi$. Since $B^* \cap C' = \bigcup_{n=-\infty}^{\infty} (B^* \cap T^n A^* \cap C')$ we have $T^n A^* \cap B^* \cap C' \neq \phi$ for some n and $T^i A^* \cap B^* \cap C' \subset T^i A^{-n} \cap B^{-n} = B_{-n}$. This contradicts to the definition of B^*, and so $B^* \cap C' = \phi$. Since a set $B \cap C' = \sum_{i=-\infty}^{\infty} B_i \cap C'$ is T-equivalent to a set $A \cap C' - A^* \cap C' = \sum_{i=-\infty}^{\infty} A_i \cap C'$ and $A^* \cap C' \neq \phi$ and since a set A is T-bounded, we have property 3). Since for any T-invariant Subset D' of D a set $A \cap D' = \sum_{i=-\infty}^{\infty} A_i \cap D'$ is countably T-equivalent to $\sum_{i=-\infty}^{\infty} B_i \cap D'$ which is a subset of $B \cap D'$, we have property 4). The uniqueness of the choice of a pair $\{C, D\}$ is obvious. q.e.d.

LEMMA 2. For a pair of T-bounded sets A, B there uniquely exists a countable sequence of T-invariant sets $\Omega_1, \Omega_2, \ldots\ldots, \Omega_n, \ldots\ldots$ which satisfy the following properties.

1) $\bigcup_{n=0}^{\infty} \Omega_n = \bigcup_{n=-\infty}^{\infty} T^n A$ and $\Omega_n \cap \Omega_m = \phi$, $n \neq m$, $n, m = 1, 2, \ldots\ldots$
2) There exist countable disjoint subsets $B_1, B_2, \ldots\ldots, B_i, \ldots\ldots$ of B such that $\Omega_n \cap B_i \sim \Omega_n \cap A$, $i = 1, 2, \ldots\ldots, n$; $n = 1, 2, \ldots\ldots$ and that for any non-vacant T-invariant subset Ω'_n of Ω_n, $B_n^* \cap \Omega'_n < A \cap \Omega'_n$ and $B_n^* \cap \Omega'_n$ is not countably T-equivalent to $A \cap \Omega'_n$ for $n = 1, 2, \ldots\ldots$, where $B_0^* = B$ and $B_n^* = B - \sum_{i=1}^{n} B_i$, $n = 1, 2, \ldots\ldots$.

Proof follows from Lemma 1.

For a pair of T-bounded sets A, B we define a function which has a value n on Ω_n defined in Lemma 2 and denote it by $[B/A](\omega)$, *i.e.*, $[B/A](\omega) = n$ for $\omega \varepsilon \Omega_n$, $n = 1, 2, \ldots\ldots$. $[B/A](\omega)$ is a T-invariant measurable function defined on $\bigcup_{n=-\infty}^{\infty} T^n A$ and has the following properties.

LEMMA 3. Let A, B and C be T-bounded sets.

1) $$[A/B](\omega) \cdot [B/C](\omega) \leqq [A/C](\omega) \leqq \{[A/B](\omega)+1\}\{[B/C](\omega)+1\}$$
$$\text{for } \omega \varepsilon \left(\bigcup_{n=-\infty}^{\infty} T^n B\right) \cap \left(\bigcup_{n=-\infty}^{\infty} T^n C\right).$$
2) If A and B are mutually disjoint,
$$[A/C](\omega) + [B/C](\omega) \leqq [A+B/C](\omega) \leqq [A/C](\omega) + [B/C](\omega) + 1,$$
$$\text{for } \omega \varepsilon \bigcup_{n=-\infty}^{\infty} T\, C.$$

3) If $A\sim B$, $[A/C](\omega)=[B/C](\omega)$, for $\omega\in\bigcup_{n=-\infty}^{\infty} T C$.

Proof is omitted.

LEMMA 4. Let T be a non-singular transformation of a measure space $(\Omega, \mathscr{F}, P)$. If T is conservative, then there uniquely exists a pair of T-invariant sets Π, A with the following properties.

1) The space Ω is a disjoint union of Π and A.
2) There exist countable sets $\Pi_1, \Pi_2, \ldots\ldots, \Pi_n, \ldots\ldots$ such that

$$\Pi=\sum_{n=1}^{\infty}\left(\sum_{i=0}^{n-1} T^i \Pi_n\right) \qquad \text{(disjoint sum)}.$$

3) There exist countable sets $A_1, A_2, \ldots\ldots, A_n, \ldots\ldots$ such that

$$A=\bigcup_{i=-\infty}^{\infty} T^i A_n, \qquad n=1,2,\ldots\ldots$$

and $\qquad [A_n/A_{n+1}](\omega)\geqq 2$, for $\omega\in A$.

PROOF. We put $\mathfrak{A}=\{D\in\mathscr{F} \mid P(TD\cap D)=0\}$ and $\sup_{D\in\mathfrak{A}} P\left(\bigcup_{n=-\infty}^{\infty} T D\right)=p$. We may assume $p>0$, and then there exists a sequence of sets $D_1, D_2, \ldots\ldots$ such that $D_n\in\mathfrak{A}$ and $P\left(\bigcup_{i=-\infty}^{\infty} T^i D_n\right)>p-\frac{1}{n}$, $n=1,2,\ldots\ldots$. Put $D=\bigcup_{n=1}^{\infty}\left(D_n-\bigcup_{j=1}^{n-1}\bigcup_{i=-\infty}^{\infty} T^i D_j\right)$, then $D\in\mathfrak{A}$ and $P\left(\bigcup_{n=-\infty}^{\infty} T D\right)=0$. Put $B_1=D$, $C_1=\bigcup_{n=-\infty}^{\infty} T D$ and $Q_1=\Omega-C_1$, then $[C_1/B_1](\omega)\geqq 2$ for $\omega\in C_1$ and $TQ'_1=Q'_1$ for any subset Q'_1 of Q_1. For $n=2,3,\ldots\ldots$ applying the above discussion to an induced transformation $T_{B_{n-1}}$ on B_{n-1} we obtain sets B_n, $C_n=\bigcup_{i=-\infty}^{\infty} T^i_{B_{n-1}} B_n$ and $Q_n=B_{n-1}-C_n$ such that $T_{B_{n-1}}B_n\cap B_n=\phi$ and $T_{B_{n-1}} Q'_n=Q'_n$ for every subsets Q'_n of Q_n. Therefore, $[C_n/B_n](\omega)\geqq 2$ for $\omega\in\bigcup_{i=-\infty}^{\infty} T^i B_n$, $n=2,3,\ldots\ldots$ and there exist countable disjoint sets $Q_{n1}, Q_{n2}, \ldots\ldots$ with the following properties.

$$Q_n=\sum_{i=1}^{\infty} Q_{ni}, \qquad n=2,3,\ldots\ldots.$$

$T Q_{ni}$, $j=0,1,2,\ldots\ldots, i\text{-}1$ are mutually disjoint, for $i=1,2,\ldots\ldots$ and $n=2,3,\ldots\ldots$.

$T Q'_{ni}=Q'_{ni}$ for every subset Q'_{ni} of Q_{ni}.

Put $\Pi=\bigcup_{n=1}^{\infty}\bigcap_{i=-\infty}^{\infty} T^i Q_n$, $\Pi_1=Q_1+Q_{21}+Q_{31}+\ldots\ldots$ and $\Pi_n=Q_{2n}+Q_{3n}+\ldots\ldots$, $n=2,3,\ldots\ldots$, then property 2) is satisfied. Put $A=\bigcap_{n=1}^{\infty}\bigcup_{i=-\infty}^{\infty} T B_n$ and $A_n=B_n\cap\left(\bigcap_{i=1}^{\infty} B_i\right)$, then properties 1) and 3) are satisfied. q.e.d.

Sets Π and A are said to be the periodic part and non-periodic part, respectively.

PROPOSITION 1. (E. Hopf) Let T be a non-singular transformation of a σ-finite measure space $(\Omega, \mathscr{F}, P)$. There exists a finite T-invariant measure μ which is equivalent to P if and only if Ω is T-bounded.

PROOF. It is easy to show that if Ω is T-unbounded there does not exist a finite T-invariant measure equivalent to P.

Assume that Ω is T-bounded, then T is conservative. We construct a finite T-invariant measure μ on the periodic part Π as follow;

$$\mu(E) = \frac{P(T^{-i}E)}{n\cdot P(\Pi_n)}, \text{ for a measurable subset } E \text{ of}$$

$$T^i\Pi_n,\ i = 0, 1, 2, \ldots\ldots,\quad n = 1, 2, \ldots\ldots .$$

From Lemma 3.1), we have

$$\frac{[E/A_k](\omega)}{[A/A_k](\omega)} \leqq \frac{\{[E/A_h](\omega)+1\}\{[A_h/A_k](\omega)+1\}}{[A/A_h](\omega)\cdot[A_h/A_k](\omega)}$$

$$\text{for } h \leqq k,\ E \subset A \text{ and } \omega \in A.$$

Lemma 3.1) and Lemma 4.3) we have $[A_h/A_k](\omega) \geqq 2^{k-h}$.
Then first making $k \uparrow \infty$, and making $h \uparrow \infty$, we have

$$\varlimsup_{k\to\infty} \frac{[E/A_k](\omega)}{[A/A_k](\omega)} \leqq \varliminf_{h\to\infty} \frac{[E/A_h](\omega)}{[A/A_h](\omega)} .$$

Put $f_E(\omega) = \lim_{k\to\infty} \frac{[E/A_k](\omega)}{[A/A_k](\omega)}$, $E \subset A$, $E \in \mathscr{F}$, then the function $f_E(\omega)$ is measurable and $0 \leqq f_E(\omega) \leqq 1$. We can assume P is a finite measure. We define a finite set function μ as $\mu(E) = \int f_E(\omega)dp(\omega)$ and show that μ is σ-additive, T-invariant and equivalent to P. For any measurable set E with $P(E) > 0$ and for any positive integer n, put $C_n = \left\{\omega \in \bigcup_{i=-\infty}^{\infty} T^iE \,\middle|\, [A/E](\omega) \leqq n\right\}$. From $\{[A/E](\omega)+1\}\{[E/A_k](\omega)+1\} \geqq [A/A_k](\omega) \geqq 2^{k-1}$, we have $[E/A_k](\omega) \geqq \frac{2^{k-1}}{n+1} - 1$, $\omega \in C_n$, $k = 1, 2, \ldots\ldots$. For any integer k such as $2^{k-1}/(n+1) \geqq 2$ and $h \geqq k$ we have

$$\frac{[E/A_h](\omega)}{[A/A_h](\omega)} \geqq \frac{[E/A_k](\omega)\cdot[A_k/A_h](\omega)}{\{[A/A_k](\omega)+1\}\{[A_k/A_h](\omega)+1\}}$$

$$\geqq \frac{1}{[A/A_k](\omega)+1} \times \frac{1}{2} > 0, \quad \omega \in C_n .$$

This implies that $f_E(\omega)$ is positive for $\omega \in \bigcup_{i=-\infty}^{\infty} T^iE$ and μ is equivalent to P.

It is easy to verify that $f_E(\omega) = f_F(\omega)$ for a pair of sets E, F which are mutually countably T-equivalent, and that if $f_E(\omega) \leqq f_F(\omega)$ then $E < F$. This implies T-invariance of μ.

From Lemma 3.2) we have $f_{E+F}(\omega) = f_E(\omega) + f_F(\omega)$ for any pair of mutually disjoint sets E, F. Hence, we have $\sum_{n=1}^{\infty} f_{E_n}(\omega) \leqq f_{\sum_{n=1}^{\infty} E_n}(\omega)$ for mutually disjoint, countable sets E_n, $n=1, 2, \ldots\ldots$. Assume that $P\left(\omega \middle| f_{\sum_{n=1}^{\infty} E_n}(\omega) - \sum_{n=1}^{\infty} f_{E_n}(\omega) > \varepsilon\right) > 0$ for some positive number ε. There exists integer h such that $0 < f_{A_h}(\omega) < \varepsilon$ for $\omega \in A$, because

$$\lim_{k\to\infty} \frac{[A_h/A_k](\omega)}{[A/A_k](\omega)} \leqq \overline{\lim_{k\to\infty}} \frac{[A_h/A_k](\omega)}{[A/A_h](\omega)\cdot[A_h/A_k](\omega)}$$

$$\leqq \frac{1}{2^{h-1}}.$$

Choose an integer n such that $P(D_n) > 0$ where $D_n = \left\{\omega \middle| \sum_{i=n+1}^{\infty} f_{E_i}(\omega) < f_{A_h}(\omega)\right.$ and $\left. f_{\sum_{i=1}^{\infty} E_i}(\omega) - \sum_{i=1}^{\infty} f_{E_i}(\omega) > \varepsilon\right\}$. We obtain a subset E'_{n+1} of $A_h \cap D_n$ such that $E'_{n+1} \sim E_{n+1} \cap D_n$, because $f_{E_{n+1}\cap D_n}(\omega) \leqq f_{A_h \cap D_n}(\omega)$. Inductively we obtain mutually disjoint subsets E'_{n+2}, $E'_{n+3}, \ldots\ldots$ of $A_h \cap D_n$ such that $E'_i \sim E_i \cap D_n$, $i = n+2$, $n+3, \ldots\ldots$. It follows that

$$\begin{aligned} f_{\sum_{i=1}^{\infty} E_i}(\omega) - \sum_{i=1}^{\infty} f_{E_i}(\omega) &= f_{\sum_{i=n+1}^{\infty} E_i}(\omega) - \sum_{i=n+1}^{\infty} f_{E_i}(\omega) \\ &\leqq f_{\sum_{i=n+1}^{\infty} E_i}(\omega) \\ &\leqq f_{\sum_{i=n+1}^{\infty} E'_i}(\omega) \\ &\leqq f_{A_h}(\omega) < \varepsilon, \quad \omega \in D_n. \end{aligned}$$

This is a contradiction. Hence, $f_{\sum_{i=1}^{\infty} E_i}(\omega) = \sum_{i=1}^{\infty} f_{E_i}(\omega)$ and μ is σ-additive. This completes the proof. *q.e.d.*

§ 4. Equivalence of the Hopf's condition and the Hajian-Kakutani's condition.

Let T be a non-singular transformation of a measure space $(\Omega, \mathscr{F}, P)$. A measurable set A, $P(A) > 0$ is said to be weakly wandering if there exists an increasing sequence of positive integers $\{n_1, n_2, \ldots\ldots, n_i, \ldots\ldots\}$ such that $T^{n_i}A \cap T^{n_j}A = \phi$, $i \neq j$, i, $j = 1, 2, \ldots\ldots$.

Hajian-Kakutani [9] showed that T has a finite invariant measure if and only if there does not exist a weakly wandering set. In this section we directly prove the equivalence of the Hopf's condition (boundedness) and the Hajian-Kakutani's condition (non-existence of weakly wandering set) without using the finite invariant measure.

A measurable set A is said to be finitely T-equivalent to a measurable set B if there exist a finite partition $A_1, A_2, \ldots\ldots, A_k$ of A, a finite partition $B_1, B_2, \ldots\ldots, B_k$ of B and k integers $n_1, n_2, \ldots\ldots, n_k$ such that $T^{n_i}A_i = B_i$, $i=1, 2, \ldots\ldots$. We define set functions σ_n, $n=1, 2, \ldots\ldots$ and $\underline{\sigma}$ as follows;

$$\sigma_n(A) = \frac{1}{n}\sum_{i=1}^{n} P(T^i A), \qquad n=1, 2, \ldots\ldots,$$
$$\underline{\sigma}(A) = \varliminf_{n\to\infty} \sigma_n(A), \qquad A \in \mathscr{F}.$$

LEMMA 5. 1) $\underline{\sigma}$ is a super-additive set function, *i.e.*, for any mutually disjoint, countable sets A_i, $i=1, 2, \ldots\ldots$,

$$\underline{\sigma}\left(\sum_{i=1}^{\infty} A_i\right) \geqq \sum_{i=1}^{\infty} \underline{\sigma}(A_i).$$

2) If A is finitely T-equivalent to B, $\underline{\sigma}(A) = \underline{\sigma}(B)$.

PROOF. 1) Since σ_n is finitely additive, *i.e.*, $\sigma_n\left(\sum_{i=1}^{m} A_i\right) = \sum_{i=1}^{m} \sigma_n(A_i)$, $m=1, 2, \ldots\ldots$, $n=1, 2, \ldots\ldots$, we have

$$\sum_{i=1}^{m} \underline{\sigma}(A_i) = \sum_{i=1}^{m} \varliminf_{n\to\infty} \sigma_n(A_i) \leq \varliminf_{n\to\infty} \sigma_n\left(\sum_{i=1}^{m} A_i\right)$$
$$= \underline{\sigma}\left(\sum_{i=1}^{m} A_i\right) \leq \underline{\sigma}\left(\sum_{i=1}^{\infty} A_i\right), \quad m = 1, 2, \ldots\ldots.$$

2) Let $A = \sum_{i=1}^{k} A_i$, $B = \sum_{i=1}^{k} B_i$ and $T^{n_i}A_i = B_i$, $i=1, 2, \ldots\ldots, k$. For any integer n larger than $\max\{n_i \mid 1 \leqq i \leqq k\}$ we have

$$|\sigma_n(A) - \sigma_n(B)| = \frac{1}{n}\left|\sum_{i=1}^{n-1} P(T^i A) - \sum_{i=1}^{n-1} P(T^i B)\right|$$
$$= \frac{1}{n}\left|\sum_{i=1}^{n-1}\sum_{j=1}^{k} \{P(T^i A_j) - P(T^i B_j)\}\right|$$
$$= \frac{1}{n}\left|\sum_{j=1}^{k}\left\{\sum_{1=i}^{n-1} P(T^i A_j) - \sum_{i=1}^{n-1} P(T^{i+n_j} A_j)\right\}\right|$$
$$= \frac{1}{n}\left|\sum_{j=1}^{k}\left\{\sum_{i=1}^{n_1} P(T^i A_j) - \sum_{i=n}^{n-1+n_j} P(T^i A_j)\right|\right.$$
$$\leqq \frac{1}{n}\sum_{j=1}^{k} 2n_j P(\Omega).$$

Since the last term tends to 0 as $n\to\infty$, we have 2). *q.e.d.*

LEMMA 6. If Ω is T-unbounded there exists a measrable set E such that $P(E)>0$ and $\underline{\sigma}(E)=0$.

PROOF. Assume that $\Omega=\sum_{i=1}^{\infty}A_i$, $T^{n_i}A_i\cap T^{n_j}A_j=\phi$, $i\neq j$, i, $j=1,2,\ldots\ldots$ and $P\left(\Omega-\sum_{i=1}^{\infty}T^{n_i}A_i\right)>0$. Put $E_1=\Omega-\sum_{i=1}^{\infty}T^{n_i}A_i$ and $E_n=\left(\Omega-\sum_{j=1}^{n-1}E_j\right)-\sum_{i=1}^{\infty}T^{n_i}\left(\left(\Omega-\sum_{j=1}^{n-1}E_j\right)\cap A_i\right)$, $n=2,3,\ldots\ldots$ then $\{E_n|n=1,2,\ldots\ldots\}$ are mutually disjoint and mutually countably T-equivalent. Since $\sum_{n=1}^{\infty}\underline{\sigma}(E_n)\leqq\underline{\sigma}(\Omega)<+\infty$, $\lim_{n\to\infty}\underline{\sigma}(E_n)=0$. Since E_1 is countably T-equivalent to E_n, $n=2,3,\ldots\ldots$ there exist a subset E_{1n} of E_1 and a subset E_{n1} of E_n such that E_{1n} is finitely T-equivaent to E_{n1} and $P(E_1-E_{1n})<\frac{\alpha}{2^n}$, $n=2,3,\ldots\ldots$ where $P(E_1)=\alpha>0$. Put $E=\bigcap_{n=2}^{\infty}E_{1n}$ the we have

$$P(E)\geqq P(E_1)-\sum_{n=2}^{\infty}P(E_1-E_{1n})\geqq\alpha-\frac{\alpha}{2}>0 \qquad \text{and}$$

$$\underline{\sigma}(E)\leqq\underline{\sigma}(E_{1n})=\underline{\sigma}(E_{n1})\leqq\underline{\sigma}(E_n)\,,\quad n=2,3,\ldots\ldots\,.$$

Since the last term tends to 0 as $n\to\infty$ we have $\underline{\sigma}(E)=0$. *q.e.d.*

LEMMA 7. (Hajian-Kakutani [9]) If $P(E)>0$ and $\lim_{n\to\infty}P(T_nE)=0$, there exists a weakly wandering subset of E.

PROOF. Put $P(E)=\alpha>0$, $\alpha_i=\frac{\alpha}{i2^{i+1}}$, $i=1,2,\ldots\ldots$ and $n_0=0$. Since $\underline{\lim}_{n\to\infty}P(T^nE)=0$ we obtain a positive integer n_1 such as $P(T^{n_1}E)<\alpha_1$. By the same discussion we obtain an increasing sequence of integers n_i, $i=1,2,\ldots$ such as $P(T^{n_i-n_j}E)<\alpha_i$, $j=0,1,\ldots\ldots,\ i-1$, $i=1,2,\ldots\ldots$. Put $A=\bigcup_{i=1}^{\infty}\bigcup_{j=0}^{i-1}T^{n_i-n_j}E\cap E$, then $P(E-A)>\frac{\alpha}{2}>0$ and $E-A$ is a weakly wandering set. *q.e.d.*

PROPOSITION 2. Let T be a non-singular transformation of a measure space $(\Omega,\mathscr{F},P)$. The set Ω is T-bounded if and only if there does not exist a weakly wandering set.

PROOF. Assume that E is a weakly wandering set. Then there exist a sequence of integers $n_1,n_2,\ldots\ldots,n_i,\ldots\ldots$ such that $T^{n_i}E\cap T^{n_j}E=\phi$, $i\neq j$. We have $S\left(\bigcup_{i=1}^{\infty}T^{n_i}E\right)=\bigcup_{i=2}^{\infty}T^{n_i}E$ where $S\omega=T^{n_{i+1}-n_i}\omega$ for $\omega\in T^{n_i}E$, $i=1,2,\ldots\ldots$. So, $\bigcup_{i=1}^{\infty}T^{n_i}E$ is T-unbounded and then Ω is T-unbounded. If Ω is T-unbounded, there exists a weakly wandering set by Lemma 6 and Lemma 7. *q.e.d.*

§ 5. Associated flow and σ-finite invariant measure.

In this section we use the terminologies "Lebesgue measure space" and "measurable partition" in the sense of Rohlin [22].

Let T be a non-singular transformation of a Lebesgue measure space $(\Omega, \mathscr{F}, P)$. We define a non-singular transformation T of the product measure space $(\Omega\times R, \mathscr{F}\times\mathscr{B}(R), dP\times du)$ as follows;

$$\tilde{T}(\omega, u)=\left(T\omega, u+\log\frac{dPT}{dP}(\omega)\right), \quad (\omega, u)\in\Omega\times R.$$

Let $\zeta(\tilde{T})$ be the measurable partition which generates all $\tilde{T}$-invariant measurable subsets $\Omega\times R$. For $-\infty<s<+\infty$, put $\theta_s(\omega, u)=(\omega, u+s)$, $(\omega, u)\in\Omega\times R$, then the flow $\{\theta_s\}_{-\infty<s<+\infty}$ commute with $\tilde{T}$, *i.e.*, for $-\infty<s<+\infty$,

$$\theta_s\tilde{T}(\omega, u)=\tilde{T}\theta_s(\omega, u),\ (\omega, u)\in\Omega\times R.$$

We can define the factor flow $\{\tilde{\theta}_s\}_{-\infty<s<+\infty}$ of $\{\theta_s\}_{-\infty<s<+\infty}$ on the quotient measure space $\Omega\times R/\zeta(\tilde{T})$. This is a measurable, non-singular flow with respect to any σ-finite measure which is equivalent to the image measure of $dP\times du$.

DEFINITION 2. We call the factor flow $\{\tilde{\theta}_s\}_{-\infty<s<+\infty}$ the non-singular flow associated with the non-singular transformation T, or simply the associated flow of T.

We note that the associated flow $\{\tilde{\theta}_s\}_{-\infty<s<+\infty}$ of T is ergodic if and only if T is ergodic.

THEOREM 1. If non-singular transformations T and T' of Lebesgue measure spaces $(\Omega, \mathscr{F}, P)$ and $(\Omega', \mathscr{F}', P')$, respectively, are mutually weakly equivalent, then their associated flows are mutually strongly equivalent.

PROOF. Let ψ be a 1-1 mapping from Ω onto Ω' such that $\psi\mathscr{F}=\mathscr{F}'$, $\psi P\sim P'$ and $\psi[T]=[T']\psi$. Put $\tilde{\psi}(\omega, u)=\left(\psi\omega, u+\log\frac{dP'\psi}{dP}(\omega)\right)$. Then $\tilde{\psi}$ is a 1-1 mapping from $\Omega\times R$ onto $\Omega'\times R$ and satisfies that for $-\infty<s<+\infty$, $\tilde{\psi}\theta_s(\omega, u)=\theta_s\tilde{\psi}(\omega, u)$, *a.e.* $(\omega, u)\in\Omega\times R$. It is enough to show that $f(\tilde{\psi}(\omega, u))$ is a $[T]$-invariant measurable function for any $[T']$-invariant measurable function. For $S\in[T]$ and $S'=\psi S\psi^{-1}\in[T']$,

$$f\left(\tilde{\psi}\left(S\omega, u+\log\frac{dPS}{dP}(\omega)\right)\right)=f\left(\psi S\omega,\ u+\log\frac{dPS}{dP}(\omega)+\log\frac{dP'\psi}{dP}(S\omega)\right)$$

$$= f\left(S'\psi\omega,\ u+\log\frac{dPS}{dP}(\omega)+\log\frac{dP'S'\psi}{dPS}(\omega)\right)$$

$$= f\left(S'\psi\omega,\ u+\log\frac{dP'S'}{dP'}(\psi\omega)+\log\frac{dP'\psi}{dP}(\omega)\right)$$

$$= f\left(\tilde{S}'\left(\psi\omega,\ u+\log\frac{dP'\psi}{dP}(\omega)\right)\right)$$

$$= f\,(\psi(\omega,\ u))\ .$$

q.e.d.

THEOREM 2. Let T be a non-singular transformation of a Lebesgue measure space $(\Omega, \mathscr{F}, P)$. The following three conditions are equivalent.

(1) There exists a σ-finite T-invariant measure equivalent to P.

(2) There exists a measurabl function $f(\omega)$ such that

$$\tilde{T}\{(\omega,\ u+f(\omega))\,|\,\omega\in\Omega\} = \{(\omega,\ u+f(\omega))\,|\,\omega\in\Omega\}\quad \text{for}\quad -\infty<u<+\infty$$

(3) The associated flow $\{\tilde{\theta}_s\}_{-\infty<s<+\infty}$ of T is strongly equivalent to a flow of translation type.

PROOF. Let μ be a σ-finite T-invariant measure which is equivalent to P. Put $f(\omega)=-\log\frac{d\mu}{dP}(\omega)$ we have

$$f(\omega)+\log\frac{dPT}{dP}(\omega) = -\log\frac{d\mu}{dP}(\omega)+\log\frac{dPT}{d\mu T}(\omega)+\log\frac{d\mu T}{d\mu}(\omega)+\log\frac{d\mu}{dP}(\omega)$$

$$= \log\frac{dPT}{d\mu T}(\omega)$$

$$= f(T\omega)\ ,$$

which means that $f(\omega)$ satisfies (2). Similarly let $f(\omega)$ be a function satisfying (2) and let μ be a measure equivalent to P such that $\frac{d\mu}{dP}(\omega)=\exp(-f(\omega))$, then we have $\log\frac{d\mu T}{d\mu}(\omega)=0$ which means that μ is a T-invariant measure.

Let $f(\omega)$ be a measurable function satisfying the condition (2) and let $\zeta(T)$ be the measurable partition of Ω which generates all T-invariant subsets of Ω. Then

$$\zeta(\tilde{T}) = \{\{(\omega,\ u+f(\omega))\,|\,\omega\in C_{\zeta(T)}\}\,|\,-\infty<u<\infty\ ,\ C_{\zeta(T)}\in\zeta(T)\}\ .$$

It is easy to see that the associated flow $\{\tilde{\theta}_s\}_{-\infty<s<+\infty}$ of T is strongly equivalent to the flow $\{\phi_s\}_{-\infty<s<+\infty}$ such that

$$\phi_s(x,\ u) = (x,\ u+s)\ ,\ (x,\ u)\in(\Omega/\zeta(T))\times\mathrm{R}\ .$$

Let ψ be a strongly equivalent mapping from the associated flow $\{\tilde{\theta}_s\}_{-\infty<s<+\infty}$ of T to a flow of translation type on a product measure space $X\times R$, p be the projection mapping from $X\times R$ onto R such as $p(x, u)=u$, $(x, u)\in X\times R$ and let π be the canonical mapping from $\Omega\times R$ onto the quotient space $\Omega\times R/\zeta(T)$. We have

$$\begin{aligned} p\psi\pi(\omega, \mu) &= p\psi\pi\theta_u(\omega, 0) \\ &= p\psi\theta_u\pi(\omega, 0) \\ &= p\psi\pi(\omega, 0)+u, \ (x, u)\in\Omega\times R. \end{aligned}$$

Then for almost all $\omega\in\Omega$, there uniquely exists a real number u such as $p\psi\pi(\omega, u)=0$ and we denote it by $f(\omega)$. Furthermore we have $\pi^{-1}\psi^{-1}p^{-1}(\{0\})=\{(\omega, f(\omega))\,|\,\omega\in\Omega\}$. From $\pi\tilde{T}=\pi$, we have

$$\begin{aligned} \tilde{T}\{(\omega, f(\omega))\,|\,\omega\in\Omega\} &= \tilde{T}\pi^{-1}\psi^{-1}p^{-1}(\{0\}) \\ &= \pi^{-1}\psi^{-1}p^{-1}(\{0\}) \\ &= \{(\omega, f(\omega))\,|\,\omega\in\Omega\}. \end{aligned}$$

This completes the proof.

q.e.d.

LEMMA 8. Let T be a non-singular transformation of a Lebesgue measure space $(\Omega, \mathscr{F}, P)$. If there exists a positive number K such as $\left|\log\frac{dPT^n}{dP}(\omega)\right|<K$, $-\infty<n<+\infty$, *a.e.* $\omega\in\Omega$, then there exists a finite invariant measure μ equivalent to P.

PROOF. For a positive number a let $A=\{(\omega, u)\,|\,|u|<a\}$ and $\tilde{A}=\bigcup_{n=-\infty}^{\infty}\tilde{T}^nA$. From the asumption we have

$$\tilde{A}\subset\{(\omega, u)\,|\,|u|<a+K\}.$$

This means that for $|s|>2(a+K)$, $\theta_sA\cap A=\phi$. Since $\bigcup_{s=-\infty}^{\infty}\theta_sA=\Omega\times R$, we can easily show that the associated flow $\{\tilde{\theta}_s\}_{-\infty<s<+\infty}$ is strongly equivalent to a flow of translation type. Furthermore, since the function in (2) is bounded the T-invariant measure μ is finite.

q.e.d.

PROPOSITION 3. (L. Arnold [1]) Let T be a non-singular transformation of a Lebesgue measure space $(\Omega, \mathscr{F}, P)$. There exists a σ-finite T-invariant measure μ equivalent to P if and only if for every $\varepsilon>0$ there exists a countable decomposition $\{\Omega_i\}$ of Ω with the following properties; for each i, for each measurable subset B of Ω_i and for each integer n such that $T^nB\subset\Omega_i$, it holds that

$$\frac{1}{1+\varepsilon} < \frac{dPT^n}{dP}(\omega) < 1+\varepsilon\,, \quad a.e.\ \omega \in B\,.$$

PROOF. Assume that there exists a σ-finite T-invariant measure equivalent to P and let $f(\omega)$ be the function of (2) of Theorem 2. Put $\Omega_i = \{\omega \mid i \log(1+\varepsilon) \leqq f(\omega) < (i+1) \log(1+\varepsilon)\}$, $i=0, \pm 1, \ldots\ldots$, the decomposition $\{\Omega_i\}$ of Ω satisfies the desired property. Conversely, for each i the induced transformation T_{Ω_i} on Ω_i of T has a finite invariant measure by Lemma 8. Therefore T has a σ-finite T-invariant measure which is equivalent to P. *q.e.d.*

§6. Characterization of flows of translation type and existence of σ-finite invariant measure.

We shall use the following well-known results to prove Theorem 3.

LEMMA 9. ([4]) Let Γ be a polish group and Γ_0 be a closed subgroup of Γ. Then there exists a Borel subset B of Γ such that B intersects each right Γ_0-coset in exactly one point.

LEMMA 10. ([22]) Let ζ be a measurable partition of a Lebesgue space $(X, \mathscr{B}, m)$. Then there exists a measurable subset M of X such that M intersects each element of ζ in exactly one point mod 0.

LEMMA 11. ([19]) Let ψ be a 1-1 measurable mapping whose domain is a standerd Borel space and whose range is contained in a measurable space with a countable and separating base. Then the range of ψ is a measurable subset and ψ^{-1} is also measurable.

LEMMA 12. ([18]) Let $(\Omega, \mathscr{F}, P)$ and $(\Omega', \mathscr{F}', P')$ be σ-finite measure spaces and $f(\omega, \omega')$ be a bounded function such that

1) for each ω, it is $\mathscr{F}'$-measurable and P'-integrable,

2) for each $E' \in \mathscr{F}'$, $\int_{E'} f(\omega, \omega') dP'(\omega')$ is $\mathscr{F}$-measurable. Then there exists an $\mathscr{F} \times \mathscr{F}'$-measurable function $\tilde{f}(\omega, \omega')$ such that for almost all ω, $f(\omega, \omega') = \tilde{f}(\omega, \omega')$ except a ω'-null set.

LEMMA 13. Let $\xi(t)$ be a real measurable function defined on R such that

$$\exp i\xi(t+\tau) = \exp i\xi(t) \cdot \exp i\xi(\tau)\,, \quad a.e.(t, \tau)\,,$$

then there exists a real constant c such that

$$\exp i\xi(t) = \exp ict\,, \quad a.e.t\,.$$

Proof is omitted, because it is easy.

LEMMA 14. Let $\{\phi_s\}_{-\infty<s<+\infty}$ be a measurable, non-singular flow of a probability space $(X, \mathscr{B}, m)$. Then

(1) $\displaystyle\int\left|f(\phi_s x)\sqrt{\frac{dm\phi_s}{dm}(x)}-f(\phi_{s_0}x)\sqrt{\frac{dm\phi_{s_0}}{dm}(x)}\right|^2 dm(x)\to 0$

as $s\to s_0$ for $f\in L^2(X, \mathscr{B}, m)$.

(2) $\displaystyle\int|h(\phi_s x)-h(\phi_{s_0}x)|^2 dm(x)\to 0$

as $s\to s_0$ for $h\in L^\infty(X, \mathscr{B}, m)$.

PROOF. (1) We put $\alpha(s, x)=\dfrac{dm\phi_s}{dm}(x)$ and for $N>0$ $\alpha_N(s, x)=\alpha(s, x)$ if $0<\alpha(s, x)\leq N$ and $\alpha_N(s, x)=N$ if $\alpha(s, x)>N$ which are (s, x)-measurable functions. We denote by U_s the unitary operator $(U_s f)(x)=f(\phi_s x)\sqrt{\alpha(s, x)}$. For $|s|<1$ and $f\in L^\infty(X, \mathscr{B}, m)$ with $|f(x)|<C$,

$$\|U_s f(x)-f(x)\|_{L^2(m)}=\int_0^1\|U_{s+u}f(x)-U_u f(x)\|_{L^2(m)}du$$

$$\leq 2C\int_{-1}^2\|\sqrt{\alpha(u, x)}-\sqrt{\alpha_N(u, x)}\|_{L^2(m)}du$$

$$+\int_0^1\|f(\phi_{s+u}x)\sqrt{\alpha_N(s+u, x)}-f(\phi_s x)\sqrt{\alpha_N(s, x)}\|_{L^2(m)}du\,.$$

For any $\varepsilon>0$ there exists N such that

$$\int_{-1}^2\|\sqrt{\alpha(u, x)}-\sqrt{\alpha_N(u, x)}\|_{L^2(m)}du<\varepsilon\,.$$

From the Fubini's theorem

$$\int_0^1\|f(\phi_{s+u}x)\sqrt{\alpha_N(s+u, x)}-f(\phi_u x)\sqrt{\alpha_N(u, x)}\|_{L^2(m)}du$$

$$\leq\left\{\int dm(x)\int_0^1|f(\phi_{s+u}x)\sqrt{\alpha_N(s+u, x)}-f(\phi_u x)\sqrt{\alpha_N(u, x)}|^2du\right\}^{\frac{1}{2}}.$$

From the Riemann-Lebesgue's theorem

$$\int_0^1|f(\phi_{s+u}x)\sqrt{\alpha_N(s+u, x)}-f(\phi_u x)\sqrt{\alpha_N(u, x)}|^2du\to 0$$

as $s\to 0$, $a.e.x.$

Therefore from the Lebesgue's convergence theorem

$$\int dm(x)\int_0^1|f(\phi_{s+u}x)\sqrt{\alpha_N(s+u, x)}-f(\phi_u x)\sqrt{\alpha_N(u, x)}|^2du\to 0$$

as $s \to 0$. Since U_s is a unitary operator,

$$\| U_s f - f \|_{L^2(m)} \to 0 \quad \text{as} \quad s \to 0$$

for $f \varepsilon L^2(x, \mathscr{B}, m)$.

(2) For $h \varepsilon L^\infty(x, m)$ with $|h(x)| < C$,

$$\begin{aligned} & \| h(\phi_{s+s_0}x) - h(\phi_{s_0}x) \|_{L^2(m)} \\ & \leqq \| h(\phi_{s+s_0}x) - U_s h(\phi_{s_0}x) \|_{L^2(m)} + \| U_s h(\phi_{s_0}x) - h(\phi_{s_0}x) \|_{L^2(m)} \\ & \leqq C \| 1 - U_s 1 \|_{L^2(m)} + \| U_s h(\phi_{s_0}x) - h(\phi_{s_0}x) \|_{L^2(m)} . \end{aligned}$$

From (1), we have

$$\| h(\phi_s x) - h(\phi_{s_0}x) \|_{L^2(m)} \to 0 \quad \text{as} \quad s \to s_0 .$$

q.e.d.

THEOREM 3. Let $\{\phi_s\}_{-\infty<s<+\infty}$ be a measurable non-singular flow of a Lebesgue measure space $(X, \mathscr{B}, m)$. Then it is strongly equivalent to a flow of translation type if and only if $\sigma(\{\phi_s\}) = R$.

PROOF. We note the set of all point spectra is invariant under the strong equivalence of non-singular flows. Let $\{\phi_s\}_{-\infty<s<+\infty}$ be a flow of translation type of a product space $Y \times R$. For any $t \varepsilon R$ put $\xi(y, u) = tu$, then we have

$$\exp i\xi\,(\phi_s(y, u)) = \exp its \cdot \exp i\xi(y, u), \quad (y, u) \varepsilon Y \times R, \qquad {}_{-\infty<s<+\infty} .$$

This means that $\sigma(\{\phi_s\}) = R$.

Assume that $\sigma(\{\phi_s\}) = R$. First we show that there exists a (t, x)-measurable function $\exp i\xi(t, x)$ such that for ${}_{-\infty<s<+\infty}$

$$(*) \qquad \exp i\xi(t, \phi_s x) = \exp its \cdot \exp i\xi(t, x), \quad a.e. \ (t, x) .$$

We assume that m is a probability measure. Let Γ be the set of all complex valued measurable functions with absolute value 1 on $(X, \mathscr{B}, m)$ and Γ_0 be the set of all $\{\phi_s\}$-invariant functions of Γ. Γ is a complete separable metric space under the relative $L^2(m)$-topology on Γ. Under the multiplication, Γ is a topological group with respect to this topology and Γ_0 is its closed subgroup. From Lemma 9 there exists a Borel subset B or Γ which intersects each coset of the guotient space Γ/Γ_0 in exactly one point. We denote by π the canonical mapping from Γ onto Γ/Γ_0 and denote by $\pi|_B$ the restriction to B. For each $-\infty < s < +\infty$ and $E \varepsilon \mathscr{B}$ we denote by $\tau_{s,E}$ a function

$$\exp i\xi(\cdot)\Gamma_0 \to \int_E \exp i\{\xi(\phi_s x) - \xi(x)\} \sqrt{\frac{dm\phi_s}{dm}(x)}\, dm(x)$$

defined on Γ/Γ_0. Since

$$\int_E \{\exp i\{\xi'(\phi_s x)-\xi'(x)\}-\exp i\{\xi(\phi_s x)-\xi(x)\}\}\sqrt{\frac{dm\phi_s}{dm}(x)}\,dm(x)$$

$$\leqq 2\left(\int |\exp i\xi'(x)-\exp i\xi(x)|^2 dm(x)\right)^{\frac{1}{2}}$$

the function $\tau_{s,E}\cdot\pi|_B$ defined on B is continuous under the relative $L^2(m)$-topology on B. Let $\mathcal{E}$ be the smallest σ-algebra of Γ/Γ_0 such that every function $\tau_{s,E}$, $-\infty<s<+\infty$, $E\in\mathscr{B}$ is measurable. We prove that $\mathcal{E}$ has a countably separating base. It is enough to show that for a countably separating base $\{E_n\}_{n\geq 1}$ of $\mathscr{B}$ and a countable dense set $\mathcal{K}$ of R, $\mathcal{E}$ is generated by $\{\tau_{s,E_n}|s\in\mathcal{K},\ n\geqq 1\}$. From Lemma 14, for $s\in R$ and $E\in\mathscr{B}$, there exist $s_n\in\mathcal{K}$ and E_{m_n} such that

$$\tau_{s_n,E_{m_n}}(\exp i\xi(\cdot)\Gamma_0)\to\tau_{s,E}(\exp i\xi(\cdot)\Gamma_0)\ ,\quad \text{as}\quad n\to\infty\ .$$

Since $\tau_{s,E}\cdot\pi|_B$, $-\infty<s<+\infty$, $E\in\mathscr{B}$ is continuous, $\pi|_B$ is measurable under the σ-algebra generated by the relative $L^2(m)$-topology of B and the σ-algebra $\mathcal{E}$. From Lemma 11 the inverse mapping $\pi|_B^{-1}$ is also measurable. For each $t\in R$, let Γ_t be the set of all measurable solutions $\exp i\xi(\cdot)$ of the equation, for $-\infty<s<+\infty$,

$$\exp i\xi(\phi_s x)=\text{expits}\cdot\exp i\xi(x)\ ,\quad a.e.x\ .$$

Then Γ_t is a coset in Γ/Γ_0. By α we denote a mapping $t\to\Gamma_t$ from R into Γ/Γ_0. Since the function

$$\tau_{s,E}\cdot\alpha(t)=\int_E \text{expits}\sqrt{\frac{dm\phi_s}{dm}(x)}\,dm(x)$$

is t-continuous for each $-\infty<s<+\infty$, $E\in\mathscr{B}$, the mapping α is measurable. For each $E\in\mathscr{B}$ we denote by γ_E a function

$$\exp i\xi(\cdot)\to\int_E \exp i\xi(x)\ dm(x)$$

defined on B. It is continuous under the $L^2(m)$-topology. Since α, $\pi|_B^{-1}$ and γ_E are all measurable, the function $\gamma_E\cdot\pi|_B^{-1}\cdot\alpha(t)=\int_E \exp i\xi_t(x)dm(x)$ is t-measurable for each $E\in\mathscr{B}$, where $\exp i\xi_t(\cdot)=\pi|_B^{-1}\Gamma_t$. From Lemma 12 there exists a (t,x)-measurable function $\exp i\xi(t,x)$ such that for almost all t, $\exp i\xi(t,x)=\exp i\xi_t(x)$ holds except an x-null set. Then the function $\exp i\xi(t,x)$ satisfies the equation (*).

From Lemma 10 there exists a measurable subset M of X such that M intesects each element of a measurable partition $\zeta(\{\phi_s\})$ which generates all $\{\phi_s\}$-invariant subsets of X, in exactly one point mod 0. We denote by V the canonical mapping from X onto M. Then the function

$$\exp i\bar{\xi}(t,x)=\exp i\{\xi(t,x)-\xi(t,Vx)\}$$

belongs to the coset Γ_t and satisfies for almost all x

$$\exp i\bar{\xi}(t+\tau, x) = \exp i\bar{\xi}(t, x)\cdot\exp i\bar{\xi}(\tau, x)\ , \quad a.e.\ (t, \tau)\ .$$

From Lemma 13 there exists a real measurable function $\xi(x)$ such that for almost all x

$$\exp i\bar{\xi}(t, x) = \exp it\xi(x)\ , \quad a.e.t\ .$$

Since $\exp i\xi(t, \cdot)$ belongs to the coset Γ_t, we have for $-\infty<s<+\infty$ and for almost all x

$$\exp it\xi(\phi_s x) = \exp its\cdot\exp it\xi(x)\ , \quad a.e.t\ .$$

Therefore we have for $-\infty<s<+\infty$

$$\xi(\phi_s x) = \xi(x)+s\ , \quad a.e.x\ .$$

Define a mapping ψ from X onto $M\times R$ such that

$$\psi x = (Vx, \xi(x))\ , \quad x\in X\ ,$$

then ψ is 1-1, bi-measurable and for $-\infty<s<+\infty$

$$\psi\phi_s x = (V\phi_s x, \xi(\phi_s x)) = (Vx, \xi(x)+s)\ .$$

This completes the proof.

q.e.d.

THEOREM 4. Let T be a non-singular transformation of a Lebesgue measure space $(\Omega, \mathscr{F}, P)$. Then there exists a σ-finite T-invariant measure which is equivalent to P if and only if for any real number t there exists a real measurable function $\xi(\omega)$ such that

$$\exp i\xi(T\omega) = \exp it\log\frac{dPT}{dP}(\omega)\cdot\exp i\xi(\omega)\ , \quad a.e.\omega\ .$$

PROOF. Let t be a point spectrum of the asociated flow $\{\phi_s\}_{-\infty<s<+\infty}$ of T. Then there exists a $\tilde{T}$-invariant measurable function $\xi(\omega, u)$ such that for $-\infty<s<+\infty$

$$\exp i\xi(\omega, u+s) = \exp its\cdot\exp i\xi(\omega, u)\ , \quad a.e.(\omega, u)\ .$$

Using the $\tilde{T}$-invariance of the function we have

$$\exp i\xi(\omega, u) = \exp i\xi\Big(T\omega,\ u+\log\frac{dPT}{dP}(\omega)\Big)$$

$$= \exp it\log\frac{dPT}{dP}(\omega)\cdot\exp i\xi(T\omega, u)\ ,\ a.e.\ (\omega, u)\ .$$

This means that $-\xi(\omega, u)$ satisfies the equation of Theorem 4 for almost all u.

Conversely, let $\xi(\omega)$ be a measurable function satisfying the equation of Theorem 4. The function $\exp i\{tu-\xi(\omega)\}$ is $\tilde{T}$-invariant and satisfies the equation which means that t is a point spectrum of the associated fiow $\{\phi_s\}_{-\infty<s<+\infty}$. Then the proof follows frow Theorem 2 and Theorem 3.

q.e.d.

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